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Canonical parametrizations of metric surfaces of higher topology

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Abstract. We give an alternate proof to the following generalization of the uniformization theorem by Bonk and Kleiner. Any linearly locally connected and Ahlfors 2-regular closed metric surface is quasisymmetrically equivalent to a model surface of the same topology. Moreover, we show that this is also true for surfaces as above with non-empty boundary and that the corresponding map can be chosen in a canonical way. Our proof is based on a local argument involving the existence of quasisymmetric parametrizations for metric discs as shown in a paper of Lytchak and Wenger.

Korkeamman topologian metristen pintojen kanoninen parametrisointi

Tiivistelmä. Esitämme vaihtoehtoisen todistuksen seuraavalle Bonkin ja Kleinerin yhdenmukaisuuslauseelle: jokainen lineaarisesti paikallisesti yhtenäinen ja Ahlforsin 2-säännöllinen suljettu metrinen pinta on kvasisymmetrisesti yhtäpitävä sellaisen mallipinnan kanssa, jolla on sama topologia. Lisäksi osoitamme, että tämä pätee myös pinnoille, joilla on epätyhjä reuna, ja että vastaava kuvaus voidaan valita kanonisesti. Todistuksemme perustuu paikalliseen tarkasteluun, joka hyödyntää Lytchakin ja Wengerin osoittamaa metristen kiekkojen kvasisymmetristen parametrisointien olemassaoloa.

1. Introduction and statement of main results

1.1. Introduction. The classical uniformization theorem states that any oriented Riemannian 2-manifold is conformally diffeomorphic to a model surface of constant curvature. The corresponding map provides a canonical parametrization of said Riemannian surface. An appropriate generalized notion of conformal diffeomorphisms in a non-smooth setting is given by quasisymmetric mappings. A homeomorphism $f\colon X \to Y$ between metric spaces is *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$d_Y(f(x), f(y)) \le \eta(t) \cdot d_Y(f(x), f(z))$$

for all points $x, y, z \in X$ with $d_X(x,y) \le t \cdot d_X(x,z)$. The quasisymmetric uniformization problem in the field of analysis on metric spaces then asks under which conditions on a metric space X topologically equivalent to some model space M one may identify X with M via a quasisymmetric homeomorphism.

A breakthrough result due to Bonk and Kleiner [3] asserts that if X is an Ahlfors 2-regular metric space homeomorphic to the 2-sphere S^2, then there exists a quasisymmetric homeomorphism between X and S^2 if and only if X is linearly locally connected. For definitions of Ahlfors 2-regularity and linear local connectedness we refer to Section 2.1.

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Lytchak and Wenger provide in [16] an alternate proof of the theorem of Bonk–Kleiner using a theory of energy and area minimizing discs in metric spaces admitting a quadratic isoperimetric inequality established in [13, 15]. The aim of this paper is to use the existence result in [16] locally to obtain canonical parametrizations of metric surfaces of higher topology with possibly non-empty boundary.

Let X be a metric space homeomorphic to a smooth surface M. Here, a *smooth surface* refers to a smooth compact oriented and connected Riemannian 2-manifold with possibly non-empty boundary. Define $\Lambda(M,X)$ to be the family of Newton–Sobolev maps $u \in N^{1,2}(M,X)$ such that u is a uniform limit of homeomorphisms from M to X and let $E_+^2(u,g)$ be the *Reshetnyak energy* of a map $u \in N^{1,2}(M,X)$ with respect to the Riemannian metric g; for definitions see Section 2.2. Our main result is the following version of [16, Theorem 1.1] for metric surfaces of higher topology. Note that the definition of $\Lambda(M,X)$ is different from [16].

Theorem 1.1. *Let X be a geodesic metric space which is Ahlfors 2-regular, linearly locally connected and homeomorphic to a smooth surface M. Then, there exist a map $u \in \Lambda(M,X)$ and a Riemannian metric g on M such that*

$$E_+^2(u,g) = \inf\{E_+^2(v,h)\colon v \in \Lambda(M,X),\ h \text{ a smooth Riemannian metric on } M\}.$$

Any such u is a quasisymmetric homeomorphism from M to X and the pair (u,g) is uniquely determined up to a conformal diffeomorphism $\varphi\colon (M,g) \to (M,h)$.

Moreover, the metric g can be chosen to be of constant sectional curvature -1, 0 or 1 and such that ∂M is geodesic (if non-empty). Note that the assumption of X being geodesic is natural and can be dropped if X is closed, see Remark 2.3.

The theorem of Bonk–Kleiner has been extended for example in [20, 22, 16, 17, 19]. In the setting of X being an Ahlfors 2-regular and linearly locally connected metric surface, there exist quasisymmetric uniformization results if $X \backslash \partial X$ is a domain in S^2, see [25, 18, 21], and if X is closed, see [7, 10]. Theorem 1.1 is a strengthening of these results in the sense that it states the existence of *canonical* quasisymmetric homeomorphism, regardless of X being closed or having non-empty boundary. A different canonical quasisymmetric homeomorphism was previously only provided by [10] for X being closed. Note that the statement of [10] also holds for non-orientable surfaces. Furthermore, in contrast to some results mentioned above, e.g. [7, 10], we do not obtain a quantitative statement in the sense that the quasisymmetric distortion function is not necessarily controlled by the Ahlfors 2-regularity and linear local connectedness constants of X.

As a corollary of Theorem 1.1, we obtain the following generalization of the result of Bonk–Kleiner, which seems to be new for surfaces having non-empty boundary as well as higher genus.

Corollary 1.2. *Let X be a geodesic Ahlfors 2-regular metric space homeomorphic to a smooth surface M with possibly non-empty boundary. Then, X is quasisymmetrically equivalent to M if and only if X is linearly locally connected.*

1.2. Elements of proof. We briefly sketch some of the arguments needed for proving Theorem 1.1. For arbitrary M and X as in the paragraph before the theorem, the set $\Lambda(M,X)$ can be empty. A crucial step in this work is to show the existence of a map $u \in \Lambda(M,X)$ in the setting of Theorem 1.1.

