Annales Fennici Mathematici
Volumen 48, 2023, 153–161

On the Karlsson–Nussbaum conjecture for resolvents of nonexpansive mappings

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Abstract. Let $D \subset \mathbb{R}^n$ be a bounded convex domain and $F\colon D \to D$ a 1-Lipschitz mapping with respect to the Hilbert metric d on D satisfying condition $d(sx + (1-s)y, sz + (1-s)w) \leq \max\{d(x,z), d(y,w)\}$. We show that if F does not have fixed points, then the convex hull of the accumulation points (in the norm topology) of the family $\{R_\lambda\}_{\lambda>0}$ of resolvents of F is a subset of ∂D. As a consequence, we show a Wolff–Denjoy type theorem for resolvents of nonexpansive mappings acting on an ellipsoid D.

Karlssonin–Nussbaumin konjektuuri venyttämättömien kuvausten resolventeille

Tiivistelmä. Olkoon $D \subset \mathbb{R}^n$ rajallinen konveksi alue ja $F\colon D \to D$ Lipschitzin kuvaus vakiolla 1 alueen D Hilbertin metriikan d suhteen, joka toteuttaa ehdon $d(sx + (1-s)y, sz + (1-s)w) \leq \max\{d(x,z), d(y,w)\}$. Osoitamme, että jos kuvauksella F ei ole kiintopisteitä, niin sen resolventtiperheen $\{R_\lambda\}_{\lambda>0}$ (normitopologian määräämien) kasautumispisteiden konveksi verho on reunan ∂D osajoukko. Tämän seurauksena osoitamme Wolffin–Denjoyn-tyyppisen lauseen ellipsoidin D venyttämättömien kuvausten resolventeille.

1. Introduction

The study of dynamics of nonlinear mappings started by considering iterates of holomorphic mappings on one-dimensional bounded domains. In this field, one of the first theorem is the classical Wolff–Denjoy theorem which describes dynamics of iteration of holomorphic self-mappings on the unit disc of the complex plane. It asserts that if $f\colon \Delta \to \Delta$ is a holomorphic map of the unit disc $\Delta \subset \mathbb{C}$ without a fixed point, then there is a point $\xi \in \partial\Delta$ such that the iterates f^n converge locally uniformly to ξ on Δ. Generalizations of this theorem in different directions have been obtained by numerous authors (see [1, 6, 9, 15, 13] and references therein). One such generalization was formulated by Beardon who noticed that the Wolff–Denjoy theorem can be considered in a purely geometric way depending only on the hyperbolic properties of a metric and gave its proof using geometric methods (see [4]). In [5], Beardon extended his approach for strictly convex bounded domains with the Hilbert metric. Considering the notion of the omega limit set $\omega_f(x)$ as the set of accumulation points of the sequence $f^n(x)$ and the notion of the attractor $\Omega_f = \bigcup_{x\in D} \omega_f(x)$, we can formulate a generalization of the Wolff–Denjoy theorem known as the Karlsson–Nussbaum conjecture, which was formulated independently by Karlsson and Nussbaum (see [10, 15]). This conjecture states that if D is a bounded convex domain in a finite-dimensional real vector space and $f\colon D \to D$ is a fixed point free nonexpansive mapping acting on the Hilbert metric space (D, d_H),

https://doi.org/10.54330/afm.126009
2020 Mathematics Subject Classification: Primary 53C60; Secondary 37C25, 47H09, 51M10.
Key words: Karlsson–Nussbaum conjecture, Wolff–Denjoy theorem, geodesic space, Hilbert's projective metric, resolvent, nonexpansive mapping.

then there exists a convex set $\Omega \subseteq \partial D$ such that for each $x \in D$, all accumulation points $\omega_f(x)$ of the orbit $O(x, f)$ lie in Ω.

The aim of this note is to show a variant of the Karlsson–Nussbaum conjecture for resolvents of nonexpansive (1-Lipschitz) mappings. For this purpose we construct in Section 3 the family of resolvents of a nonexpansive mapping and prove its main properties: nonexpansivity and the resolvent identity. In the literature, the resolvents usually occur in the context of Banach spaces or geodesic spaces that are Busemann convex, see e.g., [3, 17]. Then their construction is based on the Banach contraction principle. Since a Hilbert metric space (D, d_H) is in general not Busemann convex, our construction of resolvents is a little more complicated and exploits the argument related to Edelstein's theorem [8].

