

Additive properties of fractal sets on the parabola

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Abstract. Let $0 \leq s \leq 1$, and let $\mathbb{P} := \{(t, t^2) \in \mathbb{R}^2 : t \in [-1, 1]\}$. If $K \subset \mathbb{P}$ is a closed set with $\dim_{\mathbb{H}} K = s$, it is not hard to see that $\dim_{\mathbb{H}}(K + K) \geq 2s$. The main corollary of the paper states that if $0 < s < 1$, then adding K once more makes the sum slightly larger:

$$\dim_{\mathbb{H}}(K + K + K) \geq 2s + \epsilon,$$

where $\epsilon = \epsilon(s) > 0$. This information is deduced from an L^6 bound for the Fourier transforms of Frostman measures on \mathbb{P} . If $0 < s < 1$, and μ is a Borel measure on \mathbb{P} satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{P}$ and $r > 0$, then there exists $\epsilon = \epsilon(s) > 0$ such that

$$\|\hat{\mu}\|_{L^6(B(R))}^6 \leq R^{2-(2s+\epsilon)}$$

for all sufficiently large $R \geq 1$. The proof is based on a reduction to a δ -discretised point-circle incidence problem, and eventually to the $(s, 2s)$ -Furstenberg set problem.

Paraabelille sisältyvien fraktaalijoukkojen summaominaisuuksia

Tiivistelmä. Olkoon $0 \leq s \leq 1$ ja $\mathbb{P} := \{(t, t^2) \in \mathbb{R}^2 : t \in [-1, 1]\}$. Jos $K \subset \mathbb{P}$ on suljettu ja $\dim_{\mathbb{H}} K = s$, on suoraviivaista nähdä, että $\dim_{\mathbb{H}}(K + K) \geq 2s$. Paperin pääkorollaari kertoo, että jos $0 < s < 1$, joukon K lisääminen vielä kerran kasvattaa summaa:

$$\dim_{\mathbb{H}}(K + K + K) \geq 2s + \epsilon,$$

missä $\epsilon = \epsilon(s) > 0$. Väite päätellään seuraavasta L^6 -arviosta Frostman-mittojen Fourier-muunnoksille. Olkoon $0 < s < 1$, ja olkoon μ on Borel-mitta joukossa \mathbb{P} , joka toteuttaa ehdon $\mu(B(x, r)) \leq r^s$ kaikille $x \in \mathbb{P}$ ja $r > 0$. Silloin on olemassa $\epsilon = \epsilon(s) > 0$ ja $R_0 \geq 1$, joille seuraava epäyhtälö pätee kaikille $R \geq R_0$:

$$\|\hat{\mu}\|_{L^6(B(R))}^6 \leq R^{2-(2s+\epsilon)}.$$

Todistuksen keskeinen idea on muotoilla ongelma uudelleen sopivana δ -diskreetoituna pisteiden ja ympyröiden välisenä insidenssiongelmana. Tämä geometrinen pulma palautuu lopulta $(s, 2s)$ -Furstenberg-joukko-ongelmaan.

1. Introduction

The main result of this paper investigates the L^p -norms of Fourier transforms of fractal measures on the truncated parabola $\mathbb{P} = \{(t, t^2) : t \in [-1, 1]\}$:

Theorem 1.1. *Let $0 \leq s \leq 1$, and let μ be a Borel measure on \mathbb{P} satisfying the Frostman condition $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^2$ and $r > 0$. Then,*

$$(1.2) \quad \|\hat{\mu}\|_{L^4(B(R))} \lesssim R^{(2-2s)/4}, \quad R \geq 1.$$

If $0 < s < 1$, and $p > 4$, then there exists a constant $\epsilon = \epsilon(p, s) > 0$ such that

$$\|\hat{\mu}\|_{L^p(B(R))} \leq C_{p,s} R^{(2-(2s+\epsilon))/p}, \quad R \geq 1.$$

The function $s \mapsto \epsilon(p, s)$ is bounded away from 0 on any compact subset of $(0, 1)$.

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The inequality (1.2) means that for every $\eta > 0$, there exists a constant $C_{\eta,s} > 0$ such that $\|\hat{\mu}\|_{L^4(B(R))} \leq C_{\eta,s} R^{(2-2s)/4+\eta}$ for all $R \geq 1$. The exponent $2 - 2s$ in (1.2) is sharp, and this part of the theorem is not hard to prove. In fact, the classical *L^4 -orthogonality method* immediately reduces the oscillatory problem to a Kakeya type problem regarding families of wave packets arising from the s -dimensional measure μ . Identifying the sharp non-concentration condition for such wave packet families takes some work, see Lemma 3.3. After this has been accomplished, however, the Kakeya problem can be solved with the standard L^2 method, see the proof of Theorem 3.8.

The second part of Theorem 1.1 is more complicated, and has the following corollary regarding the dimension of triple sums of fractal subsets of \mathbb{P} :

Corollary 1.3. *For every $0 < s < 1$, there exists $\epsilon = \epsilon(s) > 0$ such that the following holds. Let $K \subset \mathbb{P}$ be a Borel set with $\dim_{\mathbb{H}} K = s$. Then,*

$$\dim_{\mathbb{H}}(K + K + K) \geq 2s + \epsilon.$$

Here $\dim_{\mathbb{H}}$ is Hausdorff dimension. For two summands instead of three, the lower bound $\dim_{\mathbb{H}}(K + K) \geq 2s$ is sharp and simple to prove. In fact, this follows directly from the fact that $(x, y) \mapsto x + y$ is bilipschitz on compact subsets of $(\mathbb{P} \times \mathbb{P}) \setminus \{x = y\}$. The main point in Corollary 1.3 is the ϵ -improvement over this “trivial” bound.

1.1. Connection to discrete problems. Corollary 1.3 is a “continuous” version of the following discrete question: if $P \subset \mathbb{P}$ is a finite set, how large is (at least) the cardinality of $|P + P + P|$, or more generally $|kP|$ for $k \geq 3$? A variant of the problem asks for upper bounds for the n^{th} *additive energy*

$$E_n(P) = |\{(x_1, \dots, x_k, y_1, \dots, y_k) \in P^{2n} : x_1 + \dots + x_n = y_1 + \dots + y_n\}|.$$

This formulation is the discrete analogue of the problem studied in Theorem 1.1. The problems are related by the inequality $E_n(P)|nP| \geq |P|^{2n}$, an easy consequence of Cauchy–Schwarz. Bourgain and Demeter [3] showed that $E_3(P) \lesssim_{\epsilon} |P|^{7/2+\epsilon}$ for all $\epsilon > 0$, and asked [3, Question 2.13] if the estimate can be improved to $E_3(P) \lesssim |P|^{3+\epsilon}$ for all $\epsilon > 0$. The positive result yields $|P + P + P| \gtrsim_{\epsilon} |P|^{5/2-\epsilon}$, and a positive answer to the question would yield the optimal result $|P + P + P| \gtrsim_{\epsilon} |P|^{3-\epsilon}$. If $\delta \in (0, 1]$, and $P \subset \mathbb{P}$ is assumed to be δ -separated, then the optimal bound follows from the sharp ℓ^2 -decoupling theorem: $E_3(P) \lesssim_{\epsilon} \delta^{-\epsilon} |P|^3$, see [5, Theorem 13.21].

For $E_3(P)$ and $|P + P + P|$, these are the best current results, as far as I know. However, Mudgal [13, Corollary 1.2] has recently obtained an improvement for higher energies: $E_k(P) \lesssim_k |P|^{2k-3+\epsilon(k)}$, where $\epsilon(k) = (1/4 - 1/7246) \cdot 2^{-k+4}$ for $k \geq 4$. In particular, it follows that $|kP| \geq |P|^{3-o_k(1)}$, where $o_k(1) \rightarrow 0$ as $k \rightarrow \infty$.

In analogy, one might hope that $\lim_{n \rightarrow \infty} \dim_{\mathbb{H}}(nK) \geq 3 \dim_{\mathbb{H}} K$, but this is clearly false if $\dim_{\mathbb{H}} K > 2/3$. A more plausible conjecture might be that $\lim_{n \rightarrow \infty} \dim_{\mathbb{H}}(nK) \geq \min\{3 \dim_{\mathbb{H}} K, \dim_{\mathbb{H}} K + 1\}$. This is closely connected to the discussion in Section 1.3.

1.2. Connection to Borel subrings and proof of Corollary 1.3. The ϵ -improvements in the second part of Theorem 1.1 and Corollary 1.3 are closely connected with the Borel subring problem. This problem, solved independently by Edgar–Miller [7] and Bourgain [2] around 2000, asked to show that every Borel subring $R \subset \mathbb{R}$ has $\dim_{\mathbb{H}} R \in \{0, 1\}$. This follows immediately from Corollary 1.3, as we will discuss in this section. The point here is not to announce a new solution (indeed the argument of Theorem 1.1 relies on previous solutions), but rather to shed light on the problem of bounding $\|\hat{\mu}\|_{L^p}$.

To deduce Corollary 1.3 from Theorem 1.1, we need a standard lemma:

Lemma 1.4. *Let μ be a non-trivial finite Borel measure on \mathbb{R}^d , and let $\mu_\delta := \mu * \psi_\delta$, $\delta > 0$, where $\psi_\delta(x) = \delta^{-d}\psi(x/\delta)$ is a standard approximate identity; $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi \geq 0$ and $\int \psi = 1$. Let $s \in [0, d]$, $\delta_0 > 0$, and assume that*

$$\|\mu_\delta\|_{L^2(\mathbb{R}^d)}^2 \leq \delta^{s-d}, \quad 0 < \delta \leq \delta_0.$$

Then $|(\text{spt } \mu)_\delta| \gtrsim_\psi \|\mu\|^2 \cdot \delta^{-s}$ for all $0 < \delta \leq \delta_0$, and $\dim_{\text{H}}(\text{spt } \mu) \geq s$. Here $|\cdot|_\delta$ refers to the δ -covering number.

Proof. Note that $|\widehat{\psi_{c\delta}}(\xi)| \gtrsim 1$ for all $|\xi| \leq \delta^{-1}$ if $c > 0$ is sufficiently small (depending on the choice of ψ). Now, if $0 < \sigma < s$, then we have

$$\int |\widehat{\mu}(\xi)|^2 |\xi|^{\sigma-d} d\xi \lesssim \sum_{j \geq 0} 2^{j(\sigma-d)} \int_{B(2^j)} |\widehat{\mu}(\xi)|^2 d\xi \lesssim \sum_{j \geq 0} 2^{j(\sigma-d)} \int |\widehat{\mu}(\xi)|^2 |\widehat{\psi_{c2^{-j}}}(\xi)|^2 d\xi.$$

The integral on the right is $\|\mu * \psi_{c2^{-j}}\|_2^2 \leq (c2^{-j})^{s-d}$, so the sum is finite for $\sigma < s$. It is well-known that this implies $\dim_{\text{H}}(\text{spt } \mu) \geq s$, see [12, Theorem 8.7 & Lemma 12.12]. The claim about $|(\text{spt } \mu)_\delta|$ is even simpler, being based on the inequality $\|f\|_{L^1}^2 \leq \text{Leb}(\text{spt } f) \|f\|_{L^2}^2$ applied to $f = \mu_\delta$. \square

Corollary 1.3 follows from Theorem 1.1, and Lemma 1.4. Indeed, if $K \subset \mathbb{P}$ is Borel with $\dim_{\text{H}} K = s \in (0, 1)$, then for any $0 \leq \sigma < s$, Frostman's lemma (see [12, Theorem 8.8]) yields a non-trivial measure μ with $\text{spt } \mu \subset K$ and $\mu(B(x, r)) \leq r^\sigma$ for all $x \in \mathbb{P}$ and $r > 0$. Then

$$\|(\mu * \mu * \mu)_\delta\|_{L^2}^2 = \|\widehat{\mu}_\delta\|_{L^6}^6 \leq C_\sigma \delta^{2\sigma + \epsilon - 2}, \quad \delta \in (0, 1],$$

and consequently $\dim_{\text{H}}(K + K + K) \geq \dim_{\text{H}} \text{spt}(\mu * \mu * \mu) \geq 2\sigma + \epsilon$. Since ϵ is bounded away from zero for σ sufficiently close to s , Corollary 1.3 follows by letting $\sigma \rightarrow s$.

Next, let us see why Corollary 1.3 implies the non-existence of Borel subrings of intermediate dimension. Let $A \subset \mathbb{R}$ be any Borel set with $\dim_{\text{H}} A = s$, where $s \in (0, 1)$. Then $K := K_A := \{(t, t^2) : t \in \mathbb{R}\}$ is a Borel subset of \mathbb{P} with $\dim_{\text{H}} K = s$. Evidently $K \subset A \times A^2$, so

$$(1.5) \quad K + K + K \subset (A + A + A) \times (A^2 + A^2 + A^2).$$

If A were a ring, then the right hand side would be contained in $A \times A$, and hence $\dim_{\text{H}}(K + K + K) \leq \dim_{\text{H}}(A \times A)$. We finally claim that $\dim_{\text{H}}(A \times A) = 2s$, which will contradict Corollary 1.3. This follows from a folklore result on the dimension of orthogonal projections: if $K \subset \mathbb{R}^2$ is Borel, then $\dim_{\text{H}}\{e \in S^1 : \dim_{\text{H}} \pi_e(K) < \frac{1}{2} \dim_{\text{H}} K\} = 0$. For a proof, see [14, Theorem 1.2]. In particular, since $\dim_{\text{H}} A > 0$, and A is a ring, we have $s = \dim_{\text{H}} A = \dim_{\text{H}}(A + aA) \geq \frac{1}{2} \dim_{\text{H}}(A \times A)$ for some $a \in A$.

We close this section by mentioning a related result of Raz and Zahl [19, Theorem 1.14]. A special case of their theorem shows that if $A \subset [0, 1]$ is a $(\delta, s, \delta^{-\epsilon})$ -set with $s \in (0, 1)$ (see Definition 2.1) and $|A|_\delta \geq \delta^{-s+\epsilon}$, and $\epsilon = \epsilon(s) > 0$ is small enough, then

$$\max\{|A + A|_\delta, |A^2 + A^2|_\delta\} \geq \delta^{-s-\epsilon}.$$

The proof of Corollary 1.3 gives an alternative argument for this fact. Indeed, if $|A + A|_\delta \leq \delta^{-s-\epsilon} \leq \delta^{-2\epsilon} |A|_\delta$ and $|A^2 + A^2|_\delta \leq \delta^{-s-\epsilon} \leq \delta^{-2\epsilon} |A|_\delta$, then it follows from the Plünnecke's inequality (see [18] for the original reference, or [20, Corollary 6.28] for a textbook), that also

$$|A + A + A|_\delta \lesssim \delta^{-s-O(\epsilon)} \quad \text{and} \quad |A^2 + A^2 + A^2|_\delta \lesssim \delta^{-s-O(\epsilon)}.$$

Consequently, $|(A + A + A) \times (A^2 + A^2 + A^2)|_\delta \lesssim \delta^{-2s-O(\epsilon)}$. Given the inclusion (1.5), this contradicts a δ -discretised version of Corollary 1.3 for $\epsilon > 0$ small enough, depending only on $s \in (0, 1)$ (the required δ -discretised version follows from Theorem 1.1, using the easier part of Lemma 1.4.) We note, however, that the theorem of Raz and Zahl concerns far more general non-linear images of $A \times A$ than just $A^2 + A^2$.