Proposition 1.3. *Let M be a smooth surface and (X, d) a metric space which is geodesic, Ahlfors 2-regular, linearly locally connected and homeomorphic to M. Then the family $\Lambda(M, X)$ is non-empty and contains a quasisymmetric homeomorphism.*

The proposition follows by a dissection of M and X into appropriate disc-type subdomains, consequently applying [16, Theorem 6.1] yielding quasisymmetric parametrizations for each subdomain in X and finally gluing all these mappings together in order to obtain a global quasisymmetric homeomorphism $M \to X$. Note that Proposition 1.3 already establishes Corollary 1.2. Moreover, this procedure also works for non-orientable surfaces, see Remark 3.9.

The map u provided by Proposition 1.3 is not necessarily canonical, i.e. possibly not of minimal energy. In order to find an energy minimizer in $\Lambda(M, X)$, we will use similar arguments as in the proofs of [16, Theorem 6.1] and [6, Theorem 8.2]. In particular, we need to ensure that a family of mappings in $\Lambda(M, X)$ of uniformly bounded energies is equicontinuous.

The paper is structured as follows. In Section 2 we provide necessary definitions and some results on Newtonian Sobolev spaces that will be of use later on. Section 3 is devoted to the proof of Proposition 1.3. In Section 4 we will show equicontinuity of energy bounded almost homeomorphisms. And finally, the proof of Theorem 1.1 is given in Section 5.

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2. Preliminaries

2.1. Basic definitions and notations. Let (X, d) be a metric space. The *open ball* in X of radius $r > 0$ centered at a point $x \in X$ is denoted by $B_X(x, r)$ or simply $B(x, r)$. Consider the Euclidean space $(\mathbb{R}^2, |\cdot|)$, where $|\cdot|$ is the Euclidean norm. The *open and closed unit discs* in $\mathbb{R}^2$ are given by

$$D := \{z \in \mathbb{R}^2 \colon |z| < 1\}, \qquad \overline{D} := \{z \in \mathbb{R}^2 \colon |z| \le 1\}.$$

An open set $\Omega \subset X$ homeomorphic to the unit disc D is a *Jordan domain in X* if its completion $\overline{\Omega} \subset X$ is homeomorphic to $\overline{D}$. A *Jordan curve* in X is a subset of X homeomorphic to S^1 and it is called *chord-arc* if it is biLipschitz equivalent to S^1. The *image* of a curve c in X is denoted by $|c|$ and the *length* by $\ell_X(c)$ or $\ell(c)$. A curve $c\colon [a, b] \to X$ is called *geodesic* if $\ell(c) = d(c(a), c(b))$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic.

A *metric surface* X is a metric space homeomorphic to a smooth surface M. We say that a metric surface X is of *T-type* if X is homeomorphic to a canonical topological surface T. By ∂M we denote the topological boundary of the smooth surface M, which is homeomorphic to a finite disjoint union of S^1. The boundary of X, denoted ∂X, is the subset of X that is homeomorphic to ∂M.

For $s \ge 0$, we denote the *s-dimensional Hausdorff measure* of a set $A \subset X$ by $\mathcal{H}^s_X(A)$ or simply $\mathcal{H}^s(A)$. The normalizing constant is chosen in such a way that if X is the Euclidean space $\mathbb{R}^n$, the Lebesgue measure agrees with $\mathcal{H}^n_X$. If (M, g) is a Riemannian manifold of dimension n then the n-dimensional Hausdorff measure $\mathcal{H}^n_g := \mathcal{H}^n_{(M,g)}$ on (M, g) coincides with the Riemannian volume.

Let g be a smooth Riemannian metric on a smooth surface M such that the boundary of M is geodesic with respect to g. We call the metric g *hyperbolic* if it is of constant sectional curvature -1, and *flat* if it has vanishing sectional curvature as well as an associated Riemannian 2-volume satisfying $\mathcal{H}^2_g(M) = 1$.

Definition 2.1. A metric space X is said to be *Ahlfors 2-regular* if there exists $K > 0$ such that for all $x \in X$ and $0 < r < \operatorname{diam} X$, we have

$$K^{-1} \cdot r^2 \leq \mathcal{H}^2_X(B(x,r)) \leq K \cdot r^2.$$

Definition 2.2. We say that a metric space X is *linearly locally connected (LLC)* if there exists a constant $\lambda \geq 1$ such that for all $x \in X$ and $r > 0$, every pair of distinct points in $B(x,r)$ can be connected by a continuum in $B(x,\lambda r)$ and every pair of distinct points in $X \setminus B(x,r)$ can be connected by a continuum in $X \setminus B(x, r/\lambda)$.

Here, *a continuum* refers to a non-empty compact connected subset of X.

Remark 2.3. If X is a closed surface, it follows by [3, Lemma 2.5] that linear local connectivity is equivalent to linear local contractibility, meaning that there exists $\lambda \geq 1$ such that every ball $B(x,r)$ of radius $0 < r < \lambda^{-1} \operatorname{diam}(X)$ is contractible in $B(x, \lambda r)$. Now, every Ahlfors 2-regular and linear local contractible metric surface is quasiconvex (see [23, Theorem B.6]) and thus geodesic up to a biLipschitz change of metric.

2.2. Metric space valued Sobolev maps. In this subsection we give a brief overview over some basic concepts used in the theory of metric space valued Sobolev maps based on upper gradients. Note that several other equivalent definitions of Sobolev spaces exist. For more details consider e.g. [9].