In Section 4 we formulate and prove the main theorem of this work. We show that if $D \subset \mathbb{R}^n$ is a bounded convex domain and $F\colon D \to D$ is a fixed point free nonexpansive mapping with respect to the Hilbert metric d_H on D satisfying condition

$$\text{(D)} \qquad d_H(sx + (1-s)y, sz + (1-s)w) \le \max\{d_H(x, z), d_H(y, w)\},$$

then the convex hull of the accumulation points of the family $\{R_\lambda\}_{\lambda>0}$ of resolvents of F is a subset of ∂D. Since a Hilbert metric space (D, d_H) is Busemann convex if and only if D is ellipsoid, we obtain as a corollary a Wolff–Denjoy type theorem for resolvents of nonexpansive mappings acting on an ellipsoid D.

2. Preliminaries

Let V be a finite dimensional real vector space, $D \subset V$ a convex bounded domain and (D, d) a metric space. A curve $\sigma\colon [a, b] \to D$ is said to be *geodesic* if $d(\sigma(t_1), \sigma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [a, b]$. We will use the same name for the image $\sigma([a, b]) \subset D$ of σ, denoted by $[\sigma(a), \sigma(b)]$. We say that D is a *geodesic space* if every two points of D can be joined by a geodesic. A map $F\colon D \to D$ is called *contractive* if $d(F(x), F(y)) < d(x, y)$ for any distinct points $x, y \in D$. A map $F\colon D \to D$ is called *nonexpansive* if for any $x, y \in D$, $d(F(x), F(y)) \le d(x, y)$.

We recall the definition of *the Hilbert metric space*. If $x, y \in D$, consider the straight line passing through x and y that intersects the boundary of D in precisely two points a and b. Assuming that x is between a and y, and y is between x and b, we define the cross-ratio metric

$$d_H(x, y) = \log\left(\frac{\|y-a\|\,\|x-b\|}{\|x-a\|\,\|y-b\|}\right), \quad x \ne y.$$

Furthermore, we put $d_H(x, y) = 0$ if $x = y$.

Following Beardon [5] we consider the subsequent lemmas.

Lemma 2.1. *Let $D_1, D_2 \subset V$, $D_1 \subset D_2$ be bounded convex domains and (D_1, d_1), (D_2, d_2) be Hilbert metric spaces, then $d_2 \le d_1$. Furthermore, for distinct points $x, y \in D_1$, $d_1(x, y) = d_2(x, y)$ iff the segment $L_{xy} \cap D_1$ coincides with $L_{xy} \cap D_2$.*

Lemma 2.2. *Suppose that (D, d_H) is a Hilbert metric space, $x_0 \in D$ and $l \in [0, 1)$. Then the mapping $g(x) = x_0 + l(x - x_0)$ is contractive.*

Proof. Fix $x_0 \in D$ and $l \in [0, 1)$. Let $x, y \in D$ and consider the straight line passing through x and y that intersects ∂D in two points x' and y' such that x is between x' and y, and y is between x and y'. Take two points $z' = (1-l)x_0 + lx' \in \partial g(D)$, $w' = (1-l)x_0 + ly' \in \partial g(D)$, and note that the points z', $g(x)$, $g(y)$, w'

are collinear such that $g(x)$ is between z' and $g(y)$, and $g(y)$ is between $g(x)$ and w'. Since $g(D)$ lies in a compact subset of D, it follows from Lemma 2.1 that $d_H(g(x), g(y)) < d_2(g(x), g(y))$, where d_2 denotes the Hilbert metric in $g(D)$. By definition of the Hilbert metric space we have

$$\begin{aligned} d_H(x, y) &= \log\left(\frac{\|x' - y\| \, \|x - y'\|}{\|x' - x\| \, \|y - y'\|}\right) = \log\left(\frac{\|z' - g(y)\| \, \|w' - g(x)\|}{\|z' - g(x)\| \, \|w' - g(y)\|}\right) \\ &= d_2(g(x), g(y)). \end{aligned}$$