1.3. Value of ϵ ? We do not know what the precise value of “ ϵ ” should be for every pair (p, s) with $s \in [0, 1]$ and $p > 4$. However, the following conjecture seems plausible:

Conjecture 1.6. *For every $0 \leq s \leq 1$ and $\epsilon > 0$, there exists $p = p(\epsilon, s) \geq 1$ such that the following holds. Let μ be a Borel measure on \mathbb{P} satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^2$ and $r > 0$. Then,*

$$(1.7) \quad \|\hat{\mu}\|_{L^p(B(R))} \leq C_{\epsilon, s} R^{[2 - \min\{3s, 1+s\}]/p + \epsilon}, \quad R \geq 1.$$

It is not hard to see that the threshold $\min\{3s, 1+s\}$ cannot be further improved:

Example 1.8. Consider a set $A \subset [0, 1] \cap (\delta^s \mathbb{Z})(\delta)$ which is a union of $\sim \delta^{-s}$ equally spaced intervals of length $\delta > 0$ in arithmetic progression. Then $A^2 \subset [0, 1] \cap (\delta^{2s} \mathbb{Z})(\delta)$ can be covered by a union of $\sim \delta^{-2s}$ intervals with spacing δ^{2s} , again in arithmetic progression. Therefore kA^2 can also be covered by $\lesssim_k \delta^{-2s}$ intervals with the same spacing δ^{2s} . It follows that

$$(1.9) \quad |kA \times kA^2|_\delta \lesssim_k \delta^{-3s}, \quad k \geq 1.$$

If $s \geq \frac{1}{2}$, we can do better: then we note that kA^2 is trivially covered by $\lesssim_k \delta^{-1}$ intervals of length δ , hence

$$(1.10) \quad |kA \times kA^2|_\delta \lesssim_k \delta^{-1-s}, \quad k \geq 1.$$

The bounds (1.9)–(1.10) show (with the assistance of Lemma 1.4, and a version of (1.5) for k -fold sums) that the exponent $\min\{3s, 1+s\}$ in (1.7) cannot be lowered.

1.4. Proof and paper outlines. Below Theorem 1.1, we already explained the key points needed to prove the first part of Theorem 1.1. The details are contained in Section 3. The proof of the second part is based on the following ingredients:

- Knowing that the first part is true: the exponent “ $2 - 2s$ ” will serve as a “base camp” from which we reach out for the ϵ -improvement for $p > 4$.
- An observation due to Bombieri, Bourgain, and Demeter, [1, 3]: the 3rd additive energy of subsets of \mathbb{P} is closely related to an incidence-counting problem between points in \mathbb{R}^2 , and circles centred along the x -axis. A δ -discretised version of this result is formulated at the beginning of Section 4.
- If the exponent “ $2 - 2s$ ” were sharp for some $p > 4$, it turns out that we could construct an $(s, 2s)$ -Furstenberg set of dimension $2s$. Furstenberg sets are introduced in Section 2.2. In particular, it is known (due to Bourgain, and Héra–Shmerkin–Yavicoli) that $(s, 2s)$ -Furstenberg sets of dimension $2s$ do not exist (for $s \in (0, 1)$). This is where the ϵ -improvement in Theorem 1.1 comes from.
- Based on the hypothetical sharpness of the exponent “ $2 - 2s$ ” for $p > 4$, and the relation between additive energies and circle incidences, we first construct an “ $(s, 2s)$ -Furstenberg of circles”. This is done in Section 4.3. Luckily, the ensuing circles are all centred along the x -axis. It turns out that the incidence geometry of such circles is equivalent to the incidence geometry of planar lines.

This is because there exists a well-behaved map between the *Poincaré half-plane model* and the *Beltrami–Klein model* of hyperbolic geometry. This was explained to me by Josh Zahl. The details of the transformation are contained in Section 4.4, where the proof of Theorem 1.1 is finally concluded.

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2. Preliminaries

2.1. Notation. The notation $B(x, r)$ stands for a closed ball of radius $r > 0$ and centre $x \in X$, in a metric space (X, d) . If $A \subset X$ is a bounded set, and $r > 0$, we write $|A|_r$ for the δ -covering number of A , that is, the smallest number of closed balls of radius r required to cover A . Cardinality is denoted $|A|$, and Lebesgue measure $\text{Leb}(A)$. The closed r -neighbourhood of A is denoted $A(r)$. Throughout the article, \mathbb{P} denotes the truncated parabola $\mathbb{P} = \{(t, t^2) : t \in [-1, 1]\}$, and $S^1 \subset \mathbb{R}^2$ is the unit circle. The notation “ π ” (without subindex) refers to the projection $\pi(x, y) = x$.

2.2. Furstenberg sets and (δ, s) -sets. We have already seen that Theorem 1.1 implies the non-existence of Borel subrings of intermediate dimension. This suggests that the non-existence proofs could be useful in establishing Theorem 1.1. This turns out to be the case, although not in a completely straightforward fashion.

In fact, we will prove Theorem 1.1 by applying the non-existence of $(s, 2s)$ -Furstenberg sets of dimension $2s$. The connection between the Furstenberg set problem and the Borel subring problem was initially discovered by Katz and Tao [11], and has been thereafter applied several times to make progress in the Furstenberg set problem, see [6, 10, 15, 17]. In the current paper, we may use the best current estimates on $(s, 2s)$ -Furstenberg sets as a black box. We record the necessary preliminaries now.

Definition 2.1. ((δ, s, C) -set) Let (X, d) be a metric space, let $\delta \in (0, 1]$, and $C, s > 0$. A finite set $P \subset X$ is called a (δ, s, C) -set if

$$|P \cap B(x, r)| \leq C \left(\frac{r}{\delta}\right)^s, \quad x \in X, \quad r \geq \delta.$$

If the value of the constant “ C ” is not relevant, we may also write “ (δ, s) -set” in place of “ (δ, s, C) -set”. We will need this notion in $X = \mathbb{R}^d$, and also in the space of all affine lines in \mathbb{R}^2 , denoted $\mathcal{A}(2, 1)$. We denote by $d_{\mathcal{A}(2, 1)}$ the following metric on $\mathcal{A}(2, 1)$:

$$d_{\mathcal{A}(2, 1)}(\ell_1, \ell_2) := \|\pi_1 - \pi_2\| + |a_1 - a_2|,$$

where π_1, π_2 are the orthogonal projections to the subspaces L_1, L_2 parallel to ℓ_1, ℓ_2 , $\|\cdot\|$ is the operator norm, and a_j is the unique point on $\ell_j \cap L_j^\perp$. The only property of the metric $d_{\mathcal{A}(2, 1)}$ we need explicitly is that if $\ell_1, \ell_2 \in \mathcal{A}(2, 1)$ are at distance

$r := d_{\mathcal{A}(2,1)}(\ell_1, \ell_2)$, then

$$(2.2) \quad \ell_1 \cap B(1) \subset \ell_2(Cr)$$

for some absolute constant $C > 0$. Implicitly, we also need that the results quoted below for Furstenberg sets are valid with the choice of metric $d_{\mathcal{A}(2,1)}$, but this requirement is compatible with (2.2).

Definition 2.3. ((δ, t, C) -set of lines) A set of lines $\mathcal{L} \subset \mathcal{A}(2, 1)$ is a (δ, t, C) -set if it is a (δ, s, C) -set in the metric space $(\mathcal{A}(2, 1), d_{\mathcal{A}(2,1)})$.

We may now define δ -discretised Furstenberg sets:

Definition 2.4. (Discretised (s, t) -Furstenberg set) Let $0 < s \leq 1$ and $0 < t \leq 2$, $C > 0$, and $\delta \in (0, 1]$. A set $F \subset \mathbb{R}^2$ is a δ -discretised (s, t, C) -Furstenberg set if there exists a (δ, t, C) -set of lines with $|\mathcal{L}| \geq C^{-1}\delta^{-t}$ with the property that $F \cap \ell(\delta)$ contains a (δ, s, C) -set of cardinality $\geq C^{-1}\delta^{-s}$ for all $\ell \in \mathcal{L}$.

It is known that if $0 < s < 1$, $s < t \leq 2$, and $\epsilon = \epsilon(s, t) > 0$ is small enough, then every δ -discretised $(s, t, \delta^{-\epsilon})$ -Furstenberg set $E \subset \mathbb{R}^2$ satisfies $|E|_\delta \geq \delta^{-2s-\epsilon}$. This is a recent result of the first author with Shmerkin [17]. In the present paper, we only need the special case $t = 2s$, which was known much earlier: the case $s = \frac{1}{2}$ is due to Bourgain [2] (modulo a slightly different definition of discretised Furstenberg sets), and the general case $s \in (0, 1)$ is due to Héra, Shmerkin, and Yavicoli [10]. In the case $t = 2s$, the best known constant “ ϵ ” is actually fairly large, due to recent work of Di Benedetto and Zahl [6]. We quote their version of the result below:

Theorem 2.5. Let $c(s) := s(1-s)/(6(155+68s))$. Then, for every $0 < s < 1$ and every $c < c(s)$ there exists $\epsilon > 0$ such that the following holds for all $\delta > 0$ sufficiently small. Let $F \subset \mathbb{R}^2$ be a δ -discretised $(s, 2s, \delta^{-\epsilon})$ -Furstenberg set. Then, $|F|_\delta \geq \delta^{-2s-c}$.

All the results on Furstenberg sets mentioned above are based on reductions to the *discretised sum-product theorem*: if $A \subset [0, 1]$ is a (δ, s) -set of cardinality $\sim \delta^{-s}$, $s \in (0, 1)$, then either $|A + A|_\delta \geq \delta^{-s-\epsilon}$ or $|A \cdot A|_\delta \geq \delta^{-s-\epsilon}$ for some $\epsilon = \epsilon(s) > 0$. This result is originally due to Bourgain [2] from the early 2000s, but a more quantitative version was proven recently by Guth, Katz, and Zahl [9]. This result (combined with a more efficient reduction) enabled Di Benedetto and Zahl to prove Theorem 2.5. We do not attempt to quantify the constant “ ϵ ” appearing in Theorem 1.1. Theorem 1.1 is based on a reduction to Theorem 2.5, but not a particularly straightforward one. The value of “ ϵ ” we obtain in Theorem 1.1 is anyway much smaller than the constant “ $c(s)$ ” in Theorem 2.5.

3. First part of Theorem 1.1

In this section, we establish the L^4 -bound in Theorem 1.1. The proof is based on the use of wave packet decompositions, which we briefly define in the next section.

3.1. Wave packet decomposition. The material in this section is standard, and we follow Demeter’s book [5, Exercise 2.7]. For full details, although in slightly different notation, see the lecture notes [16]. We use the notation $w(x) := (1+|x|)^{-100}$. If $T \subset \mathbb{R}^2$ is a rectangle, we write $w_T := w \circ A_T$, where A is the affine map taking T to $[-1, 1]^2$.

Proposition 3.1. (Definition of wave packet decomposition) Let μ be a finite Borel measure with $\text{spt}(\mu) \subset \mathbb{P}$, and let $\delta \in (0, 1]$. Let Θ be a partition of \mathbb{P} into arcs

of length $\sim \delta^{1/2}$. Write $\mu_\theta := \mu \cdot \mathbf{1}_\theta$ for $\theta \in \Theta$, so that

$$\mu = \sum_{\theta \in \Theta} \mu_\theta \quad \text{and} \quad \hat{\mu} = \sum_{\theta \in \Theta} \hat{\mu}_\theta.$$

Fix $\theta \in \Theta$, and let $R_\theta \subset \mathbb{R}^2$ be a rectangle of dimensions $\sim \delta \times \delta^{1/2}$ such that $\theta \subset \frac{1}{4}R_\theta$. Let \mathcal{T}_θ be a tiling of \mathbb{R}^2 by rectangles “ T ” dual to R_θ , with dimensions roughly $\delta^{-1} \times \delta^{-1/2}$. Then, for each $T \in \mathcal{T}_\theta$ we can associate a function $W_T \in \mathcal{S}(\mathbb{R}^2)$ with the following properties:

(W1) $\text{spt } \widehat{W}_T \subset R_\theta$, $\|W_T\|_{L^2} \sim 1$, and

$$|W_T| \lesssim_N \text{Leb}(T)^{-1/2} \cdot (w_T)^N, \quad N \geq 1.$$

(W2) If $F \in L^2(\mathbb{R}^2)$ with $\text{spt } F \subset \frac{1}{2}R_\theta$, then

$$\widehat{F} = \sum_{T \in \mathcal{T}_\theta} \langle \widehat{F}, W_T \rangle W_T \quad \text{and} \quad \|F\|_{L^2}^2 \sim \sum_{T \in \mathcal{T}_\theta} |\langle \widehat{F}, W_T \rangle|^2,$$

where $\langle \cdot, \cdot \rangle$ refers to inner product in L^2 .

In particular, property (W2) can be applied to $\mu_{\theta,\delta} := \mu_\theta * \psi_\delta$, where $\psi_\delta = \delta^{-2}\psi(x/\cdot)$ is an approximate identity: we only need that $\text{spt}(\mu_{\theta,\delta}) \subset \frac{1}{2}R_\theta$, and this can be arranged (by choosing ψ with sufficiently small support to begin with), since $\text{spt}(\mu_\theta) \subset \theta \subset \frac{1}{4}R_\theta$. Therefore, by (W2),

$$(3.2) \quad \widehat{\mu}_\delta = \sum_{\theta \in \Theta} \widehat{\mu}_{\theta,\delta} = \sum_{\theta \in \Theta} \sum_{T \in \mathcal{T}_\theta} a_T W_T, \quad a_T = \langle \widehat{\mu}_{\theta,\delta}, W_T \rangle \quad \text{for } T \in \mathcal{T}_\theta.$$

The representation (3.2) is known as the wave packet decomposition of μ at scale δ .

3.2. A non-concentration estimate for wave packets. The following lemma shows that if μ is an s -dimensional Frostman measure on \mathbb{P} , then the wave packet decomposition of μ at scale $\delta \in (0, 1]$ satisfies a 1-dimensional non-concentration condition, regardless of $s \in [0, 1]$. A more precise statement is (3.4): the “1-dimensionality” is visible in the exponent $\Delta = \Delta^1$. Why 1-dimensional, and not s -dimensional? The heuristic (and imprecise) reason is the following. Since the measure μ is s -dimensional, the directions of the wave packets indeed satisfy an s -dimensional non-concentration condition. However, since μ_θ is s -dimensional for each $\theta \in \Theta$, the wave packets associated to a fixed $\theta \in \Theta$ satisfy a $(1 - s)$ -dimensional non-concentration condition (see (3.5)). Finally, $1 = s + (1 - s)$.