Let (X,d) be a complete metric space and M a smooth compact 2-dimensional manifold, possibly with non-empty boundary. Fix a Riemannian metric g on M and consider a domain $\Omega \subset M$. Let $u\colon \Omega \to X$ be a map and $\rho\colon \Omega \to [0,\infty]$ a Borel function. Then, ρ is called *(weak) upper gradient of u with respect to g* if

$$d(u(\gamma(a)), u(\gamma(b))) \leq \int_\gamma \rho(s)\, ds \tag{1}$$

for (almost) every rectifiable curve $\gamma\colon [a,b] \to \Omega$. A weak upper gradient ρ of u is said to be *minimal* if $\rho \in L^2(\Omega)$ and for every weak upper gradient ρ' of u in $L^2(\Omega)$ it holds that $\rho \leq \rho'$ almost everywhere on Ω. Denote by $L^2(\Omega, X)$ the family of measurable essentially separably valued maps $u\colon \Omega \to X$ such that the distance function $u_x(z) := d(u(z), x)$ is in the space $L^2(\Omega)$ of 2-integrable functions for some and hence any $x \in X$. A sequence $(u_k) \subset L^2(\Omega, X)$ is said to *converge in $L^2(\Omega, X)$* to a map $u \in L^2(\Omega, X)$ if

$$\int_\Omega d^2(u_k(z), u(z))\, d\mathcal{H}^2_g(z) \to 0$$

as k tends to infinity. The *(Newton–)Sobolev space* $N^{1,2}(\Omega, X)$ is the collection of maps $u \in L^2(\Omega, X)$ such that u has a weak upper gradient in $L^2(\Omega)$. Every such u has a minimal weak upper gradient denoted by ρ_u, which is unique up to sets of measure zero (see e.g. [9, Theorem 6.3.20]). Note also that the definition of $N^{1,2}(\Omega, X)$ is independant of the chosen metric g on M.

Definition 2.4. The *Reshetnyak energy* of a map $u \in N^{1,2}(\Omega, X)$ with respect to g is defined by

$$E^2_+(u, g) := \int_\Omega |\rho_u(z)|^2\, d\mathcal{H}^2_g(z).$$

This definition of energy agrees with the one given in [6, Definition 2.2]; in particular, E_+^2 is invariant under precompositions with conformal diffeomorphisms.

3. Noncanonical quasisymmetric parametrizations

The aim of this section is to prove Proposition 1.3, which strongly depends on the following variant of [16, Theorem 6.1].

Theorem 3.1. *Let X be an Ahlfors 2-regular geodesic metric space homeomorphic to a 2-dimensional manifold. Let $J \subset X$ be a Jordan domain with $\ell(\partial J) < \infty$ and such that $\overline{J}$ is LLC. Then any quasisymmetric homeomorphism $f\colon S^1 \to \partial J$ extends to a quasisymmetric homeomorphism $\overline{f} \in \Lambda(\overline{D}, \overline{J})$.*

Note that the conclusion of Theorem 3.1 also holds if X is not geodesic and the boundary of J is not rectifiable, see [25, Theorem 1.2] and [10, Theorem 1.4].

The proof of [16, Theorem 6.1] depends on the existence and regularity of energy and area minimizing Sobolev discs in metric spaces developed by Lytchak and Wenger in [13, 14, 15]. In the following we describe the main steps in the proof of Theorem 3.1. Let X and J be as in the hypotheses of Theorem 3.1. We denote by $\Lambda(\partial J, \overline{J})$ the family of maps $v \in N^{1,2}(D, \overline{J})$ whose trace has a continuous representative which is a uniform limit of homeomorphisms $S^1 \to \partial J$, where the trace of $v \in N^{1,2}(D, \overline{J})$ is defined by $\operatorname{tr}(v)(s) := \lim_{t \nearrow 1} v(ts)$ for almost every $s \in S^1$. It can be shown that $\overline{J}$ admits a quadratic isoperimetric inequality, which implies that $\Lambda(\partial J, \overline{J})$ is not empty. The existence of a map $u \in \Lambda(\partial J, \overline{J})$ which minimizes the Reshetnyak energy $E_+^2(u, g_{\mathrm{Eucl}})$ among all maps in $\Lambda(\partial J, \overline{J})$ follows from [13, Theorem 7.6]. By [14, Theorem 4.4], u has a continuous representative, denoted again by u, which extends continuously to the boundary and by [14, Lemmas 3.2 and 4.1], the map u is infinitesimally isotropic and thus infinitesimally quasiconformal (see [16, Definition 3.3] and the comment thereafter). After equipping $\overline{J}$ with the intrinsic length metric, it can be shown that u is a homeomorphism, see [16, Theorems 1.2 and 3.6]. Moreover, using the Ahlfors 2-regularity and LLC-condition on X, it follows that the map u is a quasisymmetry, compare to [16, Proposition 3.5 and Theorem 2.5].

The quasisymmetry $f^{-1} \circ u|_{S^1}\colon S^1 \to S^1$ extends to a quasisymmetry $g\colon \overline{D} \to \overline{D}$ after applying the extension result [2, Theorem 1]. The map $\overline{f} := u \circ g^{-1}$ then satisfies all desired properties.

A *cylinder* and *Y-piece* are connected topological surfaces of genus 0 with two and three boundary components, respectively. Furthermore, we refer to a metric space homeomorphic to a cylinder or a Y-piece as a *metric cylinder* or a *metric Y-piece*, respectively. In order to prove Proposition 1.3, we will first decompose M and X into cylinders and Y-pieces, each of which can be further decomposed into suitable Jordan domains. This will be the content of Subsection 3.1. Note that the Jordan domains in X should in particular satisfy the hypotheses of Theorem 3.1. For a Jordan domain J adjacent to the boundary of X, we do not know how to ensure that $\overline{J}$ is LLC. Hence, we will prove a version of Theorem 3.1 for boundary cylinders in Subsection 3.2. In a last step we apply a quasisymmetric gluing theorem of Aseev, Kuzin and Tetenov [1, Theorem 3.1] to construct the desired quasisymmetry from M to X. A rigorous proof of Proposition 1.3 can be found in Subsection 3.3.

3.1. Decompositions of metric Y-pieces and cylinders into Jordan domains. A crucial ingredient in our decomposition of a metric surface is [16, Lemma 4.2], stated next.