Therefore we get $d_H(g(x), g(y)) < d_H(x, y)$. □

Note that if $D \subset V$ is a bounded convex domain, then the Hilbert metric d_H is locally equivalent to the euclidean norm in V. Furthermore, for any $w \in D$, if $\{x_n\}$ is a sequence in D converging to $\xi \in \partial D = \overline{D} \setminus D$, then

$$d_H(x_n, w) \to \infty$$

(see [5, 9]). The above property is equivalent to properness of D, that is, every closed and bounded subset of (D, d_H) is compact. It is not difficult to show that for $x, y, z \in D$ and $s \in [0, 1]$,

$$\text{(C)} \qquad d_H(sx + (1 - s)y, z) \leq \max\{d_H(x, z), d_H(y, z)\}.$$

In what follows, we will assume a more restrictive condition: for all $x, y, z, w \in D$ and $s \in [0, 1]$,

$$\text{(D)} \qquad d_H(sx + (1 - s)y, sz + (1 - s)w) \leq \max\{d_H(x, z), d_H(y, w)\}.$$

3. Resolvents of nonexpansive mappings

In this section we describe the construction of a resolvent of a nonexpansive mapping acting on a Hilbert metric space. Let $D \subset V$ be a convex bounded domain and $F : D \to D$ a nonexpansive mapping with respect to the Hilbert metric d on D. Recall that the topology of (D, d) coincides with the Euclidean topology and (D, d) is proper metric space, that is, every closed ball $\bar{B}(x_0, r)$, $x_0 \in D$, is compact. We fix $x \in D$, $\lambda > 0$, and define a mapping

$$G_{x,\lambda}(y) = \frac{1}{1 + \lambda} x + \frac{\lambda}{1 + \lambda} F(y), \quad y \in D.$$

It follows from Lemma 2.2 that $G_{x,\lambda}$ is contractive.

We show that $G_{x,\lambda}(D)$ is bounded. For this purpose, we select $w \in G_{x,\lambda}(D)$. Note that there exists $y \in D$ such that $w = \frac{1}{1+\lambda} x + \frac{\lambda}{1+\lambda} F(y)$. We show that $B(w, \frac{1}{1+\lambda} d) \subset D$, where $d = \inf_{v \in \partial D} \|v - x\|$. Choose any $w' \in B(w, \frac{1}{1+\lambda} d)$. Then there exists $z \in D$ such that $w' = \frac{1}{1+\lambda} z + \frac{\lambda}{1+\lambda} F(y)$. Note that

$$\|w - w'\| = \left\| \frac{1}{1 + \lambda} x + \frac{\lambda}{1 + \lambda} F(y) - \frac{1}{1 + \lambda} z + \frac{\lambda}{1 + \lambda} F(y) \right\| = \frac{1}{1 + \lambda} \|x - z\|.$$

If $\|w - w'\| < \frac{1}{1+\lambda}$, then $\|x - z\| = (1 + \lambda)\|w - w'\| < d$. It implies that $z \in D$ and hence $w' \in D$. It follows that for all $w \in G_{x,\lambda}(D)$,

$$\text{(3.1)} \qquad \inf_{v \in \partial D} \|v - w\| \geq \frac{1}{1 + \lambda} d.$$

Take a sequence $\{w_n\} \subset G_{x,\lambda}(D)$. Since $\overline{D}$ is compact in the Euclidean topology, there exists a subsequence $\{w_{n_k}\}$ and $x_0 \in \overline{D}$ such that $\|w_{n_k} - x_0\| \to 0$, if $k \to \infty$. It follows from (3.1) that $x_0 \in D$, and hence $d(w_{n_k}, x_0) \to 0$ since the topology of (D, d)

coincides with the Euclidean topology. Therefore, $G_{x,\lambda}(D)$ is bounded in (D,d) and by properness of D we have that $\overline{G_{x,\lambda}(D)}$ is compact in (D,d).