Lemma 3.3. *Let $s \in [0, 1]$, $\delta \in (0, 1]$, and let μ be a Borel measure on parabola \mathbb{P} satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{P}$ and $r > 0$. Let $\delta > 0$, and let $\{a_T\}_{T \in \mathcal{T}_\theta}$, $\theta \in \Theta$, be the coefficients in the wave packet decomposition of μ at scale δ . Fix $\epsilon > 0$, and let $S \subset \mathbb{R}^2$ be any rectangle with dimensions $\Delta \times R^{1+\epsilon}$, where $R = \delta^{-1}$, and $R^{1/2} \leq \Delta \leq R^{1+\epsilon}$. Then,*

$$(3.4) \quad \sum_{T \subset S} |a_T|^2 \lesssim_\epsilon \Delta \cdot \delta^{s-1-4\epsilon}.$$

Proof. For this proof, $\delta \in (0, 1]$ is fixed, so we abbreviate $\mu_{\theta,\delta} =: \mu_\theta$. Fix a rectangle $S \subset \mathbb{R}^2$ as in the statement, and let $\Theta_S \subset \Theta$ be the caps $\theta \in \Theta$ with the property that $T \subset S$ for at least one rectangle $T \in \mathcal{T}_\theta$. Since the rectangles $T \in \mathcal{T}_\theta$ have longer side of length R , every $\theta \in \Theta_S$ lies inside a an arc $J_S \subset \mathbb{P}$ of diameter

$\text{diam}(J_S) \lesssim \Delta/R = \delta\Delta$ by elementary geometry. Therefore,

$$\sum_{T \subset S} |a_T|^2 = \sum_{\substack{\theta \subset \Theta \\ \theta \subset J_S}} \sum_{T \in \mathcal{T}_\theta} |a_T|^2.$$

To proceed, fix $\theta \in \Theta$ with $\theta \subset J_S$. We create some separation between the rectangles $T \in \mathcal{T}_\theta$ with the following trick. Partition \mathcal{T}_θ into $N \sim R^\epsilon = \delta^{-\epsilon}$ collections $\mathcal{T}_{\theta,j}$, $1 \leq j \leq N$, such that $\text{dist}(T, T') \gtrsim R^\epsilon$ for all distinct $T, T' \in \mathcal{T}_{\theta,j}$ (for $1 \leq j \leq N$ fixed). We claim that

$$(3.5) \quad \sum_{\substack{T \in \mathcal{T}_{\theta,j} \\ T \subset S}} |a_T|^2 \lesssim_\epsilon \Delta^{1-s} \cdot \delta^{-1-3\epsilon} \mu(R_\theta), \quad 1 \leq j \leq N,$$

where $R_\theta \supset \theta$ is the rectangle mentioned in Proposition 3.1. These rectangles have bounded overlap as $\theta \subset J_S$ varies, and their union is contained in the 100δ -neighbourhood of J . Once (3.5) has been established, the proof of (3.4) is concluded by summing over $1 \leq j \leq N$ and $\theta \in J_S$, and finally using the s -Frostman estimate $\mu(J_S(100\delta)) \lesssim (\delta\Delta)^s$:

$$\sum_{T \subset S} |a_T|^2 \lesssim_\epsilon \Delta^{1-s} \cdot \delta^{-1-4\epsilon} \sum_{\substack{\theta \in \Theta \\ \theta \subset J_S}} \mu(R_\theta) \lesssim \Delta^{1-s} \cdot \delta^{-1-4\epsilon} \mu(J_S(100\delta)) \lesssim \Delta \cdot \delta^{s-1-4\epsilon}.$$

To prove (3.5), let \mathcal{S} be a rectangle which is concentric with S , but inflated by a factor R^ϵ in both directions. Thus \mathcal{S} is a rectangle of dimensions $R^\epsilon\Delta \times R^{1+2\epsilon}$. Then, let $\eta_{\mathcal{S}} \in C_c^\infty(\mathbb{R}^2)$ be a bump function satisfying $\mathbf{1}_{\mathcal{S}} \leq \eta_{\mathcal{S}} \leq \mathbf{1}_{2\mathcal{S}}$. We observe that if $T \subset S$, then

$$|a_T| = |\langle \hat{\mu}_\theta, W_T \rangle| \leq |\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle| + \|W_T\|_{L^1(\mathbb{R}^2 \setminus \mathcal{S})} \lesssim_\epsilon |\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle| + \delta,$$

using that $\|\hat{\mu}_\theta\|_{L^\infty} \leq \|\mu_\theta\| \leq 1$, and the rapid decay $|W_T| \lesssim_\epsilon (w_T)^{1/\epsilon}$ outside $T \subset S$, stated in Proposition 3.1(W1). Taking further into account the (very crude) estimate $|\{T \in \mathcal{T}_\theta : T \subset S\}| \lesssim R = \delta^{-1}$, we find that

$$(3.6) \quad \sum_{\substack{T \in \mathcal{T}_{\theta,j} \\ T \subset S}} |a_T|^2 \lesssim_\epsilon \sum_{\substack{T \in \mathcal{T}_{\theta,j} \\ T \subset S}} |\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle|^2 + \delta.$$

To estimate the the main term in (3.6), we apply the abstract inequality

$$\sum |\langle x, e_k \rangle|^2 \leq \|x\|^2 + \sum_{k \neq l} \langle x, e_k \rangle \langle x, e_l \rangle \langle e_k, e_l \rangle,$$

valid for all inner product spaces $(H, \langle \cdot, \cdot \rangle)$, all $x \in H$, and all finite sequences of unit vectors $\{e_k\}_k \subset H$. (Proof: expand the inequality $0 \leq \|x - \sum \langle x, e_k \rangle e_k\|^2$.) Applying this to $x = \hat{\mu}_\theta \eta_{\mathcal{S}} \in L^2$, and $e_T := W_T / \|W_T\|_2$, and recalling that $\|W_T\|_2 \sim 1$, the result is

$$(3.7) \quad \sum_{\substack{T \in \mathcal{T}_{\theta,j} \\ T \subset S}} |\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle|^2 \lesssim \|\hat{\mu}_\theta \eta_{\mathcal{S}}\|_{L^2}^2 + \sum_{\substack{T, T' \in \mathcal{T}_{\theta,j} \\ T, T' \subset S, T \neq T'}} |\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle \langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_{T'} \rangle \langle W_T, W_{T'} \rangle|.$$

The number of terms in the second sum is $\lesssim R^2$, and also $|\langle \hat{\mu}_\theta \eta_{\mathcal{S}}, W_T \rangle| \lesssim \|W_T\|_{L^1} \lesssim R^2$. These factors are negligible compared to the fact that $|\langle W_T, W_{T'} \rangle| \lesssim_\epsilon R^{-10}$,

which follows from $\text{dist}(T, T') \geq R^\epsilon$ for distinct $T, T' \in \mathcal{T}_{\theta, j}$, and the rapid decay of W_T outside T . Combining (3.7) with (3.6), and using Plancherel, we find that

$$\sum_{\substack{T \in \mathcal{T}_{\theta, j} \\ T \subset \mathcal{S}}} |a_T|^2 \lesssim_\epsilon \|\mu_\theta * \widehat{\eta}_{\mathcal{S}}\|_{L^2}^2 + \delta.$$

To estimate $\|\mu_\theta * \widehat{\eta}_{\mathcal{S}}\|_{L^2}^2$, we first record that $\widehat{\eta}_{\mathcal{S}}$ is essentially supported in the dual rectangle \mathcal{S}^* of \mathcal{S} , which has dimensions $\delta^\epsilon \Delta^{-1} \times \delta^{1+2\epsilon}$. In particular, \mathcal{S}^* fits inside the ball $B(\Delta^{-1})$ with room to spare. From this, we first deduce that

$$\|\widehat{\eta}_{\mathcal{S}}\| \lesssim_\epsilon \text{Leb}(\mathcal{S}) \cdot w_{B(\Delta^{-1})} \sim \Delta \cdot R^{1+3\epsilon} \cdot w_{B(\Delta^{-1})}.$$

(The “ w ”-notation was introduced at the head of Section 3.1.) Since $\|\mu_\theta * w_{B(\Delta^{-1})}\|_{L^\infty} \lesssim \Delta^{-s}$ by the s -Frostman assumption of μ , we arrive at

$$\|\mu_\theta * \widehat{\eta}_{\mathcal{S}}\|_{L^\infty} \lesssim \Delta \cdot R^{1+3\epsilon} \|\mu_\theta * w_{B(\Delta^{-1})}\|_{L^\infty} \lesssim \Delta^{1-s} \cdot R^{1+3\epsilon},$$

and finally

$$\|\mu_\theta * \widehat{\eta}_{\mathcal{S}}\|_{L^2}^2 \leq \|\mu_\theta * \widehat{\eta}_{\mathcal{S}}\|_{L^\infty} \|\widehat{\eta}_{\mathcal{S}}\|_{L^1} \|\mu_\theta\|_{L^1} \lesssim \Delta^{1-s} \cdot R^{1+3\epsilon} \mu(R_\theta).$$

This concludes the proof of (3.5), and the lemma. \square

3.3. Proof of the L^4 -estimate. In this section, we complete the proof of the first part of Theorem 1.1.

Theorem 3.8. *Let $0 \leq s \leq 1$, and let μ be a Borel measure on $\Gamma = \{(t, t^2) : t \in [-1, 1]\}$ satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{P}$ and $r > 0$. Then, for every $\epsilon > 0$, there exists $C = C_{\epsilon, s} > 0$ such that*

$$(3.9) \quad \|\widehat{\mu}\|_{L^4(B(R))} \leq CR^{(1-s)/2+\epsilon}, \quad R \geq 1.$$

Proof. Fix $R \geq 1$, write $\delta := R^{-1}$. Let ψ_δ be the approximate identity appearing in the wave packet decomposition, Proposition 3.1. Moreover, choose ψ in such a way that $\widehat{\psi} \gtrsim \mathbf{1}_{B(1)}$, and thus $\varphi_\delta := \widehat{\psi}_\delta \gtrsim \mathbf{1}_{B(R)}$. Therefore, $\|\widehat{\mu}\|_{L^4(B(R))}^4 \lesssim \|\widehat{\mu}_\delta\|_{L^4}^4$, where $\mu_\delta = \mu * \psi_\delta$. We then expand $\widehat{\mu}_\delta$ as in (3.2):

$$\widehat{\mu}_\delta = \sum_{\theta \in \Theta} \widehat{\mu_{\theta, \delta}} = \sum_{\theta \in \Theta} \sum_{T \in \mathcal{T}_\theta} a_T W_T, \quad a_T = \langle \widehat{\mu_{\theta, \delta}}, W_T \rangle \text{ for } T \in \mathcal{T}_\theta.$$

We record at this point that, by Proposition 3.1(W2), we have

$$(3.10) \quad \sum_{\theta \in \Theta} \sum_{T \in \mathcal{T}_\theta} |a_T|^2 \sim \sum_{\theta \in \Theta} \|\widehat{\mu_{\theta, \delta}}\|_{L^2}^2 \lesssim \|\mu_\delta\|_{L^2}^2 \leq \|\mu_\delta\|_{L^\infty} \|\mu_\delta\|_{L^1} \lesssim \delta^{s-2},$$

recalling that $\mu_\delta = \mu * \psi_\delta$, where $\|\psi_\delta\|_{L^\infty} \lesssim \delta^{-2}$.

All the “oscillation” in our problem can be removed by an appeal to the Córdoba–Fefferman L^4 -orthogonality lemma, see [4, 8] for original references. The form we need is recorded in [5, Proposition 3.3] or [16, Theorem 7.14]:

$$(3.11) \quad \|\widehat{\mu}_\delta\|_{L^4}^4 \lesssim \int \left(\sum_{\theta \in \Theta} |\widehat{\mu}_\theta|^2 \right)^2 = \sum_{\theta, \theta'} \int \left| \sum_{T \in \mathcal{T}_\theta} a_T W_T \right|^2 \left| \sum_{T' \in \mathcal{T}_{\theta'}} a_{T'} W_{T'} \right|^2.$$

To begin estimating the right hand side, we note that the integral over $\mathbb{R}^2 \setminus B(R^2)$ is $\lesssim \delta^{10}$. The reason is that $a_T = \langle \widehat{\mu_{\theta, \delta}}, W_T \rangle = \langle \widehat{\mu}_\theta \varphi_\delta, W_T \rangle$, and φ_δ has rapid decay outside $B(R)$, whereas W_T is essentially supported on T . So, if we split

$$\sum_{T \in \mathcal{T}_\theta} a_T W_T(\xi) = \sum_{\text{dist}(T, 0) \geq R^{3/2}} a_T W_T(\xi) + \sum_{\text{dist}(T, 0) \leq R^{3/2}} a_T W_T(\xi),$$

then both terms will be $\lesssim \delta^{10}$ (and rapidly decaying) for $\xi \in \mathbb{R}^2 \setminus B(R^2)$: the first one because the coefficients a_T are small for $\text{dist}(0, T) \geq R^{3/2}$, and the second one because all the functions W_T with $\text{dist}(0, T) \leq R^{3/2}$ are small on $\mathbb{R}^2 \setminus B(R^2)$. We leave the full details to the reader. Thus, the integral in (3.11) can be restricted to $B(R^2)$ at the cost of adding δ^{10} , which is harmless in view of our aim (3.9).

If the supports of the functions W_T , $T \in \mathcal{T}_\theta$ were disjoint, we could expand the right hand side as

$$(3.12) \quad \sum_{T, T' \in \mathcal{T}} |a_T|^2 |a_{T'}|^2 \int |W_T W_{T'}|^2,$$

where \mathcal{T} stands for the union of all the families \mathcal{T}_θ , $\theta \in \Theta$. This is not quite accurate, and we resort to a trick we already employed in the proof of Lemma 3.3, namely splitting the collections \mathcal{T}_θ into $N \sim R^\epsilon$ sub-collections $\mathcal{T}_{\theta, j}$ where the tubes $T \in \mathcal{T}_{\theta, j}$ are separated by at least $\gtrsim R^\epsilon$. Using the trivial inequality $|c_1 + \dots + c_N|^2 \leq N^2 \max |c_j|^2$, we first estimate

$$(3.13) \quad \int \left| \sum_{T \in \mathcal{T}_\theta} a_T W_T \right|^2 \left| \sum_{T' \in \mathcal{T}_{\theta'}} a_{T'} W_{T'} \right|^2 \lesssim R^{4\epsilon} \max_{i, j} \int \left| \sum_{T \in \mathcal{T}_{\theta, i}} a_T W_T \right|^2 \left| \sum_{T' \in \mathcal{T}_{\theta', j}} a_{T'} W_{T'} \right|^2.$$

Now, using the rapid decay of the functions W_T outside T , and the (crude) uniform bound $|a_T| \leq \|W_T\|_{L^1} \lesssim R$, the integrand satisfies the following pointwise bound:

$$(3.14) \quad \left| \sum_{T \in \mathcal{T}_{\theta, i}} a_T W_T \right|^2 \lesssim_\epsilon \sum_{T \in \mathcal{T}_{\theta, i}} |a_T|^2 |W_T|^2 + \delta^{10},$$

and a similar estimate holds for the second factor of the integrand. The right hand side of (3.14) is further $\lesssim \delta^{-5}$ by (3.10) (a much better bound is easy to obtain, but this suffices.) When these bounds are plugged back into (3.13), and then (3.11), we arrive at the following substitute for (3.12):

$$(3.15) \quad \|\widehat{\mu}_\delta\|_{L^4}^4 \lesssim \int_{B(R^2)} \left(\sum_{\theta \in \Theta} |\widehat{\mu}_\theta|^2 \right)^2 + \delta^{10} \lesssim_\epsilon R^{4\epsilon} \sum_{T, T' \in \mathcal{T}} |a_T|^2 |a_{T'}|^2 \int |W_T W_{T'}|^2 + \delta.$$

The localisation to $B(R)$ was used to make sure that $\delta^5 \in L^1(B(R^2))$ with norm $\lesssim \delta$. To estimate the right hand side of (3.15), we imitate the proof of the L^2 -Kakeya maximal function bound, with the only non-trivial addition of inserting Lemma 3.3 at a suitable point. We fix $T \in \mathcal{T}_{\theta_0} \subset \mathcal{T}$, and we decompose

$$(3.16) \quad \sum_{T' \in \mathcal{T}} |a_{T'}|^2 \int |W_T W_{T'}|^2 = \sum_{\delta^{1/2} \lesssim \alpha \lesssim 1} \sum_{\substack{T' \in \mathcal{T} \\ \angle(T, T') \sim \alpha}} |a_{T'}|^2 \int |W_T W_{T'}|^2.$$

To be precise, the summation over $\delta^{1/2} \lesssim \alpha \lesssim 1$ runs over dyadic rationals in the indicated range, and the summation $\{T' \in \mathcal{T} : \angle(T, T') \sim \alpha\}$ runs over the tubes in those families \mathcal{T}_θ with $\alpha \leq |\theta - \theta_0| \leq 2\alpha$. Fix $\delta^{1/2} \lesssim \alpha \lesssim 1$. Fix also $T' \in \mathcal{T}$ with $\angle(T, T') \sim \alpha$, and let \mathbf{T}, \mathbf{T}' be tubes which are concentric with T, T' , but fattened by a factor R^ϵ in both directions. If $\mathbf{T} \cap \mathbf{T}' = \emptyset$, then

$$\int |W_T W_{T'}|^2 \lesssim_\epsilon \delta^{10}$$

by the rapid decay of $W_T, W_{T'}$. Therefore, the part of the sum (3.16) over such $T' \in \mathcal{T}$ is bounded from above by $\lesssim \delta^{10} \sum_{T' \in \mathcal{T}} |a_{T'}|^2 \lesssim \delta^{8+s} \leq \delta$, applying also (3.10).