Lemma 3.2. *Let X be a geodesic metric space, and let $\Gamma \subset X$ be a topological arc connecting two points $a, b \in X$. Then for every $\varepsilon > 0$ there exists a bi-Lipschitz curve contained in the ε-neighbourhood of Γ and connecting a and b.*

A similar statement also holds for Jordan curves, compare to the proof of [17, Lemma 4.2]. One can prove Lemma 3.2 by choosing a piecewise geodesic injective curve Γ' in a small neighbourhood of Γ and modifying Γ' in the vicinity of every vertex by applying the following claim [16, Claim 4.3].

Claim 3.3. Let $s > 0$ and $\eta\colon [-s, s] \to X$ be an injective curve such that the restrictions $\eta|_{[0,s]}$ and $\eta|_{[-s,0]}$ are geodesics parametrized by their arc-length. Then there exist arbitrarily small $t \in (0, s)$ such that after replacing $\eta|_{[-t,t]}$ by a geodesic from $\eta(-t)$ to $\eta(t)$ we obtain a biLipschitz curve.

Lemma 3.4. *Let X be a geodesic metric surface and $\Sigma \subset X$ a metric cylinder or metric Y-piece such that each connected component of $\partial\Sigma$ can be parametrized by a piecewise geodesic chord-arc curve. Then there exist Jordan domains $J_1, J_2 \subset \Sigma$ with*

(i) $\Sigma = \overline{J_1} \cup \overline{J_2}$,
(ii) $J_1 \cap J_2 = \emptyset$,
(iii) *J_1, J_2 are both bounded by a biLipschitz curve.*

Proof. We give a proof for Σ being a metric Y-piece, the case of a metric cylinder only needing minor adaptations in the following arguments. Denote by $\eta_i\colon S^1 \to \partial\Sigma$ the piecewise geodesic biLipschitz curves parametrizing the three components of $\partial\Sigma$. Choose three disjoint injective curves γ_i in Σ, each one connecting two boundary components such that Σ is separated into two Jordan domains when cutting along these curves. By Lemma 3.2 and its proof, we may assume that each γ_i is biLipschitz and piecewise geodesic. Denote the endpoints of γ_i by a_i^1, a_i^2.

Choose $\varepsilon > 0$ so small that the balls $B(a_i^j, 2\varepsilon)$ are disjoint. We modify γ_i within $B(a_i^j, 2\varepsilon)$ with the following procedure. Without loss of generality assume $a_1^1 \in |\eta_1|$. Choose a point $x_1 \in B(a_1^1, \varepsilon) \cap |\gamma_1|$ distinct from a_1^1 and let $y_1 \in |\eta_1|$ be such that

$$d(x_1, y_1) = d(x_1, |\eta_1|), \tag{2}$$

where d denotes the metric on X. Let $c_1\colon I \to \Sigma$ be a geodesic segment connecting x_1 with y_1. Thus, $|c_1| \subset B(a_1^1, 2\varepsilon)$. Then consider the concatenation of c_1 with one of the subcurves of η_1 emanating from y_1. Let $s > 0$ be such that the following holds. Subcurves of η_1 and c_1 with common endpoint y_1 can be reparametrized by arc-length on $[-s, 0]$ and $[0, s]$, respectively, such that $\eta_1(0) = c_1(0) = y_1$. Denote this concatenation defined on $[-s, s]$ by η. Equality (2) implies that for $r \in [0, s]$

$$d(\eta_1(-r), c_1(r)) \geq r.$$

It follows from the proof of Claim 3.3 that η is a biLipschitz curve. Redefine γ_1 by replacing the subcurve from x_1 to a_1^1 by c_1. Analogously, construct segments $c_2, \dots, c_6$ in the vicinities of the other a_i^j and modify every γ_i near its endpoints in this way. By choosing appropriate subcurves, we have that all redefined γ_i are still injective. Moreover, Claim 3.3 shows that if γ_i is not biLipschitz at a vertex in the interior of the curve, we can change it in an arbitrarily small ball around this vertex to obtain a global biLipschitz curve.

Finally, Σ is separated into Jordan domains J_1 and J_2 by cutting along redefined γ_i. Moreover, the boundaries ∂J_1 and ∂J_2 are parametrized by biLipschitz concatenations of the redefined γ_i with respective subcurves of η_j. $\square$

The following lemma will be useful in the proof of Proposition 3.6. A *metric disc* is a metric space homeomorphic to the closed unit disc $\overline{D}$.

Lemma 3.5. *Let X be an Ahlfors 2-regular and LLC metric surface. Consider a subset $\Sigma \subset X$ that is either a metric disc bounded by a chord-arc curve in X, or that is a metric cylinder such that one component of $\partial\Sigma$ is contained in ∂X and the other component of $\partial\Sigma$ can be parametrized by a chord-arc curve in X. Then Σ equipped with the subspace metric is Ahlfors 2-regular and LLC.*

Lemma 3.5 can be shown readily by using the LLC-property of X and replacing parts of the continua which lie in $X \setminus \Sigma$ with appropriate subcurves of the biLipschitz boundary component in order to obtain desired continua in Σ. Compare also to the proof of [16, Proposition 6.4]. The quadratic upper bound on the Hausdorff 2-measure of a ball is inherited by any subspace, while the lower bound essentially follows from the LLC condition and the coarea inequality for Lipschitz maps, see e.g. [20, p. 1369].

3.2. Parametrizations of boundary cylinders. The aim of this section is to establish the following extension result for cylindrical surfaces which is needed later in the proof of Proposition 1.3.

Proposition 3.6. *Let Z be a smooth cylinder and $\partial Z^1 \subset \partial Z$ a boundary component. Let Σ be a geodesic, Ahlfors 2-regular and LLC metric cylinder and $\partial\Sigma^1 \subset \partial\Sigma$ a boundary component. Assume furthermore that there exists a biLipschitz homeomorphism $f\colon \partial Z^1 \to \partial\Sigma^1$. Then f extends to a quasisymmetric homeomorphism $\overline{f} \in \Lambda(Z, \Sigma)$.*

As a first step in the proof of Proposition 3.6, we will perform a gluing of the metric cylinder Σ with the closed unit disc $\overline{D}$ along corresponding boundary components. We now introduce some notation and needed results concerning this gluing method.