Note that $D \supset G_{x,\lambda}(D) \supset G^2_{x,\lambda}(D) \supset \cdots$, which means that the orbits of $G_{x,\lambda}$ are bounded. Fix $y \in D$. Since $\overline{G_{x,\lambda}(D)}$ is compact, there exists a subsequence $\{G^{n_k}_{x,\lambda}(y)\}$ of $\{G^n_{x,\lambda}(y)\}$ converging to some $z \in D$. Let

$$d_n = d(G^n_{x,\lambda}(y), G^{n+1}_{x,\lambda}(y)).$$

Since $G_{x,\lambda}$ is contractive, the sequence $\{d_n\}$ is decreasing and hence it converges to some ζ, as $n \to \infty$. Hence

$$\zeta \leftarrow d_{n_k} = d(G^{n_k}_{x,\lambda}(y), G^{n_k+1}_{x,\lambda}(y)) \to d(G_{x,\lambda}(z), z),$$

and

$$\zeta \leftarrow d_{n_k+1} = d(G^{n_k+1}_{x,\lambda}(y), G^{n_k+2}_{x,\lambda}(y)) \to d(G^2_{x,\lambda}(z), G_{x,\lambda}(z)).$$

We get

$$d(G^2_{x,\lambda}(z), G_{x,\lambda}(z)) = d(G_{x,\lambda}(z), z) = \zeta.$$

Since the map $G_{x,\lambda}$ is contractive, $G_{x,\lambda}(z) = z$. Moreover, z is the unique fixed point of $G_{x,\lambda}$. Indeed, otherwise if $z_1, z_2 \in D$, $z_1 \neq z_2$ are fixed points of $G_{x,\lambda}$, then

$$d(z_1, z_2) = d(G_{x,\lambda}(z_1), G_{x,\lambda}(z_2)) < d(z_1, z_2),$$

and we obtain a contradiction. Define $z = R_\lambda(x)$. We refer to the mapping $R_\lambda \colon D \to D$ as *the resolvent of* F. We have

$$z = G_{x,\lambda}(z) = \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}F(z), \quad x \in D,\ \lambda > 0,$$

and hence

$$R_\lambda(x) = \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}F(R_\lambda(x)), \quad x \in D,\ \lambda > 0. \tag{3.2}$$

Furthermore, any converging subsequence $G^{m_k}_{x,\lambda}(y)$ has the limit z (the unique fixed point), as $k \to \infty$. This gives the formula:

$$\lim_{n\to\infty} G^n_{x,\lambda}(y) = R_\lambda(x), \quad y \in D. \tag{3.3}$$

It turns out that if (D,d) is sufficiently regular, then the resolvent of a nonexpansive mapping is also nonexpansive.

Lemma 3.1. *Let (D,d) be a Hilbert metric space satisfying condition (D), $F \colon D \to D$ a nonexpansive mapping, and $\lambda > 0$. Then the resolvent $R_\lambda \colon D \to D$ is nonexpansive.*

Proof. Fix $z_0, z_1, z_2 \in D$. First we show that $d(G^n_{z_1,\lambda}(z_0), G^n_{z_2,\lambda}(z_0)) \le d(z_1, z_2)$ for each n. We proceed by induction. For $n = 1$, it follows from condition (C) that

$$\begin{aligned} d(G_{z_1,\lambda}(z_0), G_{z_2,\lambda}(z_0)) &= d\Big(\frac{1}{1+\lambda}z_1 + \frac{\lambda}{1+\lambda}F(z_0), \frac{1}{1+\lambda}z_2 + \frac{\lambda}{1+\lambda}F(z_0)\Big) \\ &\le \max\{d(z_1,z_2), d(F(z_0),F(z_0))\} = d(z_1,z_2). \end{aligned}$$

