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On the density of S-adic integers near some projective G-varieties

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Abstract. We provide some general conditions which ensure that a system of inequalities involving homogeneous polynomials with coefficients in a S-adic field has nontrivial S-integral solutions. The proofs are based on the strong approximation property for Zariski-dense subgroups and adelic geometry of numbers. We give some examples of applications for systems involving quadratic and linear forms.

S-adisten kokonaislukujen tiheydestä projektiivisten G-varistojen läheisyydessä

Tiivistelmä. Esitämme joitakin yleisiä ehtoja sille, että homogeenisiä polynomeja käsittävällä epäyhtälösysteemillä, jonka kertoimet ovat S-adisen kunnan alkioita, on epätriviaaleja S-kokonaislukuratkaisuita. Todistukset perustuvat Zariskin topologian suhteen tiheiden aliryhmien vahvaan likiarvoistusominaisuuteen ja adeliseen geometriseen lukuteoriaan. Annamme joitakin esimerkkejä sovelluksista neliöllisiä ja lineaarisia muotoja käsittäviin systeemeihin.

1. Introduction

Given a finite set of valuations S of $\mathbb{Q}$ which contains the archimedean one, we consider a finite family of homogeneous polynomials $(f_{i,p})_{p\in S}(1 \leqslant i \leqslant r)$ where each $f_{i,p}$ has coefficients in the completion of $\mathbb{Q}$ relative to the place $p \in S$. We are interested in the following problem, given any real $\varepsilon > 0$, can we find an nonzero S-integral vector x such that

$$0 < |f_{i,p}(x)|_p \leq \varepsilon \quad \text{for every} \quad p \in S \text{ and } i = 1, \dots, r? \tag{1}$$

Despite its apparent simplicity, this question is extremely difficult to solve in general and as far as we know, only few cases have been settled. Our point of departure is the case of a single isotropic quadratic form $f_1 = Q$ for which a solution was found only quite recently by Borel and Prasad [5] for $S = \{\infty\}$ and completed in the general case as soon Ratner gave a complete solution to the Raghunathan conjecture [4] in full generality. Their result is an S-arithmetic generalization of Margulis' proof of the Oppenheim conjecture [21]. For the reader interested in such dynamical methods and the applications of Ratner's theory to number theoretical problems we refer to [14].

The main tool we are going to use in order to treat the question (1) with the highest level of generality is the *strong approximation property for algebraic groups* (see [32, 16]). In other words, we will be merely focusing on the arithmetical properties of groups actions rather than their ergodic behaviour. It is not very surprizing that strong approximation could solve the same density problems as Ratner's orbit closure theorem does since both results take place in groups generated by one-dimensional

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unipotent elements. The bridge between these two notions is the Kneser–Tits conjecture [15, 16] which asserts that any simply-connected group which is simple and isotropic over a local field, is generated by its one dimensional unipotent elements. This conjecture was solved by Platonov for such groups ([30], see also [31, §7.2]). Our main purpose is to provide a substitute in the case when Ratner's theorem fails to hold. One illustration of the great advantage of using strong approximation rather than Ratner's theory is that we get rid of the heavy task of classifying intermediate Lie groups. Indeed in the most cases such classification is unfeasible unless in the very rare cases when we are able to reduce to lower dimension. Fortunately, this is the case of the original proof of the Oppenheim conjecture which has been proved for $n = 3$ and for Gorodnik's result for pairs (Q, L) which reduces to the dimension four but for pairs the classification of intermediate, the latter classification is much more involved (see [13]).

In the same circle of ideas, recently Ghosh, Gorodnik and Nevo developped in [11, 9, 10, 12]) the metric theory of diophantine approximation on homogeneous varieties of semisimple groups in the S-arithmetic setting. Among many other results, they proved analogs of Khintchine's and Jarnik's theorems for S-adic homogeneous spaces using both ergodic theory and strong approximation for algebraic groups combined with deep concepts coming from the theory of automorphic forms and representation theory. Their method also provides a quantitative version of the strong approximation theorem in homogeneous spaces of semisimple groups.