Let (X, d_X) and (Y, d_Y) be two compact metric surfaces with non-empty boundary and let $\partial X^j \subset \partial X$, $\partial Y^k \subset \partial Y$ be two boundary components. Assume $\gamma\colon \partial X^j \to \partial Y^k$ is a biLipschitz homeomorphism and define the quotient

$$\widehat{XY} := (X \sqcup Y)/\sim,$$

where $x \sim y$ for $x \in X$, $y \in Y$ if $y = \gamma(x)$. Equip $\widehat{XY}$ with the quotient metric $\widehat{d}$, which for $[x], [y] \in \widehat{XY}$ is defined by

$$\widehat{d}([x],[y]) := \inf \left\{ \sum_{i=1}^{k} d(p_i, q_i)\colon [p_{i+1}] = [q_i], p_1 = x, q_k = y, k \in \mathbb{N} \right\}.$$

Consider X and Y as subsets of $\widehat{XY}$ and set $X \cap Y := \{[x]\colon x \in \partial X^j\}$. It follows immediately that the identity maps $(X, d_X) \to (X, \hat{d}|_{X\times X})$ and $(Y, d_Y) \to (Y, \hat{d}|_{Y\times Y})$ are 1-Lipschitz. The next lemma is a consequence of the compactness of $X \cap Y$ and the biLipschitz property of γ.

Lemma 3.7. *The identity maps $\mathrm{id}_X\colon (X, \hat{d}|_{X\times X}) \to (X, d_X)$ and $\mathrm{id}_Y\colon (Y, \hat{d}|_{Y\times Y}) \to (Y, d_Y)$ are L-Lipschitz, where $L \geq 1$ denotes the biLipschitz constant of γ. In* *particular, the restrictions $\hat{d}|_{X\times X}$ and $\hat{d}|_{Y\times Y}$ are L-biLipschitz equivalent to d_X and d_Y.*

Moreover, we have the following geometric property of the space $(\widehat{XY}, \hat{d})$.

Lemma 3.8. *If (X, d_X) and (Y, d_Y) are Ahlfors 2-regular and LLC, then so is $(\widehat{XY}, \hat{d})$.*

The proof of Lemma 3.8 can be found in the appendix. A similar gluing procedure with quantitative versions of Lemmas 3.7 and 3.8 was studied in [18, Section 9].

We are now able to provide a proof of Proposition 3.6.

Proof of Proposition 3.6. Consider the quotient space

$$\widehat{\Sigma D} := (\Sigma \sqcup \overline{D})/\sim$$

defined as above for some biLipschitz homeomorphism $\partial\Sigma^1 \to \partial D = S^1$ and equipped again with the quotient metric $\widehat{d}$. By definition, the metric disc $(\widehat{\Sigma D}, \widehat{d})$ is geodesic and from Lemma 3.8 it follows that $(\widehat{\Sigma D}, \widehat{d})$ is Ahlfors 2-regular and LLC. Theorem 3.1 implies the existence of a quasisymmetric homeomorphism $v \in \Lambda(\overline{D}, \widehat{\Sigma D})$. Consider $\widehat{D} = D$ as a subset of $\widehat{\Sigma D}$ and define

$$\Omega := \overline{D} \backslash v^{-1}(\widehat{D}).$$

By the annulus conjecture (see [24, Theorem 3.12]) there exists a quasisymmetric homeomorphism $g\colon A \to \Omega$, where

$$A := \{p \in \mathbb{R}^2 \colon 1/2 \le |p| \le 1\} \subset \overline{D}$$

denotes the standard annulus equipped with the Euclidean metric. Without loss of generality, we may assume that g maps the unit circle onto $\partial(v^{-1}(\widehat{D}))$. Let $\varphi\colon Z \to A$ be a biLipschitz homeomorphism with $\varphi(\partial Z^1) = S^1$. Then, the mapping $u \in N^{1,2}(Z, \Sigma)$ defined by $u := \mathrm{id}_\Sigma \circ v \circ g \circ \varphi$ is a quasisymmetric homeomorphism with $u(\partial Z^1) = \partial\Sigma^1$. Moreover, the composition

$$h := \varphi \circ u^{-1} \circ f \circ \varphi^{-1}|_{S^1} \colon S^1 \to S^1$$

is a quasisymmetric homeomorphism, which we may assume to be orientation-preserving. By [24, Theorem 3.14], the map h extends to a quasisymmetric homeomorphism $\overline{h}\colon \overline{D} \to \overline{D}$ such that $\overline{h}$ restricted to the ball $B(0, 1/2)$ is the identity map. Hence

$$\overline{f} := u \circ \varphi^{-1} \circ \overline{h} \circ \varphi$$

is a desired quasisymmetric homeomorphism from Z to Σ with $\overline{f}|_{\partial Z^1} = f$. □

3.3. Noncanonical quasisymmetric parametrizations. Using the extension result established in the previous subsection, we may obtain Proposition 1.3 mentioned in the introduction.

Proof of Proposition 1.3. The cases where M is a disc or a sphere follow from [16, Theorem 6.1] and [16, Proposition 6.4].

Depending on its topology, endow M with a hyperbolic or flat Riemannian metric (for a smooth surface M with non-empty boundary, see e.g. [11, Exercices for §4.4]). Let $h\colon M \to X$ be a homeomorphism.