Fix $n \in \mathbb{N}$ and suppose that $d(G^n_{z_1,\lambda}(z_0), G^n_{z_2,\lambda}(z_0)) \leq d(z_1, z_2)$. Then it follows from (D) that

$$\begin{aligned}&d(G^{n+1}_{z_1,\lambda}(z_0), G^{n+1}_{z_2,\lambda}(z_0))\\&= d\left(\frac{1}{1+\lambda}z_1 + \frac{\lambda}{1+\lambda}F(G^n_{z_1,\lambda}(z_0)), \frac{1}{1+\lambda}z_2 + \frac{\lambda}{1+\lambda}F(G^n_{z_2,\lambda}(z_0))\right)\\&\leq \max\{d(z_1, z_2), d(G^n_{z_1,\lambda}(z_0), G^n_{z_2,\lambda}(z_0))\} = d(z_1, z_2).\end{aligned}$$

Now the formula (3.3) yields

$$d(R_\lambda(z_1), R_\lambda(z_2)) = \lim_{n\to\infty} d\left(G^n_{z_1,\lambda}(z_0), G^n_{z_2,\lambda}(z_0)\right) \leq d(z_1, z_2),$$

which shows that R_λ is a nonexpansive mapping. □

We will also use the following property called the resolvent identity.

Proposition 3.2. *Suppose that $F\colon D \to D$ is a nonexpansive mapping. Then its resolvent R_λ satisfies*

$$R_\lambda(x) = R_\mu\left(\frac{\lambda-\mu}{\lambda}R_\lambda(x) + \frac{\mu}{\lambda}x\right), \quad x \in D,$$

for all $\lambda > \mu > 0$.

Proof. Fix $x \in D$ and $\lambda, \mu > 0$ such that $\lambda > \mu$. Define

$$y := \frac{\lambda-\mu}{\lambda}R_\lambda(x) + \frac{\mu}{\lambda}x. \tag{3.4}$$

It follows from (3.2) that there exists the unique point

$$z := R_\mu(y) = \frac{1}{1+\mu}y + \frac{\mu}{1+\mu}F(R_\mu(y)). \tag{3.5}$$

On the other hand, we have

$$\tilde{z} := R_\lambda(x) = \frac{1}{1+\lambda}x + \frac{\lambda}{1+\lambda}F(R_\lambda(x)). \tag{3.6}$$

From the above and (3.4) we get $\lambda y - \mu\tilde{z}(1+\lambda) = (\lambda-\mu)\tilde{z} - \lambda\mu F(\tilde{z})$, which implies

$$\tilde{z} = \frac{1}{1+\mu}y + \frac{\mu}{1+\mu}F(\tilde{z}).$$

Therefore, from the uniqueness of the construction of the point z and by (3.5) and (3.6), we have

$$R_\mu(y) = z = \tilde{z} = R_\lambda(x).$$ □

For any $x \in D$, $F\colon D \to D$, the set of accumulation points (in the norm topology) of the sequence $\{F^n(x)\}$ is called the *omega limit set of* x and is denoted by $\omega_F(x)$. In a similar way, if $R_\lambda\colon D \to D$, $\lambda > 0$, is a family of resolvents of F, we define

$$\begin{aligned}\omega_{\{R_\lambda\}_{\lambda>0}}(x) = \{y \in \overline{D}\colon\ &\|R_{\lambda_n}(x) - y\| \to 0\\&\text{for some increasing sequence } \{\lambda_n\},\ \lambda_n \to \infty\},\end{aligned}$$

and the *attractor* of $\{R_\lambda\}_{\lambda>0}$,

$$\Omega_{\{R_\lambda\}_{\lambda>0}} = \bigcup_{x\in D}\omega_{\{R_\lambda\}_{\lambda>0}}(x).$$

Lemma 3.3. *Suppose that $F\colon D \to D$ is a nonexpansive mapping without fixed points and $R_\lambda\colon D \to D$, $\lambda > 0$ is a family of resolvents of F. Then $\Omega_{\{R_\lambda\}_{\lambda>0}} \subset \partial D$.*