Assume then that $\mathbf{T} \cap \mathbf{T}' \neq \emptyset$. Recall that \mathbf{T}, \mathbf{T}' are rectangles of dimensions $R^{1/2+\epsilon} \times R^{1+\epsilon}$. Therefore $\text{Leb}(\mathbf{T} \cap \mathbf{T}') \lesssim R^{1+2\epsilon}/\alpha$. Using this, and Proposition 3.1(W1), we first deduce that

$$(3.17) \quad \int |W_T W_{T'}|^2 \lesssim_\epsilon \int_{\mathbf{T} \cap \mathbf{T}'} |W_T W_{T'}|^2 + \delta \lesssim \text{Leb}(T)^{-2} \cdot \frac{R^{1+2\epsilon}}{\alpha} = \frac{R^{2\epsilon-2}}{\alpha}.$$

Moreover, since $\angle(T, T') = \angle(\mathbf{T}, \mathbf{T}') \sim \alpha$, the non-empty intersection of \mathbf{T}, \mathbf{T}' implies that $T' \subset \mathbf{T}' \subset \mathbf{T}(\Delta) =: S$, where $\Delta \sim \alpha \cdot R^{1+\epsilon} \geq R^{1/2}$. As usual, the notation stands $\mathbf{T}(\Delta)$ stands for the Δ -neighbourhood of \mathbf{T} , which is a rectangle of dimensions roughly $\Delta \times R^{1+\epsilon}$, noting that $\Delta \lesssim R^{1+\epsilon}$. Consequently, applying Lemma 3.3, we have

$$\begin{aligned} \sum_{\substack{T' \in \mathcal{T} \\ \angle(T, T') \sim \alpha, \mathbf{T} \cap \mathbf{T}' \neq \emptyset}} |a_{T'}|^2 \int |W_T W_{T'}|^2 &\stackrel{(3.17)}{\lesssim_\epsilon} \frac{R^{2\epsilon-2}}{\alpha} \cdot \sum_{\substack{T' \in \mathcal{T} \\ T' \subset S}} |a_{T'}|^2 \\ &\stackrel{\text{L. 3.3}}{\lesssim_\epsilon} \frac{R^{2\epsilon-2}}{\alpha} \cdot (\Delta \cdot \delta^{s-1-4\epsilon}) \sim R^{7\epsilon-s}. \end{aligned}$$

Plugging this back into (3.15)–(3.16), we see that

$$\|\hat{\mu}\|_{L^4(B(R))}^4 \lesssim_\epsilon R^{4\epsilon} \sum_{T \in \mathcal{T}} |a_T|^2 \sum_{\delta^{1/2} \lesssim \alpha \lesssim 1} R^{7\epsilon-s} \lesssim R^{11\epsilon-s} \sum_{T \in \mathcal{T}} |a_T|^2 \stackrel{(3.10)}{\lesssim} R^{2-2s+11\epsilon}.$$

This completes the proof of the proposition. □

4. Second part of Theorem 1.1

The purpose of this section is to prove the second part of Theorem 1.1, concerning exponents $p > 4$. In fact, since we already know the $p = 4$ endpoint from Theorem 3.8, it suffices to establish the ϵ -improvement in Theorem 1.1 for $p = 6$. Namely, if this is already known, and $4 < p \leq 6$, then $p = 4\theta + 6(1 - \theta)$ for some $\theta < 1$, and hence

$$\|\hat{\mu}\|_{L^p(B(R))}^p \leq \|\hat{\mu}\|_{L^4(B(R))}^{4\theta} \|\hat{\mu}\|_{L^6(B(R))}^{6(1-\theta)} \stackrel{\text{T. 3.8}}{\lesssim} R^{(2-2s)\theta} \cdot R^{(2-2s-\epsilon)(1-\theta)} = R^{2-2s-\epsilon(1-\theta)}.$$

The cases $p > 6$ follow from the trivial estimate $\|\hat{\mu}\|_{L^p(B(R))}^p \lesssim_p \|\hat{\mu}\|_{L^6(B(R))}^6$, using only that $\|\hat{\mu}\|_{L^\infty} \leq \mu(\mathbb{P}) \leq 2$ for every measure as in the hypothesis of Theorem 1.1. We then restate the case $p = 6$ of Theorem 1.1:

Theorem 4.1. *For every $s \in (0, 1)$, there exist $C = C(s) > 0$ and $\epsilon = \epsilon(s) > 0$ such that the following holds. Let μ be a Borel measure on \mathbb{P} satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{P}$ and $r > 0$. Then,*

$$(4.2) \quad \|\hat{\mu}\|_{L^6(B(R))}^6 \leq CR^{2-2s-\epsilon}, \quad R \geq 1.$$

4.1. Auxiliary results. The $L^6(B(R))$ -norm of $\hat{\mu}$ equals the L^2 -norm of the convolution $\mu * \mu * \mu$, roughly speaking mollified at scale R^{-1} . This quantity, on the other hand, counts R^{-1} -approximate solutions to the equation $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$, with $(x_1, \dots, x_6) \in \text{spt } \mu =: P$ (see Lemma 4.9). It is well-known that if $P \subset \mathbb{P}$ is a finite set, then the problem of counting such (exact) solutions is connected to an incidence-counting problem for circles in the plane. The connection was discovered by Bourgain and Bombieri [1] (for $P \subset S^1$) and then Bourgain and Demeter [3] (for $P \subset \mathbb{P}$). The connection is captured by the following lemma. The case $\delta = 0$ is sketched in [3, Proposition 2.15], but we give the details (we anyway need the details to prove the “approximate” version):

Lemma 4.3. *Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, and write*

$$(4.4) \quad (\xi_1, \xi_1^2) + (\xi_2, \xi_2^2) + (\xi_3, \xi_3^2) =: (a, b).$$

Then, the point

$$A(\xi_1, \xi_2) := (3(\xi_1 + \xi_2), \sqrt{3}(\xi_1 - \xi_2))$$

is contained on the circle $S_{a,b} := \partial B((2a, 0), \sqrt{6b - 2a^2})$. The following approximate version also holds. Assume that $\delta \in (0, 1]$, $\xi_1, \xi_2, \xi_3 \in [-1, 1]$, and (4.4) is replaced by

$$(4.5) \quad |(\xi_1, \xi_1^2) + (\xi_2, \xi_2^2) + (\xi_3, \xi_3^2) - (a, b)| \leq \delta.$$

If $r := 6b - 2a^2 \leq \delta$, then $A(\xi_1, \xi_2) \in B((2a, 0), C\sqrt{\delta})$ for an absolute constant $C > 0$. Otherwise, $A(\xi_1, \xi_2)$ is contained in the annulus $S_{a,b}(C\delta/\sqrt{r})$.

Remark 4.6. The proof below also shows that $6b - 2a^2 \geq 0$, whenever (a, b) arises as in (4.4). If the conclusion of Lemma 4.3 seems unintuitive at first, the following remark might be helpful: by (4.4), the point $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ lies on the circle in \mathbb{R}^3 obtained by intersecting the sphere $\xi_1^2 + \xi_2^2 + \xi_3^2 = b$ with the plane $\xi_1 + \xi_2 + \xi_3 = a$. The fact that $A(\xi_1, \xi_2)$ lies on the planar circle $S_{a,b}$ could be “derived” from this observation with some effort, but in the following proof it is simpler to just “check” the conclusion.

Proof of Lemma 4.3. We first record that the equation (4.4) yields

$$(4.7) \quad (a - \xi_3)^2 \stackrel{(4.4)}{=} (\xi_1 + \xi_2)^2 = \xi_1^2 + \xi_2^2 + 2\xi_1\xi_2 \stackrel{(4.4)}{=} b - \xi_3^2 + 2\xi_1\xi_2,$$

or in other words

$$(4.8) \quad 2\xi_1\xi_2 = a^2 - 2a\xi_3 + 2\xi_3^2 - b.$$

After this observation, the rest of the argument is rather straightforward. To check that $A(\xi_1, \xi_2) \in S((2a, 0), \sqrt{6b - 2a^2})$, we simply calculate the distance

$$\begin{aligned} |A(\xi_1, \xi_2) - (2a, 0)|^2 &= (3(\xi_1 + \xi_2) - 2a)^2 + 3(\xi_1 - \xi_2)^2 \\ &\stackrel{(4.4)}{=} (a - 3\xi_3)^2 + 3(\xi_1^2 + \xi_2^2 - 2\xi_1\xi_2) \\ &\stackrel{(4.4)}{=} (a^2 - 6a\xi_3 + 9\xi_3^2) + 3(b - \xi_3^2 - 2\xi_1\xi_2) \\ &\stackrel{(4.8)}{=} (a^2 - 6a\xi_3 + 9\xi_3^2) + 3(b - \xi_3 - a^2 + 2a\xi_3 - 2\xi_3^2 + b) \\ &= 6b - 2a^2. \end{aligned}$$

This is what we claimed in the first part of the statement.

The second part follows by inspecting the calculation above. Using $\max\{|\xi_1|, |\xi_2|, |\xi_3|\} \leq 1$, the calculation (4.7) combined with (4.5) shows that

$$2\xi_1\xi_2 = a^2 - 2a\xi_3 + 2\xi_3^2 - b + O(\delta).$$

This leads to

$$(3(\xi_1 + \xi_2) - 2a)^2 + 3(\xi_1 - \xi_2)^2 = 6b - 2a^2 + O(\delta).$$

In case $r = 6b - 2a^2 \leq \delta$, we may conclude that $(3(\xi_1 + \xi_2), \sqrt{3}(\xi_1 - \xi_2)) \in B((2a, 0), C\sqrt{\delta})$. In the opposite case we use $|\sqrt{r} - \sqrt{s}| \leq |r - s|/\sqrt{r}$ to estimate

$$|\sqrt{(3(\xi_1 + \xi_2) - 2a)^2 + 3(\xi_1 - \xi_2)^2} - \sqrt{6b - 2a^2}| \lesssim \delta/\sqrt{r},$$

so $A(\xi_1, \xi_2) \in S_{a,b}(C\delta/\sqrt{r})$ as claimed. (The latter estimates are also valid if $0 < r < \delta$, but in this case the bound δ/\sqrt{r} is not very useful.) \square

We next formalise the connection of Fourier transforms and additive energies:

Lemma 4.9. *Let $P_1, \dots, P_6 \subset \mathbb{R}^d$ be δ -separated sets, and let $\mu_1, \dots, \mu_6 \in C_c^\infty(\mathbb{R}^d)$ be functions satisfying $0 \leq \mu_j \leq \mathbf{1}_{P_j(\delta)}$. Then,*

$$\begin{aligned} & \int \widehat{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} \\ & \lesssim \delta^{5d} |\{(x_1, \dots, x_6) \in P_1 \times \dots \times P_6 : |(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)| \leq 6\delta\}|. \end{aligned}$$

Proof. By Plancherel,

$$\int \widehat{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6} = \int (\mu_1 * \mu_2 * \mu_3)(\mu_4 * \mu_5 * \mu_6).$$

For $r > 0$, write

$$m(z) := |\{(x_1, x_2, x_3) \in P_1 \times P_2 \times P_3 : |(x_1 + x_2 + x_3) - z| \leq r\}|$$

and

$$n(z) := |\{(x_4, x_5, x_6) \in P_4 \times P_5 \times P_6 : |(x_4 + x_5 + x_6) - z| \leq r\}|.$$

Then,

$$\begin{aligned} (\mu_1 * \mu_2 * \mu_3)(z) &= \iint \mu_1(z - x_2 - x_3) \mu_2(x_2) \mu_3(x_3) dx_2 dx_3 \\ &\lesssim \delta^{2d} \sum_{(x_2, x_3) \in P_2 \times P_3} \mathbf{1}_{P_1(3\delta)}(z - x_2 - x_3) \\ &\leq \delta^{2d} \sum_{x_1 \in P_1} |\{(x_2, x_3) \in P_2 \times P_3 : |z - (x_1 + x_2 + x_3)| \leq 3\delta\}| \\ &= \delta^{2d} m_{3\delta}(z), \end{aligned}$$

and similarly $(\mu_4 * \mu_5 * \mu_6)(z) \lesssim \delta^{2d} n_{3\delta}(z)$. Therefore,

$$\begin{aligned} & \int (\mu_1 * \mu_2 * \mu_3)(\mu_4 * \mu_5 * \mu_6) \lesssim \delta^{4d} \int m_{3\delta}(z) n_{3\delta}(z) dz \\ &= \delta^{4d} \sum_{x_1, \dots, x_6} \text{Leb}(\{z \in \mathbb{R}^d : |z - (x_1 + x_2 + x_3)| \leq 3\delta \text{ and } |z - (x_4 + x_5 + x_6)| \leq 3\delta\}). \end{aligned}$$

The sum runs over $(x_1, \dots, x_6) \in P_1 \times \dots \times P_6$, and it can evidently be restricted to those 6-tuples with $|(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)| \leq 6\delta$. For such 6-tuples, on the other hand, the possible $z \in \mathbb{R}^d$ lie in a ball of radius $\sim \delta$, and their Lebesgue measure contributes the 5th factor of “ δ^{4d} ”. This completes the proof of the lemma. \square

Finally, we record the following consequence of transversality:

Lemma 4.10. *Let $0 < \delta \leq \tau \leq 1$. Let $I, J \subset \mathbb{P}$ be arcs with $\text{dist}(I, J) \geq \tau$, and let $P_I \subset I$ and $P_J \subset J$ be δ -separated sets. Then, for all $x_0, y_0 \in \mathbb{R}^2$, and $C > 0$, we have*

$$|\{(x, y) \in P_I \times P_J : |(x + x_0) \pm (y + y_0)| \leq C\delta\}| \lesssim C^2/\tau.$$

Proof. First, we record that

$$(4.11) \quad \text{diam}((x_0 + I)(C\delta) \cap \pm(y_0 + J)(C\delta)) \lesssim C\delta/\tau.$$

This follows by parametrising the arcs $x_0 + I =: G(f_I)$ and $\pm(y_0 + J) =: G(f_J)$ as graphs of some quadratic functions “ f_I ” and “ f_J ”, and noting that $(f_I - f_J)' \gtrsim \tau$ (or $(f_J - f_I)' \gtrsim \tau$) by assumption. We extend $x_0 + I$ and $y_0 + J$ so that $f, g \in C^1(\mathbb{R})$, f, g are 2-Lipschitz, and the inequality $(f_I - f_J)'(x) \gtrsim \tau$ remains valid for all $x \in \mathbb{R}$. Clearly $\text{diam}(\{x \in \mathbb{R} : |(f - g)(x)| \leq r\}) \lesssim r/\tau$. Finally, it follows from the

fact that f, g are 2-Lipschitz that every point $(x, y) \in G(f_I)(C\delta) \cap G(f_J)(C\delta)$ has $|(f - g)(x)| \lesssim C\delta$. This gives (4.11).