We first give a proof in the special case when X has either empty boundary or else is bounded by piecewise geodesic chord-arc curves. Choose a collection of simple closed geodesics $\{\gamma_i\colon S^1 \to M\}$ decomposing M into smooth Y-pieces or cylinders M_k, respectively. Using [16, Lemma 4.2], we may partition X into Y-pieces/cylinders

X_k such that each X_k is homotopic to $h(M_k) \subset X$ and bounded by piecewise geodesic chord-arc curves. We then further decompose M_k and X_k into Jordan domains: if M_k is a Y-piece, then it is a standard result from hyperbolic geometry that M_k is isometric to the partial gluing of the boundary of two copies $\Omega_{k,1}, \Omega_{k,2}$ of a right-angled hexagon in $\mathbb{H}$, see e.g. [4, Proposition 3.1.5]. If M_k is of cylindrical type, then a similar decomposition into isometric rectangles in the Euclidean plane, again denoted $\Omega_{k,1}$ and $\Omega_{k,2}$, is possible. Note that in either case $\Omega_{k,1}$ and $\Omega_{k,2}$ are biLipschitz equivalent to the closed unit disc $\overline{D}$. In X we decompose each X_k into Jordan domains $J_{k,1}$, $J_{k,2}$ as in Lemma 3.4. After possibly inverting the notation of $J_{k,1}$ and $J_{k,2}$, let

$$f\colon \bigcup_{\substack{j=1,2\\k}} \partial\Omega_{k,j} \to \bigcup_{\substack{j=1,2\\k}} \partial J_{k,j}$$

be a biLipschitz homeomorphism satisfying $f(\partial\Omega_{k,j}) = \partial J_{k,j}$ for each j, k. By Lemma 3.5 and Theorem 3.1, there exists for each k a quasisymmetric homeomorphism $g_{k,j}\colon \overline{\Omega_{k,j}} \to \overline{J_{k,j}}$ with $g_{k,j}|_{\partial\Omega_{k,j}} = f|_{\partial\Omega_{k,j}}$. The map $u\colon M \to X$ agreeing with $g_{k,j}$ on $\Omega_{k,j}$ satisfies the hypotheses of the quasisymmetric gluing theorem [1, Theorem 3.1] as each $\Omega_{k,j}$ is bounded and has biLipschitz boundary and every $g_{k,j}$ is a quasisymmetric homeomorphism. Therefore, the map u itself is a quasisymmetric homeomorphism. This shows the proposition in the special case.

We now turn to the general case, where X might be bounded by curves of unknown regularity. For each boundary component ∂X^i, define a piecewise geodesic biLipschitz curve $c_i\colon S^1 \to X$ which is homotopic to an oriented parametrization of ∂X^i, but disjoint from it. Furthermore, we may assume that the curves $\{c_i\}$ are all pairwise disjoint. Let $\Sigma_i \subset X$ be the metric cylinder bounded by $c_i(S^1)$ and ∂X^i, and let $\Sigma \subset X$ be the subsurface bounded by $\bigcup_i c_i(S^1)$. Note that Σ is homeomorphic to X. The first part of the proof then shows that there exists a quasisymmetric homeomorphism $u\colon M \to \Sigma$. Then embed M smoothly into a surface $\tilde{M}$ such that for each i, there exists exactly one boundary component $\partial\tilde{M}^i$ which together with $\partial Z_i^1 := u^{-1}(c_i(S^1)) \subset \partial M$ bounds a smooth cylinder $Z_i \subset \tilde{M}$. Finally, use Lemma 3.5 and Proposition 3.6 to obtain quasisymmetric extensions $u_i\colon Z_i \to \Sigma_i$ of $u|_{\partial Z_i^1}$. Once again, the gluing result [1, Theorem 3.1] ensures that the map $u\colon \tilde{M} \to X$ agreeing with u on M and with u_i on Z_i is a quasisymmetric homeomorphism. The proof of the proposition is complete. □

Remark 3.9. The proof of Proposition 1.3 can be adapted to show a version of the proposition for non-orientable and homeomorphic surfaces M and X. In particular, we obtain a generalization of Bonk–Kleiner to all compact surfaces, meaning surfaces of arbitrary genus that are not necessarily orientable and possibly possess non-empty boundary; compare to Corollary 1.2.

4. Equicontinuity of energy bounded almost homeomorphisms

The map provided by Proposition 1.3 does not need to be canonical, i.e. of minimal energy. In order to obtain such a parametrization in Section 5, we will apply a direct variational method for which we need to know equicontinuity of a given energy-minimizing sequence of parametrizations. More explicitly, we prove the following statement in this section.

Proposition 4.1. *Let M be a smooth surface endowed with a Riemannian metric g and which is neither of disc- nor of sphere-type. Let X be a metric surface* *homeomorphic to M and such that ∂X is rectifiable. Then the family*

$$\mathcal{F} := \{v \in \Lambda(M,X) \colon E_+^2(v,g) \leq K\}$$

is equicontinuous.

In order to show Proposition 4.1, we need the following elementary lemma. Its proof is left to the reader.

Lemma 4.2. *Let X be a metric surface which is not of sphere-type. Then for every $\varepsilon > 0$ there exists $\rho > 0$ such that the following holds. Every embedding $u \colon \overline{D} \to X$ with $\operatorname{diam}(u(S^1)) < \rho$ satisfies $\operatorname{diam}(u(\overline{D})) < \varepsilon$.*

By continuity, the statement holds for any uniform limit of embeddings from $\overline{D}$ to X.

Proof of Proposition 4.1. Let $\varepsilon > 0$ and define

$$\eta := \inf\{\ell(c) \mid c \colon S^1 \to X \text{ is a non-contractible curve in } X\} > 0.$$

By Lemma 4.2, there exists $0 < \rho < \min\{\varepsilon, \eta\}$ such that for any uniform limit of embeddings $u \colon \overline{D} \to X$ with $\operatorname{diam}(u(S^1)) < \rho$ there holds $\operatorname{diam}(u(\overline{D})) < \varepsilon$. Similarly, there exists $0 < \rho' < \rho/2$ such that the following is true. If $x, x' \in \partial X$ satisfy $d(x,x') < \rho'$, then they lie on the same component $\partial X^i \subset \partial X$ and the shorter of the two subcurves of ∂X^i connecting x and x' has length at most $\rho/2$. Since M is compact, there exists $0 < \delta < 1$ so small that

$$\pi \cdot \left(\frac{8K}{|\log(\delta)|}\right)^{1/2} < \rho'$$

and such that every point $p \in M$ is contained in a neighbourhood in M which is the image of the set $B := B_{\mathbb{R}^2}(q, \sqrt{\delta}) \cap \overline{D}$ under a map ψ that is 2-biLipschitz and takes the point $q \in [0,1] \subset \overline{D}$ to p, where q is chosen to be 1 if $p \in \partial M$ and 0 whenever the distance between p and ∂M is big enough. In particular, if the set $B_{\mathbb{R}^2}(q, \sqrt{\delta}) \cap S^1$ is not empty, then it is mapped onto a subcurve of ∂M.