Proof. On the contrary, we suppose that there exists $y \in D$ such that $\|R_{\lambda_n}(x) -y\| \to 0$ for some $x \in D$ and an increasing sequence $\{\lambda_n\}$, $\lambda_n \to \infty$. Then

$$\|R_{\lambda_n}(x) - F(R_{\lambda_n}(x))\| = \frac{1}{1+\lambda_n}\|x - F(R_{\lambda_n}(x))\| \to 0, \tag{3.7}$$

as $n \to \infty$. Since the topology of (D, d) coincides with the norm topology, $F\colon D \to D$ is norm-continuous, and hence

$$\|F(y) - y\| \leq \|F(y) - F(R_{\lambda_n}(x))\| + \|F(R_{\lambda_n}(x)) - R_{\lambda_n}(x)\| + \|R_{\lambda_n}(x) - y\| \to 0,$$

as $n \to \infty$. Thus $F(y) = y$, and we obtain a contradiction. □

4. Main theorem

We begin by recalling one of the fundamental properties of a Hilbert metric space that allows Karlsson and Noskov to extend Beardon's Wolff–Denjoy theorem for bounded strictly convex domains (see [11, Theorem 5.5],[14, Proposition 8.3.3]).

Lemma 4.1. *Let $D \subseteq V$ be an open bounded convex set and d a Hilbert metric on D. If $\{x_n\}$ and $\{y_n\}$ are convergent sequences in D with limits x and y in ∂D, respectively, and the segment $[x, y] \nsubseteq \partial D$, then for each $z \in D$ we have*

$$\lim_{n\to\infty}[d(x_n, y_n) - \max\{d(x_n, z), d(y_n, z)\}] = \infty.$$

We also need the following standard argument that can be found for example in [7, Lemma 5.4].

Lemma 4.2. *Let (D, d) be a separable metric space and let $a_n\colon D \to \mathbb{R}$ be a nonexspansive mapping for each $n \in \mathbb{N}$. If for every $x \in D$, the sequence $\{a_n(x)\}$ is bounded, then there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $\lim_{j\to\infty} a_{n_j}(x)$ exists for every $x \in D$.*

Fix $x_0 \in D$ and consider a sequence $\{x_n \in D\colon n \in \mathbb{N}\}$ contained in D. Define $a_n(x) = d(x, x_n) - d(x_n, x_0)$ for any $n \in \mathbb{N}$. Note that

$$|a_n(y) - a_n(x)| \leq d(x, y),$$

i.e., a_n is nonexpansive and the sequence $\{a_n(y)\}$ is bounded (by $d(y, x_0)$) for every $y \in D$. It follows from Lemma 4.2 that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} a_{n_j}(x)$ exists for any $x \in D$, i.e.,

$$\lim_{j\to\infty} d(x, x_{n_j}) - d(x_{n_j}, x_0) \tag{4.1}$$

exists for every $x \in D$.

Now we are in a position to prove a variant of the Karlsson–Nussbaum conjecture for resolvents of nonexpansive mappings.

Theorem 4.3. *Let $D \subset V$ be a bounded convex domain. Suppose that (D, d) is a Hilbert metric space satisfying condition (D) and $R_\lambda\colon D \to D, \lambda > 0$, is a family of resolvents of a nonexpansive mapping $F\colon D \to D$ without fixed points. Then $\operatorname{co}\Omega_{\{R_\lambda\}_{\lambda>0}} \subseteq \partial D$.*

Proof. Suppose on the contrary that there exist $z_1, \ldots, z_m \in D$, $\zeta^1 \in \omega_{\{R_\lambda\}_{\lambda>0}}(z_1)$, $\ldots, \zeta^m \in \omega_{\{R_\lambda\}_{\lambda>0}}(z_m)$ and $0 < \alpha_1, \ldots, \alpha_m < 1$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\sum_{i=1}^m \alpha_i\zeta^i \in D$. Since F does not have fixed points, it follows from Lemma 3.3 that the omega limit sets $\omega_{\{R_\lambda\}_{\lambda>0}}(z_i) \subseteq \partial D$, $i = 1, \ldots, m$, and, following [14, Theorem 8.3.11], we can assume that $m \geq 2$ is minimal with the property that