Now, if $(x, y) \in P_I \times P_J$ satisfies $|(x + x_0) \pm (y + y_0)| \leq C\delta$, then certainly

$$x \in -x_0 + ((x_0 + I)(C\delta) \cap \pm(y_0 + J)(C\delta)).$$

Since P_I is δ -separated, it follows from (4.11) that the number of admissible “ x ” is $\lesssim C/\tau$. Finally, for every admissible $x \in P_I$, the number possible $y \in P_J$ satisfying $|(x + x_0) \pm (y + y_0)| \leq C\delta$ is $\lesssim C$, by the δ -separation of P_J . This completes the proof. \square

4.2. Proof of Theorem 4.1: initial reductions. Let $s \in (0, 1)$, and let μ be a measure as in Theorem 4.1, satisfying $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{P}$ and $r > 0$. In this section, the implicit constants in the “ \lesssim ” notation are allowed to depend on “ s ”. We claim that

$$(4.12) \quad \int |\hat{\mu}(\xi)|^6 \chi_R(\xi) d\xi \lesssim R^{2-2s-\epsilon}, \quad R \geq 1,$$

for some $\epsilon = \epsilon(s) > 0$, where $\chi_R \in \mathcal{S}(\mathbb{R}^2)$ a Schwartz function satisfying $\mathbf{1}_{B(R)} \lesssim \chi_R \lesssim 1$ (constants independent of R), decaying rapidly outside $B(2R)$, and with $\text{spt } \widehat{\chi_R} \subset B(\delta)$ (as usual $\delta = R^{-1}$). Concretely, it will be useful to take χ_R of the form

$$(4.13) \quad \chi_R = (\widehat{\varphi_\delta})^6,$$

where $\varphi_\delta(x) = \delta^{-2}\varphi(x/\delta)$, and $\varphi \in C^\infty(B(1))$, and $\widehat{\varphi_\delta} \geq 0$.

Here is a brief and informal description of the proof. We fix a small parameter $\epsilon = \epsilon(s) > 0$. We will first reduce the proof of (4.12) to an “extremal” situation where the measure μ is concentrated on $\lesssim \delta^{-s-\epsilon}$ arcs $I \subset \mathbb{P}$ of length δ , each satisfying $\delta^{s+\epsilon} \lesssim \mu(I) \leq \delta^s$. Roughly speaking, if the measure μ fails to look like this, the estimate (4.12) will readily follow from Theorem 3.8. After this reduction, in the next section, we will make the counter assumption that (4.12) fails for some measure of the kind mentioned above. This information is then used to construct a δ -discretised $(s, 2s, \delta^{-C\epsilon})$ -Furstenberg set $F \subset \mathbb{R}^2$ with $|F|_\delta \lesssim \delta^{-2s-C\epsilon}$, for some absolute constant $C > 0$. Choosing $\epsilon > 0$ sufficiently small will finally violate Theorem 2.5, and the proof of (4.12) will be complete.

We turn to the details. We start by reducing the proof of (4.12) to the case where μ is “essentially constant” at scale δ . This is a simple consequence of pigeonholing, but let us make the statement precise. Given a dyadic rational $r \in 2^{-\mathbb{N}}$, let \mathcal{D}_r be the partition of $[-1, 1)$ to dyadic intervals of length r . For $I \in \mathcal{D}_r$, we also write $\tilde{I} \subset \mathbb{P}$ for the arc “above” I on \mathbb{P} . We define $\mu_I := \mu|_{\tilde{I}}$. Now, we claim that in order to prove (4.12), it suffices to do so for measures μ with the following extra property: there exists a constant $\kappa \in 2^{-\mathbb{N}}$ such that if $I \in \mathcal{D}_\delta$, then either

$$(4.14) \quad \mu_I \equiv 0 \quad \text{or} \quad \kappa \leq \mu_I(\mathbb{P}) = \mu(\tilde{I}) \leq 2\kappa.$$

To see this, note that every measure μ , as in the statement of the theorem, can be written as a series $\mu = \sum_{\kappa \in 2^{-\mathbb{N}}} \mu_\kappa$, where μ_κ has the additional property (4.14). Moreover, let us record the following three observations: first $\|\widehat{\mu_\kappa}\|_\infty \leq \mu_\kappa(\mathbb{R}^2) \lesssim \kappa \cdot |\mathcal{D}_\delta| \sim \kappa R$, second

$$(4.15) \quad \left| \sum_{\kappa \leq \delta^2} \alpha_\kappa \right|^6 = \left| \sum_{\kappa \leq \delta^2} \kappa^{1/6} \kappa^{-1/6} \alpha_\kappa \right|^6 \leq \left(\sum_{\kappa \leq \delta^2} \kappa^{1/5} \right)^5 \left(\sum_{\kappa \leq \delta^2} \kappa^{-1} |\alpha_\kappa|^6 \right) \lesssim \kappa^{-1} \sum_{\kappa \leq \delta^2} |\alpha_\kappa|^6$$

for all $\{\alpha_\kappa\} \subset \mathbb{C}$, and third $\int \chi_R(\xi) d\xi \lesssim \text{Leb}(B(R)) \sim R^2$, using the rapid decay of χ_R outside $B(2R)$. Combining these three observations leads to the estimate

$$\int \left| \sum_{\kappa \leq \delta^2} \widehat{\mu}_\kappa(\xi) \right|^6 \chi_R(\xi) d\xi \stackrel{(4.15)}{\lesssim} \sum_{\kappa \leq \delta^2} \kappa^{-1} \int |\widehat{\mu}_\kappa(\xi)|^6 \chi_R(\xi) d\xi \lesssim R^8 \sum_{\kappa \leq \delta^2} \kappa^5 \sim R^8 \delta^{10} = R^{-2}.$$

This is much better than what we claim at (4.12). On the other hand, since the sum over $\delta^2 \leq \kappa \leq 1$ only contains $\lesssim \log(1/\delta) = \log R$ terms, we also have

$$\int \left| \sum_{\kappa \geq \delta^2} \widehat{\mu}_\kappa(\xi) \right|^6 \chi_R(\xi) d\xi \lesssim (\log R) \cdot R^{2-2s-\epsilon},$$

assuming that (4.12) has already been established for each measure μ_κ individually. Thus, (4.12) holds with “ $\epsilon/2$ ” instead of “ ϵ ” for the original measure μ .

From now on, we assume that μ satisfies the additional property (4.14) for some $\kappa \in 2^{-\mathbb{N}}$. Another simple initial reduction is this: we may assume that

$$(4.16) \quad \mu(\mathbb{R}^2) \geq \delta^\epsilon$$

for a small constant $\epsilon = \epsilon(s) > 0$ (whose value will be determined during the proof). Indeed, in the opposite case $\|\widehat{\mu}\|_{L^\infty} \leq \delta^\epsilon$, and

$$\int |\widehat{\mu}(\xi)|^6 \chi_R(\xi) d\xi \leq \delta^{2\epsilon} \int |\widehat{\mu}(\xi)|^4 \chi_R(\xi) d\xi \lesssim_\epsilon R^{2-2s-\epsilon},$$

by Theorem 3.8 (or rather a version of it with the smooth cut-off χ_R , which is easy to deduce from the proper statement).

We next reduce the proof of (4.12) to a (partially) bilinear statement. To this end, let \mathcal{W} be a Whitney decomposition of the set $\Omega := [-1, 1]^2 \setminus \{(x, x) : x \in [-1, 1]\}$ into squares of the form $Q = I \times J$, where $I, J \in \mathcal{D}_r$ for some $r \in 2^{-\mathbb{N}}$. With this notation, we may write

$$\begin{aligned} \widehat{\mu}(\xi)^2 &= \iint e^{-2\pi i(x+y)\xi} d\mu(x) d\mu(y) \stackrel{(*)}{=} \sum_{I \times J \in \mathcal{W}} \iint e^{-2\pi i(x+y)\xi} d\mu_I(x) d\mu_J(y) \\ &= \sum_{I \times J \in \mathcal{W}} \widehat{\mu}_I(\xi) \widehat{\mu}_J(\xi). \end{aligned}$$

Recall here that $\mu_I := \mu|_{\tilde{I}}$, where $\tilde{I} \subset \mathbb{P}$ is the arc “above” $I \in \mathcal{D}_r$. The equation (*) uses the fact that the “Whitney squares” $\tilde{I} \times \tilde{J}$ partition $\mu \times \mu$ almost all of $\mathbb{P} \times \mathbb{P}$. For future reference, we immediately record the estimates

$$(4.17) \quad \|\widehat{\mu}_I\|_\infty \leq \mu(\tilde{I}) \lesssim \ell(I)^s \quad \text{and} \quad \|\widehat{\mu}_J\|_\infty \leq \mu(\tilde{J}) \lesssim \ell(J)^s.$$

Now, we decompose

$$\begin{aligned} \int |\widehat{\mu}(\xi)|^6 \chi_R(\xi) d\xi &= \int |\widehat{\mu}(\xi)|^2 \widehat{\mu}(\xi)^2 \overline{\widehat{\mu}(\xi)}^2 \chi_R(\xi) d\xi \\ &= \sum_{I \times J \in \mathcal{W}} \int |\widehat{\mu}(\xi)|^2 \widehat{\mu}_I(\xi) \widehat{\mu}_J(\xi) \overline{\widehat{\mu}(\xi)}^2 \chi_R(\xi) d\xi. \end{aligned}$$

We denote the individual terms on the right hand side $\mathcal{F}(I \times J)$. To estimate these terms, fix a “separation constant” of the form $\tau := \delta^{\epsilon/100}$. Then, we write

$$\int |\widehat{\mu}(\xi)|^6 \chi_R(\xi) d\xi = \sum_{\substack{I \times J \in \mathcal{W} \\ \ell(I) = \ell(J) \leq \tau}} \mathcal{F}(I \times J) + \sum_{\substack{I \times J \in \mathcal{W} \\ \ell(I) = \ell(J) > \tau}} \mathcal{F}(I \times J) =: \mathcal{F}_{\leq \tau} + \mathcal{F}_{> \tau}.$$

The main work of the proof will be to show that $\mathcal{F}_{>\tau} \leq R^{2-2s-\epsilon}$ if $\epsilon = \epsilon(s) > 0$ is chosen sufficiently small. A much easier task, carried out immediately below, is to show that $\mathcal{F}_{\leq\tau} \lesssim R^{2-2s-\epsilon s/200}$. To do this, fix $I \times J \in \mathcal{W}$ with $\ell(I) = \ell(J) < \tau$, and start by applying (4.17) and then Theorem 3.8 (with constant $\epsilon s/200$) to estimate

$$\mathcal{F}(I \times J) \lesssim \mu(\tilde{I})\ell(J)^s \int |\hat{\mu}(\xi)|^4 \chi_R(\xi) d\xi \lesssim_\epsilon \mu(\tilde{I})\ell(J)^s R^{2-2s+\epsilon s/200}.$$

By the properties of Whitney squares in the domain Ω , if $I \times J \in \mathcal{W}$, then $J \subset CI$ for some absolute constant $C > 0$. This allows us to estimate as follows:

$$\begin{aligned} \mathcal{F}_{\leq\tau} &\lesssim_\epsilon R^{2-2s+\epsilon s/200} \sum_{\substack{I \times J \in \mathcal{W} \\ \ell(I) = \ell(J) \leq \tau}} \mu(\tilde{I})\ell(J)^s \\ &\leq R^{2-2s+\epsilon s/200} \sum_{r \leq \tau} r^s \sum_{I \in \mathcal{D}_r} \sum_{\substack{J \in \mathcal{D}_r \\ J \subset CI}} \mu(\tilde{I}) \lesssim R^{2-2s+\epsilon s/200} \tau^s = R^{2-2s-\epsilon s/200}. \end{aligned}$$

This is what we claimed regarding the term $\mathcal{F}_{\leq\tau}$, so in the sequel we focus on $\mathcal{F}_{>\tau}$. We note that the number of elements in $\{I \times J \in \mathcal{W} : \ell(I) = \ell(J) \geq \tau\}$ is $\lesssim R^{\epsilon/50}$. It now suffices to prove an upper bound of the following form for the individual terms in the definition of $\mathcal{F}_{>\tau}$:

$$(4.18) \quad \mathcal{F}(I \times J) \lesssim R^{2-2s-\epsilon}.$$

Once this has been accomplished, we may deduce that

$$\mathcal{F}_{>\tau} \lesssim R^{2-2s-\epsilon} \cdot |\{I \times J \in \mathcal{W} : \ell(I) = \ell(J) \geq \tau\}| \leq R^{2-2s-\epsilon/2}.$$

This will conclude the proof of Theorem 4.1.