Fix $p \in M$ and $v \in \mathcal{F}$. By the Courant–Lebesgue Lemma (see e.g. [13, Lemma 7.3]) there exists $r \in (\delta, \sqrt{\delta})$ such that

$$\ell(v \circ \psi \circ \gamma_r) \leq \pi \cdot \left(\frac{2E_+^2(v \circ \psi)}{|\log(\delta)|}\right)^{1/2} \leq \pi \cdot \left(\frac{8E_+^2(v)}{|\log(\delta)|}\right)^{1/2} < \rho',$$

where γ_r is an arc-length parametrization of $\{z \in B : |z - q| = r\}$.

Consider the set $A := \{z \in B \colon |z - q| < r\}$. It holds that $B_M(p, \delta/2) \subset \psi(A)$ and $\overline{A}$ is biLipschitz equivalent to $\overline{D}$ with constant only depending on r. If $\psi(A)$ does not intersect ∂M, by applying Lemma 4.2, we can conclude $\operatorname{diam}(v(\psi(A))) < \varepsilon$ and therefore $v(B_M(p, \delta/2)) \subset B_X(v(p), \varepsilon)$.

If $\psi(A) \cap \partial M$ is not empty, then $\psi(A)$ is bounded by $\psi \circ \gamma_r$ and a subarc of ∂M^i, denoted α_r. The endpoints $a_r, b_r \in \partial M^i$ of $\psi \circ \gamma_r$ satisfy $d(v(a_r), v(b_r)) < \rho' < \rho/2$. Thus, $v(a_r)$ and $v(b_r)$ lie on the same boundary component $\partial X^i \subset \partial X$ and the shorter subcurve of ∂X^i connecting $v(a_r)$ and $v(b_r)$ has length at most $\rho/2 < \eta/2$. This segment corresponds to the curve $v \circ \alpha_r$. Indeed otherwise, the concatenation of $v \circ \psi \circ \gamma_r$ with $v|_{\partial M^i \setminus \alpha_r}$ would yield a non-contractible closed curve in X of length strictly less than η, which is impossible. Again by applying Lemma 4.2 we obtain $v(B_M(p, \delta/2)) \subset v(\psi(A)) \subset B_X(v(p), \varepsilon)$. Since the choice of δ was independant of v and of p, this proves equicontinuity of $\mathcal{F}$. □

5. Proof of Main Theorem

We finally turn to the proof of Theorem 1.1. First however, we introduce some notation. Define the family

$$\Lambda_{\mathrm{metr}}(M,X) := \{(v,h)\colon v \in \Lambda(M,X),\ h \text{ a smooth Riemannian metric on M}\}.$$

An *energy minimizing sequence* in $\Lambda_{\mathrm{metr}}(M,X)$ is a sequence of pairs $(u_n, g_n) \in \Lambda_{\mathrm{metr}}(M,X)$ satisfying

$$E_+^2(u_n, g_n) \to \inf\{E_+^2(v,h)\colon (v,h) \in \Lambda_{\mathrm{metr}}(M,X)\}$$

as n tends to infinity.

Proof of Theorem 1.1. The proofs in the cases where M is of disc- or sphere-type follow from [16], and we therefore only consider M being of higher topological type. Moreover, we assume that M is equipped with a hyperbolic metric. The case where M only admits flat metrics follows analogously. In a first step, we show the existence of an energy minimizing pair in Λ_{metr}. By Proposition 1.3, the set $\Lambda(M,X)$ is not empty. Therefore, we are able to consider an energy minimizing sequence (u_n, g_n) in $\Lambda_{\mathrm{metr}}(M,X)$. We lose no generality in assuming that the metrics g_n are all hyperbolic. Observe that each u_n, being a uniform limit of homeomorphisms, satisfies the condition of cohesion for some $\eta > 0$ in the sense of [6, Definition 8.1]. Thus by [6, Proposition 8.4] there exists $\varepsilon > 0$ depending only on η and $K := \sup_{n\in\mathbb{N}} E_+^2(u_n, g_n)$ such that for every n the relative systole of (M, g_n) (see [6, Definition 3.1]) is bounded from below by ε. Then, there exist diffeomorphisms $\varphi_n\colon M \to M$ such that a subsequence of $(\varphi_n^* g_n)$ converges smoothly to a hyperbolic metric g on M (see [5, Theorem 4.4.1] if M is a closed surface; and e.g. [6, Theorem 3.3] if M has non-empty boundary). Set $v_n := u_n \circ \varphi_n$. The convergence above implies that

$$\lim_{n\to\infty} E_+^2(v_n, g) = \lim_{n\to\infty} E_+^2(u_n, g_n).$$

Thus, the sequence (v_n, g) is energy minimizing in $\Lambda_{\mathrm{metr}}(M,X)$. Now by Proposition 4.1, the sequence (v_n) is equicontinuous and the Arzelà–Ascoli theorem implies that a subsequence of (v_n) converges uniformly to some continuous map $u\colon M \to X$. It follows that u is in $N^{1,2}(M,X)$ (compare to [12, Theorem 1.6.1]) as well as a uniform limit of homeomorphisms, hence $u \in \Lambda(M,X)$. By the lower semicontinuity of $E_+^2(\cdot)$ it follows that the pair (u,g) is an energy minimizer in $\Lambda_{\mathrm{metr}}(M,X)$.