$\sum_{i=1}^m \alpha_i\zeta^i \in D$. It follows that $R_{\lambda_j^i}(z_i) \to \zeta^i \in \partial D$ for some increasing sequences $\{\lambda_j^i\}_j, \lambda_j^i \to \infty$, as $j \to \infty$, $i = 1, \ldots, m$. We put $\zeta = \zeta^1$ and $\eta = \sum_{i=2}^m \mu_i\zeta^i$, where $\mu_i = \frac{\alpha_i}{1-\alpha_1}$ for $i \in [2, m]$. Let $\eta^j = \sum_{i=2}^m \mu_i R_{\lambda_j^i}(z_i)$ for all $j \geq 1$. Since m is minimal, we have $\zeta, \eta \in \partial D$ and $\alpha_1\zeta + (1-\alpha_1)\eta \in D$. Since D is convex, we get $\alpha\zeta + (1-\alpha)\eta \in D$ for all $\alpha \in (0, 1)$. By passing to a subsequence we can assume from (4.1) that for every $x \in D$ there exists the limit

$$g(x) = \lim_{j\to\infty} d(x, R_{\lambda_j^1}(z_1)) - d(R_{\lambda_j^1}(z_1), z_1). \tag{4.2}$$

Since

$$\left\| y - \frac{\lambda_j^1 - \mu}{\lambda_j^1} y - \frac{\mu}{\lambda_j^1} z_1 \right\| = \frac{\mu}{\lambda_j^1}\|y - z_1\| \to 0, \quad \text{as } \lambda_j^1 \to \infty,$$

and topologies of (D, d) and $(\overline{D}, \|\cdot\|)$ coincide on D, we have

$$d\left(y, \frac{\lambda_j^1 - \mu}{\lambda_j^1} y + \frac{\mu}{\lambda_j^1} z_1\right) \to 0, \tag{4.3}$$

if $\lambda_j^1 \to \infty$. According to Lemma 3.1, Proposition 3.2, (4.3) and condition (C), we get

$$\begin{aligned}
g(R_\mu(y)) &= \lim_{j\to\infty} d(R_\mu(y), R_{\lambda_j^1}(z_1)) - d(R_{\lambda_j^1}(z_1), z_1) \\
&= \lim_{j\to\infty} d\left(R_\mu(y), R_\mu\left(\frac{\lambda_j^1 - \mu}{\lambda_j^1} R_{\lambda_j^1}(z_1) + \frac{\mu}{\lambda_j^1} z_1\right)\right) - d(R_{\lambda_j^1}(z_1), z_1) \\
&\leq \limsup_{j\to\infty} d\left(y, \frac{\lambda_j^1 - \mu}{\lambda_j^1} R_{\lambda_j^1}(z_1) + \frac{\mu}{\lambda_j^1} z_1\right) - d(R_{\lambda_j^1}(z_1), z_1) \\
&= \lim_{j\to\infty} d\left(\frac{\lambda_j^1 - \mu}{\lambda_j^1} y + \frac{\mu}{\lambda_j^1} z_1, \frac{\lambda_j^1 - \mu}{\lambda_j^1} R_{\lambda_j^1}(z_1) + \frac{\mu}{\lambda_j^1} z_1\right) - d(R_{\lambda_j^1}(z_1), z_1) \\
&\leq \lim_{j\to\infty} d(y, R_{\lambda_j^1}(z_1)) - d(R_{\lambda_j^1}(z_1), z_1) = g(y).
\end{aligned}$$

From the above we have $g(R_\mu(y)) \leq g(y) \leq d(y, z_1)$ for every $y \in D$ and $\mu > 0$. It follows from (C) that for any $k \in \mathbb{N}$,

$$g(\eta^k) = g\left(\sum_{i=2}^m \mu_i R_{\lambda_k^i}(z_i)\right) \leq \max_{i=2,\ldots,m} g(z_i) \leq \max_{i=2,\ldots,m} d(z_i, z_1) = M.$$

Consequently, by diagonal method, there exists a subsequence $\lambda_{j_1}^1 \leq \lambda_{j_2}^1 \leq \ldots \leq \lambda_{j_k}^1 \leq \ldots$ of $\{\lambda_j^1\}$ such that

$$\limsup_{k\to\infty} d(\eta^k, R_{\lambda_{j_k}^1}(z_1)) - d(R_{\lambda_{j_k}^1}(z_1), z_1) \leq M + 1. \tag{4.4}$$