Most of the proof of (4.18) will be contained in the next sections, but here we still reduce it to a special case where the constant “ κ ” from (4.14) satisfies $\kappa \geq \delta^{s+\epsilon}$ (the upper bound $\kappa \lesssim \delta^s$ is also true, and follows immediately from the s -Frostman condition of μ). To this end, fix $I \times J \in \mathcal{W}$ with $\ell(I) = \ell(J) \geq \tau$. Start by expanding

$$(4.19) \quad \mathcal{F}(I \times J) = \int \hat{\mu}_I \hat{\mu}_J \overline{\hat{\mu}_I \hat{\mu}_J} \cdot \chi_R \stackrel{(4.13)}{=} \int \hat{\mu}_I \hat{\mu}_J \overline{\hat{\mu}_I \hat{\mu}_J} \cdot \widehat{\varphi_\delta}^6,$$

and recalling that $\varphi_\delta = \delta^{-2}\varphi(\cdot/\delta) \in C_c^\infty(\mathbb{R}^2)$ satisfies $\text{spt } \varphi_\delta \subset B(\delta)$. Since $\mu(B(x, \delta)) \leq \kappa$ for all $x \in \mathbb{P}$ by (4.14), we have

$$(4.20) \quad \|\mu * \varphi_\delta\|_\infty \lesssim \delta^{-2}\kappa,$$

and μ_I, μ_J satisfy a similar estimate, being restrictions of μ . Now, Lemma 4.9 will be applicable to the right hand side of (4.19). To make this precise, let P, P_I, P_J be δ -nets in the supports of μ, μ_I, μ_J , respectively. Taking into account (4.20), Lemma 4.9 (with $d = 2$) applied to the functions $\mu_j \in \{\mu * \varphi_\delta, \mu_I * \varphi_\delta, \mu_J * \varphi_\delta\}$, $1 \leq j \leq 6$, implies that

$$\mathcal{F}(I \times J) \lesssim \delta^{-2}\kappa^6 |\{(x_1, \dots, x_6) \in P_I \times P_J \times P^4 : |(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)| \leq 6\delta\}|.$$

We expand the count over the 6-tuples as

$$\sum_{x_3, \dots, x_6 \in P} |\{(x_1, x_2) \in P_I \times P_J : |(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)| \leq 6\delta\}|.$$

Since $\text{dist}(I, J) \geq \tau$, it follows from Lemma 4.10 that each term in the sum here is $\lesssim \tau^{-1} = \delta^{-\epsilon/100}$. Consequently,

$$(4.21) \quad \mathcal{F}(I \times J) \lesssim \delta^{-2-\epsilon/100} \kappa^6 \cdot |P|^4 \lesssim \delta^{-2-\epsilon/100} \kappa^2.$$

In the second inequality, we used the lower bound $\mu(I) \geq \kappa$ from the almost constancy property (4.14) to deduce that $|P| \lesssim \kappa^{-1}$. From the inequality above, we finally see that if $\kappa \leq \delta^{s+\epsilon}$, then $\mathcal{F}(I \times J) \lesssim \delta^{2s-2-\epsilon/100+2\epsilon} \leq R^{2-2s-\epsilon}$, and (4.18) has been established. So, the remaining—and most substantial—case in the proof of (4.18) is where $\kappa \geq \delta^{s+\epsilon}$. In this case, we record that

$$(4.22) \quad |P| \lesssim \delta^{-s-\epsilon},$$

where we recall that P is a δ -net in $\text{spt } \mu$. We record here that P is a $(\delta, s, C\delta^{-2\epsilon})$ -set of cardinality $|P| \gtrsim \delta^{-s+\epsilon}$. Indeed, if $x \in \mathbb{P}$ and $r \geq \delta$, note that $B(x, 2r)$ contains a δ -arc \tilde{I} of μ measure $\mu(\tilde{I}) \sim \kappa \geq \delta^{s+\epsilon}$ around every point $y \in P \cap B(x, r)$ (this is because of the κ -almost constancy property of μ , and $P \subset \text{spt } \mu$). Therefore,

$$|P \cap B(x, r)| \lesssim \kappa^{-1} \mu(B(x, 2r)) \lesssim \delta^{-\epsilon} \cdot \left(\frac{r}{\delta}\right)^s, \quad x \in \mathbb{P}, r \geq \delta.$$

The lower bound $|P| \gtrsim \delta^{-s+\epsilon}$ follows from (4.16): indeed $\delta^\epsilon \leq \mu(\mathbb{R}^2) \lesssim |P|\kappa \lesssim |P|\delta^s$.

4.3. Finding an $(s, 2s)$ -Furstenberg set of circles. We then proceed to prove the inequality (4.18) under the assumption (4.22). In brief, we will show that if (4.22) fails, then we can construct a “ $2s$ -dimensional” family of circles centred along the x -axis, all of which contain an “ s -dimensional” subset of a fixed “ $2s$ -dimensional set”. This will eventually lead to a contradiction with the non-existence of $2s$ -dimensional $(s, 2s)$ -Furstenberg sets.

We have already seen above, as a consequence of Lemma 4.9, that

$$\mathcal{F}(I \times J) \lesssim \delta^{6s-2} |\{(x_1, \dots, x_6) \in P_I \times P_J \times P^4 : |(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)| \leq 8\delta\}|,$$

where we already plugged in the (trivial) upper bound $\kappa \lesssim \delta^s$. Let E_3 be the cardinality of 6-tuples on the right hand side. What remains to be done is to show that

$$(4.23) \quad E_3 \leq \delta^{-4s+\epsilon}$$

for some $\epsilon = \epsilon(s) > 0$, and for all $\delta > 0$ small enough. This will be true if (i) the separation $\text{dist}(I, J) \geq \delta^{\epsilon/100}$ is valid for $\epsilon > 0$ small enough, and (ii) the upper bound (4.22) holds for $\epsilon > 0$ small enough, both requirements only depending on $s \in (0, 1)$. To prove (4.23), we start by expanding

$$(4.24) \quad E_3 = \sum_{x_3, y_1, y_2, y_3 \in P} |\{(x_1, x_2) \in P_I \times P_J : |(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3)| \leq 6\delta\}|.$$

By the separation $\text{dist}(P_I, P_J) \geq \tau = \delta^{\epsilon/100} \geq \delta^\epsilon$, and Lemma 4.10, we have the uniform upper bound

$$(4.25) \quad |\{(x_1, x_2) \in P_I \times P_J : |x_1 + x_2 + X| \leq 6\delta\}| \lesssim \delta^{-\epsilon}, \quad X \in \mathbb{R}^2.$$

We now make the counter assumption that

$$(4.26) \quad E_3 \gtrsim \delta^{-4s}.$$

Here, and in the remainder of the argument, the notation “ \lesssim ” and “ \gtrsim ” is allowed to hide constants of the form $C\delta^{-C\epsilon}$ for an absolute constant $C > 0$. So, in particular (4.25) tells us that the left hand side is $\lesssim 1$ for all $X \in \mathbb{R}^2$. We also say that a set $P' \subset \mathbb{R}^d$ is a (δ, t) -set if P' is a (δ, t, C) -set with $C \lesssim 1$.

Now, apply (4.25) to $X = x_3 - (y_1 + y_2 + y_3)$, as in (4.24). Recall that $E_3 \gtrsim \delta^{-4s}$ by (4.26), and on the other hand the sum in (4.24) only contains $\lesssim \delta^{-4s}$ terms, by (4.22). These facts together imply that

$$(4.27) \quad |\{(x_1, x_2) \in P_I \times P_J : |(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3)| \leq 6\delta\}| \geq 1$$

for $\gtrsim \delta^{-4s}$ quadruples $(x_3, y_1, y_2, y_3) \in P^4$.

We restate this information in more convenient form. Since the number of quadruples (x_3, y_1, y_2, y_3) satisfying (4.27) is $\gtrsim \delta^{-4s}$, and $|P| \lesssim \delta^{-s}$, there exists a fixed point $Y := y_3 \in P$ such that (4.27) holds for $\gtrsim \delta^{-3s}$ triples $(x_3, y_1, y_2) \in P^3$. This point $Y \in P$ will not change during the remainder of the proof.

Further, since the number of triples (x_3, y_1, y_2) is $\gtrsim \delta^{-3s}$, and again $|P| \lesssim \delta^{-s}$, we may deduce the following: there exists a set $\mathcal{S} \subset P \times P$ of $|\mathcal{S}| \approx \delta^{-2s}$ pairs (y_1, y_2) with the property that (4.27) holds for $\gtrsim \delta^{-s}$ different choices $x_3 \in P$ (for y_1, y_2, Y fixed). In symbols, the cardinality of the set

$$(4.28) \quad P(y_1, y_2) := \{x_3 \in P : (4.27) \text{ holds for the quadruple } (x_3, y_1, y_2, Y)\}$$

is $|P(y_1, y_2)| \gtrsim \delta^{-s}$ for all $(y_1, y_2) \in \mathcal{S}$.

We briefly explain what happens next before giving the details. The set \mathcal{S} will be identified with a “ $(\delta, 2s)$ -set of circles” $S_{(y_1, y_2)} \subset \mathbb{R}^2$, all centred along the x -axis. Given a circle $S = S_{(y_1, y_2)}$ with $(y_1, y_2) \in \mathcal{S}$, the condition $|P(y_1, y_2)| \gtrsim \delta^{-s}$ will translate into the statement that the $(\approx \delta)$ -neighbourhood of S contains a (δ, s) -set of cardinality $\gtrsim \delta^{-s}$. Finally, it turns out that the union of all these (δ, s) -sets is contained in a set of the form $F := T(P \times P)$, where $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is an $O(1)$ -Lipschitz linear map. In particular, $|F|_\delta \lesssim |P \times P| \lesssim \delta^{-2s}$. These properties allow us to build (in Section 4.4) a δ -discretised $(s, 2s)$ -Furstenberg set of cardinality $\lesssim \delta^{-2s}$, and this will violate Theorem 2.5.

We then define the sets \mathcal{S} and F , which are inspired by Lemma 4.3. For every $(y_1, y_2) \in \mathcal{S}$ (or more generally $(y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2$), we write

$$(4.29) \quad y_1 + y_2 + Y =: \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2,$$

and we define the circle

$$(4.30) \quad S_{(y_1, y_2)} := S_{\sigma_1, \sigma_2} = \partial B \left((2\sigma_1, 0), \sqrt{6\sigma_2 - 2\sigma_1^2} \right).$$

The notation S_{σ_1, σ_2} is familiar from Lemma 4.3, and $6\sigma_2 - 2\sigma_1^2 \geq 0$, as observed in Remark 4.6. The definition of $S_{(y_1, y_2)}$ also depends on $Y \in P$, but this point can be viewed as “fixed forever”. We then define the (“Furstenberg”) set F as

$$(4.31) \quad F := \{(3(\pi(x) + \pi(y)), \sqrt{3}(\pi(x) - \pi(y))) : x, y \in P\},$$

where as usual $\pi(\xi_1, \xi_2) = \xi_1$. Evidently F is the image of the δ -separated set $P \times P$ under a certain $O(1)$ -Lipschitz linear map $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$. In particular,

$$(4.32) \quad |F|_\delta \lesssim |P \times P| \lesssim \delta^{-2s}.$$

The linear map T is closely connected with the map “ A ” from Lemma 4.3, indeed $T(x, y) = A(\pi(x), \pi(y))$ for all $(x, y) \in \mathbb{R}^4$.

Next, we claim that if $\mathbf{C} \approx 1$ is a suitable constant, and $(y_1, y_2) \in \mathcal{S}$ is fixed, then there exists a (δ, s) -set

$$(4.33) \quad F_{(y_1, y_2)} \subset F \cap S_{(y_2, y_2)}(\mathbf{C}\delta) \quad \text{with} \quad |F_{(y_1, y_2)}| \approx \delta^{-s}.$$

We will infer this by showing that the circle $S_{(y_1, y_2)}$ has radius ≈ 1 , and

$$(4.34) \quad [\pi(F \cap S_{(y_2, y_2)}(\mathbf{C}\delta))](C\delta) \supset 3 \cdot [\pi(\sigma) - \pi(P(y_1, y_2))],$$

where $\sigma = y_1 + y_2 + Y$, and $C > 0$ is an absolute constant. This implies (4.33). Indeed, recall that $P(y_1, y_2) \subset P$ is a subset of P of cardinality $|P(y_1, y_2)| \approx \delta^{-s}$, for every $(y_1, y_2) \in \mathcal{S}$. We observed below (4.22) that P is a (δ, s) -set of cardinality $|P| \approx \delta^{-s}$, and these properties are inherited (with slightly worse constants) by any subset of

cardinality $\approx \delta^{-s}$. In particular, $P(y_1, y_2)$ is a (δ, s) -set. Since $P(y_1, y_2) \subset \mathbb{P}$, the same is true of $\pi(P(y_1, y_2))$, and therefore the set on the right hand side of (4.34). Thus, (4.34) shows that $\pi(F \cap S_{(y_2, y_2)}(\mathbf{C}\delta))$ contains a (δ, s) -set of cardinality $\gtrsim \delta^{-s}$. But since $S_{(y_2, y_2)}$ has radius ≈ 1 (as we will prove), it follows that $F \cap S_{(y_2, y_2)}(\mathbf{C}\delta)$ itself must contain a (δ, s) -set $F_{(y_1, y_2)}$ of cardinality $|F_{(y_1, y_2)}| \approx \delta^{-s}$, as claimed.

We then verify the inclusion (4.34). Fix $(y_1, y_2) \in \mathcal{S}$, and write $\sigma := y_1 + y_2 + Y$. Denote $r := 6\sigma_2 - 2\sigma_1^2$, the square of the radius of the circle $S_{(y_1, y_2)} = S_{\sigma_1, \sigma_2}$ defined in (4.30).

Continuing with the proof of (4.34), we fix $x_3 \in P(y_1, y_2)$. By definition of $P(y_1, y_2)$ (see (4.28)), this means that the property (4.27) holds for the quadruple (x_3, y_1, y_2, Y) : there exists at least one pair $(x_1, x_2) \in P_I \times P_J$ such that

$$(4.35) \quad |(x_1 + x_2 + x_3) - \sigma| = |(x_1 + x_2 + x_3) - (y_1 + y_2 + Y)| \leq 6\delta.$$

We then apply Lemma 4.3. Write $x_j =: (\xi_j, \xi_j^2)$ for $1 \leq j \leq 3$. It follows from (4.35) and Lemma 4.3 that either

$$(4.36) \quad \begin{aligned} T(x_1, x_2) &= (3(\pi(x_1) + \pi(x_2)), \sqrt{3}(\pi(x_1) - \pi(x_2))) \\ &= A(\xi_1, \xi_2) \in S_{(y_1, y_2)}(C\delta/\sqrt{r}), \end{aligned}$$

or

$$T(x_1, x_2) \in B((2\sigma_1, 0), C\sqrt{\delta}),$$

where the latter case occurs if $r \leq \delta$. In both cases $T(x_1, x_2) \in F$, by definition (see (4.31)). We also note that in both cases $T(x_1, x_2) \in B((2\sigma_1, 0), \rho)$ where $\rho := C\sqrt{\delta}$ if $r \leq \delta$, and $\rho := \sqrt{r} + C\delta/\sqrt{r}$ if $r > \delta$. We will next infer from all of the above that $r \gtrsim 1$.

First, we use (4.35) to deduce that $\pi(\sigma) - \pi(x_3)$ lies at distance $\lesssim \delta$ from the point

$$\pi(x_1) + \pi(x_2) = \frac{1}{3} \cdot \pi(T(x_1, x_2)).$$

We have just seen that all the points $\pi(T(x_1, x_2))$ obtained this way (for various $x_3 \in P(y_1, y_2)$) lie in an interval of length $\sim \rho$ centred at $2\sigma_1$. But $P(y_1, y_2) \subset \mathbb{P}$ is a (δ, s) -set, so

$$\text{diam}(\pi(\sigma) - \pi(P(y_1, y_2))) \sim \text{diam}(P(y_1, y_2)) \approx 1.$$

This forces $\rho \approx 1$, hence also $r \approx 1$. In particular, we are safely outside the case $r \leq \delta$, and therefore (4.36) is true for all points $T(x_1, x_2)$. We infer that

$$\pi(x_1) + \pi(x_2) = \frac{1}{3} \cdot \pi(T(x_1, x_2)) \in \frac{1}{3} \cdot \pi(F \cap S_{(y_1, y_2)}(\mathbf{C}\delta)),$$

where $\mathbf{C} \approx 1$. Using once more that $\pi(\sigma) - \pi(x_3)$ lies at distance $\lesssim \delta$ from $\pi(x_1) + \pi(x_2)$ by (4.35), we may finally conclude that

$$3(\pi(\sigma) - \pi(x_3)) \in [\pi(F \cap S_{(y_1, y_2)}(\mathbf{C}\delta))](C\delta) = (\pi(F_{(y_1, y_2)}))(C\delta),$$

where $C > 0$ is absolute. This is what we claimed in (4.34).