We now show that any energy minimizing pair (u,g) in $\Lambda_{\mathrm{metr}}(M,X)$ is a quasisymmetric homeomorphism. As a uniform limit of homeomorphisms, the map u is continuous, monotone and surjective. Furthermore, by [6, Theorem 4.2], the map u is infinitesimally isotropic and hence infinitesimally $\sqrt{2}$-quasiconformal with respect to g (see [6, Definition 4.1] and the explanation thereafter). It follows from [16, Theorem 3.6] that u is a local homeomorphism. Monotonicity of u implies then that u is injective. Hence, u is a homeomorphism as it is a continuous bijection on a compact set M. Using analogous statements to Theorem 2.5 and Proposition 3.5 in [16] for the domain (M,g) instead of $(\overline{D}, g_{\mathrm{Eucl}})$, one can argue as in the proof of [16, Theorem 6.1] to obtain that u is a quasisymmetric homeomorphism with respect to g. Note that the analogue to [16, Theorem 2.5] follows since M admits a $(1,2)$-Poincaré inequality and is thus a Loewner space, see [8, Theorem 9.10].

It remains to show uniqueness of (u,g) up to precomposition with conformal diffeomorphisms. Let (u,g), (v,h) be energy minimizing pairs in $\Lambda_{\mathrm{metr}}(M,X)$. We claim that the map $\varphi := v^{-1} \circ u\colon (M,g) \to (M,h)$ is then a conformal diffeomorphism.

Indeed, for any choice of conformal charts $\psi\colon \overline{U} \to \overline{D}$ of (M,g) and $\phi\colon \overline{V} \to \overline{D}$ of (M,h), we can argue as in the last paragraph in the proof of [16, Theorem 6.1] that the transition maps

$$\phi \circ v^{-1} \circ u \circ \psi^{-1}\colon \overline{D} \to \overline{D}$$

are conformal diffeomorphisms, which implies the respective property for the mapping φ. The proof of the theorem is complete. □

6. Appendix

Proof of Lemma 3.8. Let $z \in \widehat{XY}$ and $r > 0$ be arbitrary. By symmetry, we may assume $z \in X$. Observe that there exists $y \in Y$ such that $B_{\widehat{XY}}(z,r)$ is contained in $(B_{\widehat{XY}}(z,r) \cap X) \cup (B_{\widehat{XY}}(y,2r) \cap Y)$. The Ahlfors 2-regularity of $(\widehat{XY}, \hat{d})$ now follows from Lemma 3.7 and the Ahlfors 2-regularity of X and Y.

It remains to prove that $(\widehat{XY}, \hat{d})$ is LLC. Both X and Y are quasiconvex (see [23, Theorem B.6]) with constants C_X and C_Y depending only on the LLC and Ahlfors 2-regularity constants of X and Y, respectively. Hence, the space $(\widehat{XY}, \hat{d})$ is quasiconvex with constant $\widehat{C} := \max\{C_X, C_Y\}$ implying that the first LLC condition holds with constant $\widehat{C}$.

Denote by λ_X and λ_Y the LLC-constants of X and Y, respectively, and choose

$$\hat{\lambda} \geq \max\{2, \lambda_X, \lambda_Y\}$$

such that $2\operatorname{diam}_{\hat{d}}(\widehat{XY})/\hat{\lambda} < \operatorname{diam}_{\hat{d}}(X \cap Y)$. Let $x, y \in \widehat{XY} \setminus B_{\widehat{XY}}(z,r)$. We want to prove the existence of a uniform $\lambda \geq 1$ such that x, y can be joined by a continuum in $\widehat{XY} \setminus B_{\widehat{XY}}(z, r/\lambda)$. If $x, y \in X$ or $x, y \in Y$, the statement follows from the LLC-property of X or Y and Lemma 3.7. Consider $x \in X$, $y \in Y \setminus X$ and assume for the moment that $B := B_{\widehat{XY}}(z, r/(2L\hat{\lambda}^2)) \subset X$. Choose any point $a \in (X \cap Y) \setminus B_{\widehat{XY}}(z, r/\hat{\lambda})$. Then there exists a continuum in

$$X \setminus B_X\left(z, r/\hat{\lambda}^2\right) \subset \widehat{XY} \setminus B_{\widehat{XY}}\left(z, r/(L\hat{\lambda}^2)\right) \subset \widehat{XY} \setminus B$$

connecting x with a, which can be concatenated with any continuum in Y connecting a with y to obtain a desired path between x and y in $\widehat{XY} \setminus B$. If the intersection of $B_{\widehat{XY}}(z, r/(2L\hat{\lambda}^2))$ with Y is not empty, choose a point $b \in X \cap Y \cap B$ and define

$$d := \hat{d}(b,z) < \frac{r}{2L\hat{\lambda}^2} < \frac{r}{\hat{\lambda}}.$$

It then follows from the triangle inequality that

$$d_X(b,x) \geq \hat{d}(b,x) \geq r - d \geq r - \frac{r}{\hat{\lambda}} \geq \frac{r}{\hat{\lambda}}$$

and similarly, that $d_Y(b,y) \geq r/\hat{\lambda}$. After picking a point $a \in (X \cap Y) \setminus B_{\widehat{XY}}(b, r/\hat{\lambda})$, we have the existence of continua $E \subset X \setminus B_X(b, r/\hat{\lambda}^2)$ connecting x with a respectively $F \subset Y \setminus B_Y(b, r/\hat{\lambda}^2)$ joining a with y; and therefore a continuum in

$$\widehat{XY} \setminus B_{\widehat{XY}}\left(b, r/(L\hat{\lambda}^2)\right) \subset \widehat{XY} \setminus B_{\widehat{XY}}\left(z, r/(L\hat{\lambda}^2) - d\right) \subset \widehat{XY} \setminus B$$

between x and y. We thus have proven that the space $(\widehat{XY}, \hat{d})$ is LLC with constant $\lambda := \max\{2L\hat{\lambda}^2, \widehat{C}\}$. □

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