Since $R_{\lambda_j^i}(z_i) \to \zeta^i$, as $j \to \infty$ for any $i = 1, \ldots, m$, we have

$$\|\eta^j - \eta\| = \left\|\sum_{i=2}^m \mu_i R_{\lambda_j^i}(z_i) - \sum_{i=2}^m \mu_i\zeta^i\right\| \leq \sum_{i=2}^m \mu_i\|R_{\lambda_j^i}(z_i) - \zeta^i\| \to 0, \quad j \to \infty,$$

which implies that $\|\eta^j - \eta\| \to 0, j \to \infty$. Moreover, since $[\zeta, \eta] \nsubseteq \partial D$ it follows from Lemma 4.1 that

$$\liminf_{k\to\infty} d(\eta^k, R_{\lambda_{j_k}^1}(z_1)) - d(R_{\lambda_{j_k}^1}(z_1), z_1) = \infty.$$

However, the above formula contradicts (4.4). $\square$

We can use Theorem 4.3 to show a Wolff–Denjoy type theorem for resolvents of nonexpansive mappings. Let (D,d) be a geodesic metric space and $[x,y],[x',y']$ two arbitrary geodesic segments in D. For every $\alpha\in[0,1]$, consider the point $z=\alpha x+(1-\alpha)y$ on segment $[x,y]$ such that $d(\alpha x+(1-\alpha)y,y)=\alpha d(x,y)$ and in the same way, the point $z'=\alpha x'+(1-\alpha)y'$ on segment $[x',y']$ such that $d(\alpha x'+(1-\alpha)y',y')=\alpha d(x',y')$. Recall that a geodesic space (D,d) is called *Busemann convex* if

$$d(z,z')\le(1-\alpha)d(x,x')+\alpha d(y,y')$$

for every $x,y,x',y'\in D$ and $\alpha\in[0,1]$.

Combining Corollary 3.3 and Proposition 3.4 in [2], we obtain the following proposition (see also [12], [16, p. 191]).

Proposition 4.4. *Let $D\subset V$ be a bounded convex domain. A Hilbert metric space (D,d) is Busemann convex if and only if D is an ellipsoid.*

Since in Hilbert's metric spaces every straight-line segment is a geodesic, it follows from Proposition 4.4 that (D,d) satisfies condition (D), whenever D is an ellipsoid. This leads to the following Wolff–Denjoy type theorem for resolvents of nonexpansive mappings.

Corollary 4.5. *Suppose that $D\subset V$ is an ellipsoid and $R_\lambda\colon D\to D,\lambda>0$, is the resolvent of a nonexpansive mapping $F\colon D\to D$ (with respect to Hilbert's metric) without fixed points. Then there exists $\xi\in\partial D$ such that $\{R_\lambda\}_{\lambda>0}$ converge uniformly on bounded sets of D to ξ.*

Proof. It follows from Theorem 4.3 that $\operatorname{co}\Omega_{\{R_\lambda\}_{\lambda>0}}\subseteq\partial D$. Since D is strictly convex, $\Omega_{\{R_\lambda\}_{\lambda>0}}$ consists of a single element $\xi\in\partial D$. The proof of uniform convergence on bounded sets is standard (see, e.g., [5]): suppose, on the contrary, that there exist an open neighbourhood $U\subset\overline{D}$ of ξ, a bounded set $K\subset D$ and a sequence $\{y_{\lambda_n}\}\subset K$ $(\lambda_n\to\infty)$ such that $R_{\lambda_n}(y_{\lambda_n})\notin U$ for each n. Then

$$d(R_{\lambda_n}(y_{\lambda_n}),R_{\lambda_n}(y))\le d(y_{\lambda_n},y)\le\operatorname{diam}K$$

for any $y\in K$ and, since $R_{\lambda_n}(y)\to\xi$, we deduce from Lemma 4.1 that $R_{\lambda_n}(y_{\lambda_n})\to\xi\in\overline{D}\setminus U$, a contradiction. □

Acknowledgements. The first author was partially supported by National Science Center (Poland) Preludium Grant No. UMO-2021/41/N/ST1/02968.

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Received 1 August 2022 • Revision received 12 December 2022 • Accepted 12 December 2022
Published online 15 January 2023

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