4.4. Mapping circles to lines and concluding the proof of Theorem 4.1.

We start by taking stock of what we have proven so far. We have constructed the following objects:

- (1) $P \subset \mathbb{P}$ is a (δ, s) -set of cardinality $|P| \approx \delta^{-s}$,
- (2) $\mathcal{S} \subset P \times P$ is a δ -separated set of cardinality $|\mathcal{S}| \approx \delta^{-2s}$.
- (3) $F \subset \mathbb{R}^2$ is a set with $|F|_\delta \lesssim \delta^{-2s}$.

- (4) For every $(y_1, y_2) \in \mathcal{S}$, the intersection $F \cap S_{(y_1, y_2)}(\mathbf{C}\delta)$ contains a (δ, s) -set $F_{(y_1, y_2)}$ of cardinality $\approx \delta^{-s}$, where $\mathbf{C} \approx 1$. Here $S_{(y_1, y_2)}$ is the circle

$$S_{(y_1, y_2)} = \partial B \left((2\sigma_1, 0), \sqrt{6\sigma_2 - 2\sigma_1^2} \right), \quad (\sigma_1, \sigma_2) = y_1 + y_2 + Y.$$

- (5) $6\sigma_2 - 2\sigma_1^2 \approx 1$ for all $(y_1, y_2) \in \mathcal{S}$.

In property (5) it should be understood that σ_1, σ_2 refer to the coordinates of $y_1 + y_2 + Y$. For future reference, we add one more item:

- The conclusion of (4) holds if F is replaced by $F \cap \mathbb{H} := F \cap \{(x, y) : y \geq 0\}$.

Indeed, if we are so unlucky that most of $F \cap S_{(y_1, y_2)}(\mathbf{C}\delta)$ is contained in $\mathbb{C} \setminus \mathbb{H}$, for most $(y_1, y_2) \in \mathcal{S}$, then we simply replace F by the set $F' := \{(x, -y) : (x, y) \in F\}$. We also recall here that the notation $A \lesssim B$ means: $A \leq C\delta^{-C\epsilon}B$, where $\epsilon > 0$ was the small parameter fixed in the previous section, and $C > 0$ is absolute.

This is all the data from the previous section we need to complete the proof of Theorem 4.1. The moral is: the family of all circles centred along the x -axis has the same incidence geometric properties as the family $\mathcal{A}(2, 1)$. Indeed, there is an explicit map $\mathcal{G} : \mathbb{C} \rightarrow \mathbb{C}$ which sends circles centred along the x -axis to *chords* of $B(1)$. I warmly thank Josh Zahl for finding this map! In retrospect, this map is the one which transforms the *Poincaré half-plane model* in hyperbolic geometry to the *Beltrami–Klein (disc) model*.

Roughly speaking, the set of circles $S_{(y_1, y_2)}$, $(y_1, y_2) \in \mathcal{S}$, is a “ $(\delta, 2s)$ -set of circles”, because the parameter set $\mathcal{S} \subset P \times P$ is a $(\delta, 2s)$ -set. To be more precise, we will show the set of chords $\mathcal{G}(S_{(y_1, y_2)} \cap \{y \geq 0\})$, $(y_1, y_2) \in \mathcal{S}$, spans a $(\delta, 2s)$ -subset of $\mathcal{A}(2, 1)$.

If the reader finds plausible what we wrote above, then he may believe that (after the transformation by \mathcal{G}), the set F appearing in properties (3)–(4) is essentially a $(s, 2s)$ -Furstenberg set with $|F|_\delta \lesssim \delta^{-2s}$. Such a set should not exist by Theorem 2.5, and this contradiction will eventually conclude the proof of Theorem 1.1.

We then turn to the details. We spell out the map \mathcal{G} immediately. It is a composition of the form $\mathcal{G} = \mathcal{F} \circ \mathcal{C}$, where (in complex notation)

$$\mathcal{C}(z) = \frac{z - i}{z + i} \quad \text{and} \quad \mathcal{F}(z) = \frac{2z}{1 + |z|^2}.$$

See Figure 1. The Möbius map \mathcal{C} is the *Cayley transform*. Every Möbius map sends circles to circles or lines, and \mathcal{C} sends the x -axis to the unit circle S^1 . Since $\mathcal{C}(i) = 0$, one sees that \mathcal{C} maps the upper half-plane $\mathbb{H} := \{(x, y) : y \geq 0\}$ to the closed unit disc $B(1)$. Circles along the x -axis are mapped to circles intersecting S^1 twice in straight angles. A slightly special case occurs when a circle $S = S(x, r)$, $x \in \mathbb{R}$, contains the singularity $z = -i$ of \mathcal{C} : then S also contains the point $z = i$, and $\mathcal{C}(S)$ is a line passing through $\mathcal{C}(i) = 0$.

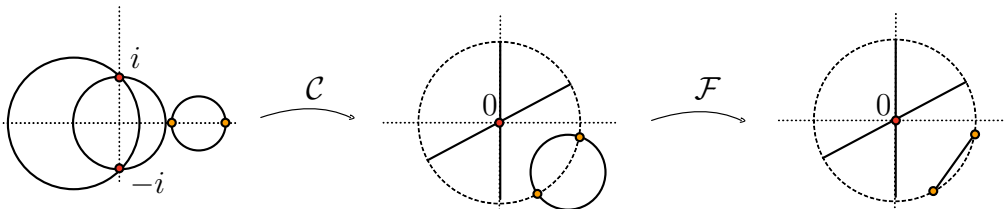


Figure 1. The maps \mathcal{C} and \mathcal{F} .

In the language of hyperbolic geometry, \mathcal{C} maps the *Poincaré half-plane model* to the *Poincaré disc model*, where the geodesics are precisely the circles intersecting S^1 in straight angles. It is more surprising that the Poincaré disc model can be further mapped (by \mathcal{F}) to the *Beltrami–Klein model*, where the geodesics are chords of S^1 . This is accomplished by the map \mathcal{F} . It is clear from the formula that

$$\mathcal{F}(B(1)) = B(1) \quad \text{and} \quad \mathcal{F}|_{S^1} = \text{id}.$$

It is a bit less easy to see that the (unique) circular arc intersecting $\{a, b\} \subset S^1$ in straight angles gets mapped to the chord $[a, b] \subset B(1)$ under \mathcal{F} . This is a standard fact of hyperbolic geometry, but it was not easy to find a simple (fully) geometric argument, so we provide one in Appendix A.

It is clear that \mathcal{C} is bilipschitz in any bounded subset of \mathbb{H} , and \mathcal{F} is certainly bilipschitz on the image $\mathcal{C}(\mathbb{H}) = B(1)$. So, the composition $\mathcal{G} = \mathcal{F} \circ \mathcal{C}$ is also bilipschitz on any bounded subset of \mathbb{H} . This implies that the neighbourhoods $S_{(y_1, y_2)}(\mathbf{C}\delta) \cap \mathbb{H}$ (see (4)) are mapped to $\mathbf{C}'\delta$ -neighbourhoods of chords inside $B(1)$, for some $\mathbf{C}' \approx 1$. In particular,

$$(4.37) \quad \mathcal{G}(S_{(y_1, y_2)}(\mathbf{C}\delta) \cap \mathbb{H}) \subset \ell_{(y_1, y_2)}(\mathbf{C}'\delta),$$

where $\ell_{(y_1, y_2)} \in \mathcal{A}(2, 1)$ is the line spanned by the chord $\mathcal{G}(S_{(y_1, y_2)} \cap \mathbb{H})$. For similar reasons, it is clear that for every $(y_1, y_2) \in \mathcal{S}$,

- the (δ, s) -subset of $F \cap S_{(y_1, y_2)}(\mathbf{C}\delta) \cap \mathbb{H}$ is mapped to a (δ, s) -subset of $\ell_{(y_1, y_2)}(\mathbf{C}'\delta)$,
- $|\mathcal{G}(F \cap \mathbb{H})|_\delta \lesssim \delta^{-2s}$.

Have we already established that $F' := \mathcal{G}(F \cap \mathbb{H})$ is a δ -discretised $(s, 2s)$ -Furstenberg set with $|F'|_\delta \lesssim \delta^{-2s}$? This would violate Theorem 2.5 and conclude the proof of Theorem 4.1. Unfortunately, the most technical piece is still missing: we need to verify that the family of lines $\mathcal{L} := \{\ell_{(y_1, y_2)} : (y_1, y_2) \in \mathcal{S}\}$, is a $(\delta, 2s)$ -set of cardinality $|\mathcal{L}| \gtrsim \delta^{-2s}$. More precisely, we will show that \mathcal{L} contains such a subset of lines.

We start with a few auxiliary results:

Lemma 4.38. *The set $\{y_1 + y_2 + Y : (y_1, y_2) \in \mathcal{S}\}$ contains a $(\delta, 2s)$ -set Σ of cardinality $\approx \delta^{-2s}$.*

Proof. Recall that $P \subset \mathbb{P}$ is a (δ, s) -set, and $\mathcal{S} \subset P \times P$ has $|\mathcal{S}| \approx \delta^{-2s}$. Because P is a (δ, s) -set, every arc $J \subset \mathbb{P}$ of length $\mathcal{H}^1(J) \leq \delta^{C\epsilon}$ satisfies $|P \cap J| \lesssim \delta^{C\epsilon s} \cdot \delta^{-s}$. In particular, if “ C ” here is chosen appropriately, at most $\frac{1}{2}|\mathcal{S}|$ pairs in \mathcal{S} are contained in $(P \cap J) \times (P \cap J)$ for some fixed arc $J \subset \mathbb{P}$ of length $\leq \delta^{C\epsilon}$. This implies that we may find two arcs $J_1, J_2 \subset \mathbb{P}$ such that $\text{dist}(J_1, J_2) \approx 1$, $|P \cap J_i| \approx \delta^{-s}$, and

$$|\mathcal{S} \cap (J_1 \times J_2)| \approx \delta^{-2s}.$$

Now, the map $g: (y_1, y_2) \mapsto y_1 + y_2 + Y$ is (≈ 1)-bilipschitz on $J_1 \times J_2$, and $\mathcal{S} \cap (J_1 \times J_2)$ is a $(\delta, 2s)$ -set. The image of $\mathcal{S} \cap (J_1 \times J_2)$ under “ g ” is a $(\delta, 2s)$ -set contained in $\{y_1 + y_2 + Y : (y_1, y_2) \in \mathcal{S}\}$, which is denoted “ Σ ” from now on. \square

Lemma 4.39. *Let $\theta > 0$, and let $\Omega_\theta \subset B(1) \subset \mathbb{R}^2$ be the set*

$$\Omega_\theta := \left\{ \sigma = (\sigma_1, \sigma_2) \in B(10) : \sqrt{6\sigma_2 - 2\sigma_1^2} \geq \theta \right\}.$$

The map

$$(\sigma_1, \sigma_2) \mapsto \Phi(\sigma_1, \sigma_2) = \left(2\sigma_1 - \sqrt{6\sigma_2 - 2\sigma_1^2}, 2\sigma_1 + \sqrt{6\sigma_2 - 2\sigma_1^2} \right)$$

is $O(\theta^{-1})$ -bilipschitz on Ω_θ .

The point of this technical lemma is that the $(\delta, 2s)$ -set Σ found in Lemma 4.38 is contained in Ω_θ for some $\theta \approx 1$ by the property (5) listed at the head of the section. The map Φ encodes the two intersection points of the circle S_{σ_1, σ_2} with the x -axis.

Proof of Lemma 4.39. It suffices to show that the map

$$(\sigma_1, \sigma_2) \mapsto \Psi(\sigma_1, \sigma_2) = \left(2\sigma_1, \sqrt{6\sigma_2 - 2\sigma_1^2} \right)$$

is $O(\theta^{-1})$ -bilipschitz on Ω_θ , because Φ is obtained by composing Ψ with the globally bilipschitz map $(x, y) \mapsto (x - y, x + y)$. Regarding Ψ , the whole argument is based on writing

$$(4.40) \quad \left| \sqrt{6\sigma_2 - 2\sigma_1^2} - \sqrt{6\eta_2 - 2\eta_1^2} \right| = \frac{|6(\sigma_2 - \eta_2) + 2(\sigma_1^2 - \eta_1^2)|}{\sqrt{6\sigma_2 - 2\sigma_1^2} + \sqrt{6\eta_2 - 2\eta_1^2}}.$$

The $O(\theta^{-1})$ -Lipschitz property on Ω_θ follows immediately. For the co-Lipschitz estimate, split into cases where $|\sigma_1 - \eta_2| \sim |(\sigma_1, \sigma_2) - (\eta_1, \eta_2)|$, and the opposite case. In the first case, observe that

$$|\Psi(\sigma_1, \sigma_2) - \Psi(\eta_1, \eta_2)| \geq |\sigma_1 - \eta_1| \sim |(\sigma_1, \sigma_2) - (\eta_1, \eta_2)|.$$

In the opposite case, observe that $|\sigma_1^2 - \eta_1^2| \ll |\sigma_2 - \eta_2|$, and use (4.40). \square

Corollary 4.41. *The set*

$$\Phi(\Sigma) = \left\{ \left(2\sigma_1 - \sqrt{6\sigma_2 - 2\sigma_1^2}, 2\sigma_1 + \sqrt{6\sigma_2 - 2\sigma_1^2} \right) : (\sigma_1, \sigma_2) \in \Sigma \right\}$$

is a $(\delta, 2s)$ -set of cardinality $\approx \delta^{-2s}$.

Proof. As discussed just before the proof of Lemma 4.39, the set Σ is contained in Ω_θ for some $\theta \approx 1$. Thus Φ is (≈ 1) -bilipschitz on Σ , and such maps preserve $(\delta, 2s)$ -sets. \square

We record here that

$$(4.42) \quad \Phi(\Sigma) \subset \{(\xi_1, \xi_2) \in [-10, 10]^2 : \xi_2 - \xi_1 \geq c\delta^{C\epsilon}\} =: [-10, 10]^2 \setminus \Delta,$$

where $c, C > 0$ are absolute constants. Indeed, recall that $6\sigma_2 - 2\sigma_1^2 \gtrsim 1$ for $(y_1, y_2) \in \mathcal{S}$ by (5). On the other hand, since $\Sigma \subset P+P+P \subset B(3)$, we have $\Phi(\Sigma) \subset [-10, 10]^2$.

We recap what \mathcal{G} does to circles centred on the x -axis. Every such circle is uniquely determined by its two intersection points with the x -axis (denoted \mathbb{R}). For $\xi_1, \xi_2 \in \mathbb{R}$, let $S(\xi_1, \xi_2) \subset \mathbb{R}^2$ be the circle centred at the x -axis with intersection points ξ_1, ξ_2 . Then, \mathcal{C} first maps $S(\xi_1, \xi_2)$ to the circle $S'(\xi_1, \xi_2)$ which intersects S^1 in straight angles at the two points

$$\mathcal{C}(\xi_j) = \frac{\xi_j - i}{\xi_j + i}, \quad j \in \{1, 2\}.$$

Next, \mathcal{F} sends $S'(\xi_1, \xi_2) \cap B(1)$ to the chord between $\mathcal{C}(\xi_1)$ and $\mathcal{C}(\xi_2)$. Consequently,

$$(4.43) \quad \mathcal{G}(S(\xi_1, \xi_2) \cap \mathbb{H}) = [\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)].$$

In the proof below, it will be useful to keep in mind that \mathcal{C} is bilipschitz $[-10, 10] \rightarrow \mathcal{C}([-10, 10]) \subset S^1$. In particular, if $(\xi_1, \xi_2) \in [-10, 10]^2 \setminus \Delta$, recall (4.42), then $[\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)]$ is a chord of length ≈ 1 .

Lemma 4.44. For $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1 \neq \xi_2$, let

$$\ell(\xi_1, \xi_2) := \text{span}([\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)]) \in \mathcal{A}(2, 1)$$

be the unique line containing the chord $[\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)]$. The map $(\xi_1, \xi_2) \mapsto \ell(\xi_1, \xi_2) \in \mathcal{A}(2, 1)$ is (≈ 1) -bilipschitz on the set $[-10, 10]^2 \setminus \Delta$ introduced in (4.42).

Proof. The inequality

$$d_{\mathcal{A}(2,1)}(\ell(\xi_1, \xi_2), \ell(\bar{\xi}_1, \bar{\xi}_2)) \lesssim |(\xi_1, \xi_2) - (\bar{\xi}_1, \bar{\xi}_2)|$$

is straightforward, and in fact holds for all $(\xi_1, \xi_2), (\bar{\xi}_1, \bar{\xi}_2) \in [-10, 10]^2$. This only uses the fact that \mathcal{C} is a Lipschitz map, and we leave the details to the reader. The trickier task is to prove that

$$(4.45) \quad d_{\mathcal{A}(2,1)}(\ell(\xi_1, \xi_2), \ell(\bar{\xi}_1, \bar{\xi}_2)) \gtrsim |(\xi_1, \xi_2) - (\bar{\xi}_1, \bar{\xi}_2)|$$

for all $(\xi_1, \xi_2), (\bar{\xi}_1, \bar{\xi}_2) \in [-10, 10]^2 \setminus \Delta$. This is clear if the left hand side is $\gtrsim 1$, so we may assume that

$$(4.46) \quad r := d_{\mathcal{A}(2,1)}(\ell(\xi_1, \xi_2), \ell(\bar{\xi}_1, \bar{\xi}_2)) \leq c_1 \delta^{C_1 \epsilon}$$

for suitable absolute constants $c_1, C_1 > 0$, to be determined in the course of the proof.

The key geometric observation is this: if $(\xi_1, \xi_2) \in [-10, 10]^2 \setminus \Delta$, and $r \in (0, 1]$, then

$$(4.47) \quad [\ell(\xi_1, \xi_2)](r) \cap S^1 \subset B(\mathcal{C}(\xi_1), \mathbf{C}r) \cup B(\mathcal{C}(\xi_2), \mathbf{C}r),$$

where $\mathbf{C} \approx 1$. This is because $[\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)] \subset B(1)$ is a chord of length ≈ 1 for $(\xi_1, \xi_2) \in [-10, 10]^2 \setminus \Delta$, and such chords intersect S^1 at angle ≈ 1 .

Now, let $(\xi_1, \xi_2), (\bar{\xi}_1, \bar{\xi}_2) \in [-10, 10]^2 \setminus \Delta$, and write $\ell := \ell(\xi_1, \xi_2)$ and $\bar{\ell} := \ell(\bar{\xi}_1, \bar{\xi}_2)$. Thus $r = d_{\mathcal{A}(2,1)}(\ell, \bar{\ell})$. This implies that

$$[\mathcal{C}(\bar{\xi}_1), \mathcal{C}(\bar{\xi}_2)] \subset \bar{\ell} \cap B(1) \subset \ell(\mathbf{C}r)$$

for some absolute constant $\mathbf{C} > 0$. As we mentioned all the way back in (2.2), the inclusion $\bar{\ell} \cap B(1) \subset \ell(\mathbf{C}r)$ is the only property of the metric $d_{\mathcal{A}(1,2)}$ we explicitly need in the paper. In particular,

$$(4.48) \quad \{\mathcal{C}(\bar{\xi}_1), \mathcal{C}(\bar{\xi}_2)\} \subset [\ell(\xi_1, \xi_2)](\mathbf{C}r) \cap S^1 \stackrel{(4.47)}{\subset} B(\mathcal{C}(\xi_1), \mathbf{C}r) \cup B(\mathcal{C}(\xi_2), \mathbf{C}r),$$

using (4.47). Formally speaking, this is possible in the following 4 ways:

- (G) $\mathcal{C}(\bar{\xi}_1) \in B(\mathcal{C}(\xi_1), \mathbf{C}r)$ and $\mathcal{C}(\bar{\xi}_2) \in B(\mathcal{C}(\xi_2), \mathbf{C}r)$, or
- (B1) $\{\mathcal{C}(\bar{\xi}_1), \mathcal{C}(\bar{\xi}_2)\} \subset B(\mathcal{C}(\xi_1), \mathbf{C}r)$, or
- (B2) $\{\mathcal{C}(\bar{\xi}_1), \mathcal{C}(\bar{\xi}_2)\} \subset B(\mathcal{C}(\xi_2), \mathbf{C}r)$, or
- (B3) $\mathcal{C}(\bar{\xi}_2) \in B(\mathcal{C}(\xi_1), \mathbf{C}r)$ and $\mathcal{C}(\bar{\xi}_1) \in B(\mathcal{C}(\xi_2), \mathbf{C}r)$.

The case (G) is good: it implies that

$$(4.49) \quad |\mathcal{C}(\bar{\xi}_j) - \mathcal{C}(\xi_j)| \lesssim r, \quad j \in \{1, 2\}.$$

Since \mathcal{C} is bilipschitz on $[-10, 10]$, this gives $|\bar{\xi}_j - \xi_j| \lesssim r$ for $j \in \{1, 2\}$, and therefore the proof of (4.45) is complete. So, it remains to show that the bad scenarios (B1)–(B3) cannot occur. In cases (B1)–(B2), we have $|\bar{\xi}_1 - \bar{\xi}_2| \lesssim \mathbf{C}r$, which is impossible by $(\bar{\xi}_1, \bar{\xi}_2) \in [-10, 10]^2 \setminus \Delta$, assuming that constants $c_1, C_2 > 0$ in the the upper bound for “ r ” were chosen correctly in (4.46) (relative to the constants in the definition of Δ). To see that the scenario (B3) is also impossible, write

$$(4.50) \quad \bar{\xi}_2 - \bar{\xi}_1 = (\bar{\xi}_2 - \xi_1) + (\xi_1 - \xi_2) + (\xi_2 - \bar{\xi}_1).$$

The middle term is negative with absolute value ≈ 1 (since $(\xi_1, \xi_2) \in [-10, 10]^2 \setminus \Delta$), and the two other terms have absolute value $\lesssim \mathbf{C}r$ in scenario (B3). Therefore, again, if the upper bound for “ r ” was chosen small enough at (4.46), we see that the right hand side of (4.50) is negative in case (B3). In particular $\bar{\xi}_2 < \bar{\xi}_1$, violating the assumption $(\bar{\xi}_1, \bar{\xi}_2) \in [-10, 10]^2 \setminus \Delta$. This proves that only scenario (G) is possible, and completes the proof of the lemma. \square

We can finally conclude that the set of lines $\{\text{span}(\mathcal{G}(S_{(y_1, y_2)} \cap \mathbb{H})) : (y_1, y_2) \in \mathcal{S}\}$ contains a $(\delta, 2s)$ -set of cardinality $\approx \delta^{-2s}$, namely the set $\{\text{span}(\mathcal{G}(S_\sigma \cap \mathbb{H})) : \sigma \in \Sigma\}$. For this final stretch, recall the set $\Sigma \subset \mathbb{R}^2$, which was a $(\delta, 2s)$ -subset of $\{y_1 + y_2 + Y : (y_1, y_2) \in \mathcal{S}\}$ of cardinality $|\Sigma| \approx \delta^{-2s}$. Recall that every $(\sigma_1, \sigma_2) \in \Sigma$ is associated to the circle S_{σ_1, σ_2} centred along the x -axis. Recall that $\mathcal{G} = \mathcal{F} \circ \mathcal{C}$ sends the intersection of each such circle with \mathbb{H} to a chord of S^1 .

Corollary 4.51. *The set*

$$\mathcal{G}(\Sigma) := \{\text{span}(\mathcal{G}(S_\sigma \cap \mathbb{H})) : \sigma \in \Sigma\} \subset \mathcal{A}(2, 1)$$

is a $(\delta, 2s)$ -set of lines with $|\mathcal{G}(\Sigma)| \approx \delta^{-2s}$.

Proof. In Corollary 4.41, we already observed that $\Phi(\Sigma)$ is a $(\delta, 2s)$ -set with $|\Phi(\Sigma)| \approx \delta^{-2s}$, and in (4.42) we recorded that $\Phi(\Sigma) \subset [-10, 10]^2 \setminus \Delta$. Now, we claim that

$$(4.52) \quad \text{span}(\mathcal{G}(S_\sigma \cap \mathbb{H})) = \ell(\Phi(\sigma)), \quad \sigma \in \Sigma,$$

where “ ℓ ” is the map from Lemma 4.44. This will complete the proof of the corollary, since the map “ ℓ ” was shown in Lemma 4.44 to be ≈ 1 -bilipschitz on the set $[-10, 10]^2 \setminus \Delta$, and in particular on $\Phi(\Sigma)$.

The proof of (4.52) is a matter of unwrapping the definitions. The circle S_σ intersects the x -axis in precisely the two points

$$\xi_1 := 2\sigma_1 - \sqrt{6\sigma_2 - 2\sigma_1^2} \quad \text{and} \quad \xi_2 := 2\sigma_1 + \sqrt{6\sigma_2 - 2\sigma_1^2},$$

which are also the coordinates of $\Phi(\sigma)$. Thus $S_\sigma = S(\xi_1, \xi_2)$ in the notation of (4.43). Therefore, recalling the definition of $\ell(\xi_1, \xi_2)$ from Lemma 4.44, we have

$$\begin{aligned} \ell(\Phi(\sigma)) &= \ell(\xi_1, \xi_2) = \text{span}([\mathcal{C}(\xi_1), \mathcal{C}(\xi_2)]) \stackrel{(4.43)}{=} \text{span}(\mathcal{G}(S(\xi_1, \xi_2) \cap \mathbb{H})) \\ &= \text{span}(\mathcal{G}(S_\sigma \cap \mathbb{H})). \end{aligned}$$

This completes the proof of (4.52). \square

We finally summarise the proof of Theorem 4.1:

Proof of Theorem 4.1. The map \mathcal{G} sends $S_{(y_1, y_2)} \cap \mathbb{H}$, $(y_1, y_2) \in \mathcal{S}$, to a certain chord of $B(1)$, which then spans a line $\ell_{(y_1, y_2)} \in \mathcal{A}(2, 1)$. The set of lines so obtained contains a $(\delta, 2s)$ -set \mathcal{L} of cardinality $|\mathcal{L}| = |\Sigma| \approx \delta^{-2s}$. This is the content of Corollary 4.51.

On the other hand, \mathcal{G} is bilipschitz on bounded subsets of \mathbb{H} , so

$$\mathcal{G}(S_{(y_1, y_2)}(\mathbf{C}\delta) \cap \mathbb{H}) \subset \ell_{(y_1, y_2)}(\mathbf{C}'\delta), \quad (y_1, y_2) \in \mathcal{S},$$

for some $\mathbf{C}' \sim \mathbf{C}$. In particular, $\ell_{(y_1, y_2)}(\mathbf{C}'\delta)$, $(y_1, y_2) \in \mathcal{S}$, contains the (δ, s) -set $\mathcal{G}(F_{(y_1, y_2)})$. Recall that $F_{(y_1, y_2)} \subset S_{(y_1, y_2)}(\mathbf{C}\delta) \cap F \cap \mathbb{H}$ was a (δ, s) -set of cardinality $|F_{(y_1, y_2)}| \approx \delta^{-s}$, see (4.33), and the remark below (5) (about why we may add the intersection with “ \mathbb{H} ”).

Therefore, $F' = \mathcal{G}(F \cap \mathbb{H}) \subset \mathbb{R}^2$ is a δ -discretised $(s, 2s)$ -Furstenberg set. The fact that $\mathcal{G}(F_{(y_1, y_2)})$ is contained in $\ell_{(y_1, y_2)}(\mathbf{C}'\delta)$, rather than $\ell_{(y_1, y_2)}(\delta)$, makes no difference: each of the thicker neighbourhoods can be covered by ≈ 1 thinner neighbourhoods, and the ensuing slightly larger family of lines is still a $(\delta, 2s)$ -set. Since $|F'|_\delta \lesssim |F|_\delta \lesssim \delta^{-2s}$, existence of F' violates Theorem 2.5, assuming that $\epsilon > 0$ was small enough, depending only on $s \in (0, 1)$. This contradiction completes the proof of Theorem 4.1. \square

Appendix A. Mapping circular arcs to chords

We give a short geometric argument for the fact that $z \mapsto 2z/(1 + |z|^2)$ maps the Poincaré disc model to the Beltrami–Klein model.

Proposition A.1. *The map $\mathcal{F}(z) = 2z/(1 + |z|^2)$ has the following property. Let $S \subset \mathbb{R}^2$ be a circle which intersects the unit circle S^1 in straight angles. Let $J := S \cap B(1)$ be the part of S inside the closed unit disc, and let $\{a, b\} := S \cap S^1$. Then $\mathcal{F}(J) = [a, b]$.*

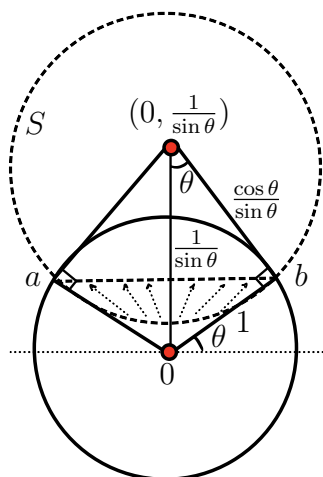


Figure 2. Objects in Proposition A.1.

Proof. It evidently suffices to consider the case where the centre of the circle S lies on the y -axis, as in Figure 2. Instead of checking that the map \mathcal{F} does the right thing, we “find” it as follows. We seek a map of the form $\mathcal{F}(z) = r(z)z$, where $r(z) \in [1, \infty)$, and which maps the arc J to the chord $[a, b]$.

Let $\theta \in (0, \pi)$ be the angle depicted in Figure 2. Using the hypothesis that S meets S^1 in straight angles, one calculates that the centre of S is the point $x := (0, \frac{1}{\sin \theta})$, and the radius of S is $r := \frac{\cos \theta}{\sin \theta}$. Moreover, the chord $[a, b]$ is contained in the set $\{y = \sin \theta\}$.

Every point on S , and in particular J , has the form

$$(A.2) \quad z = x + re = \left(\frac{\cos \theta}{\sin \theta} e_1, \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} e_2 \right), \quad e = (e_1, e_2) \in S^1.$$

Our desired map $\mathcal{F}(z) = r(z)z$ has the property of sending each $z \in J$ inside the set $\{y = \sin \theta\}$. Looking at the 2nd coordinate of z in (A.2), this gives

$$r(z) \left(\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} e_2 \right) = \sin \theta \quad \iff \quad r(z) = \frac{\sin^2 \theta}{1 + e_2 \cos \theta}.$$

On the other hand, a straightforward computation, using (A.2) and the identities $e_1^2 + e_2^2 = 1 = \cos^2 \theta + \sin^2 \theta$, shows that

$$\frac{2}{1 + |z|^2} = \frac{\sin^2 \theta}{1 + e_2 \cos \theta} = r(z).$$

Thus, $z \mapsto 2z/(1 + |z|^2)$ maps J to the chord $[a, b]$, as claimed. \square

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