Finally one should mention that for higher degrees, i.e. when the number of variables and the degrees of the f_i's are greater than the number of r of polynomials, the circle method of Hardy and Littlewood still remains the most powerful method for proving (1) in great generality. In fact, it is providing also sharp quantitative estimate of the numbers of solutions with bounded heights of the number of points lying exactly on a variety following the program promoted by Manin in the seventies.

1.1. Background and notations. Let us denote by Σ the set of all places in $\mathbb{Q}$, these are given by the set of all prime numbers and the archimedean place corresponding to ∞. Let S be a finite set of places in Σ which contains the archimedean one, and let us denote by S_f the subset of all finite (prime) places in S, thus $S = S_f \cup \{\infty\}$. For each prime p, we can define the p-adic absolute value is denoted by $|\cdot|_p$ over $\mathbb{Q}$ and we denote by $\mathbb{Q}_p$ the corresponding completion of $\mathbb{Q}$. The product $\mathbb{Q}_S$ is defined by $\prod_{p\in S}\mathbb{Q}_p$. The set of p-adic integers, denoted by $\mathbb{Z}_p$, is defined to be the set of $x \in \mathbb{Q}$ such that $|x|_p \leq 1$. The ring of S-integers of $\mathbb{Q}$ is the set $\mathbb{Z}_S$ which elements are integral outside S i.e. such that $x \in \mathbb{Z}_p$ for $p \notin S$. For each $p \in S_f$, $\mathbb{Q}_p$ is a locally compact (additive) group, hence it is equipped with a Haar measure characterized by the formula $\mu_p(a\Omega_p) = |a|_p\mu_p(\Omega_p)$ for all $a \in \mathbb{Q}_p$ and Ω_p is a measurable subset of $\mathbb{Q}_p$ of finite measure. We normalize it by prescribing the value of the measure μ_p over the basis of open sets in $\mathbb{Q}_p$ by taking $\mu_p(a + p^n\mathbb{Z}_p) = p^{-n}$, in particular, $\mu_p(\mathbb{Z}_p) = 1$. The set of adeles $\mathbb{A}$ of $\mathbb{Q}$ is the subset of the direct product $\prod_p \mathbb{Q}_p$ over all the places of $\mathbb{Q}$ consisting of those $x = (x_p)$ such that $x \in \mathbb{Z}_p$ for all but finitely many places. The set of adeles $\mathbb{A}$ is a locally compact ring with respect to the adele topology given by the base of open sets of the form $\prod_{p\in S}\mathbb{Q}_p \times \prod_{p\notin S}\mathbb{Z}_p$ where $S \subset \Sigma$ is finite with $S \supset S_\infty$. For any finite subset $S \subset \Sigma$ with $S \supset S_\infty$, the ring of S-integral adeles is defined by:

$$\mathbb{A}_S = \prod_{p\in S}\mathbb{Q}_p \times \prod_{p\notin S}\mathbb{Z}_p, \text{ thus we can see that } \mathbb{A} = \bigcup_{S\supset S_\infty}\mathbb{A}_S.$$

By definition $\mathbb{Z}_S = \mathbb{A}_S \cap \mathbb{Q}$, in addition, it can be proved that $\mathbb{Z}_S$ is a lattice in $\mathbb{Q}_S$ i.e. a discrete subgroup of finite invariant covolume. We can also realize $\mathbb{Z}_S$ as a cocompact lattice in $\mathbb{A}_S$. One of most fundamental result in arithmetic is the following fact which says that for any nonempty set of places S, the image of $\mathbb{Q}$ under the diagonal embedding is dense in $\mathbb{A}_S$ (see e.g. [19, Theorem 1, II, §1]). This property is called *the strong approximation* for the field $\mathbb{Q}$ and it can be seen as a refinement of the *Chinese remainder theorem* [19, I, §4]. For a brief review about this property in the framework of algebraic groups, we invite the reader to read the recent account about this question in [32], for more details we advice one the most complete reference about this topic [31].

Quadratic forms over local fields. A quadratic form in n variables over a local field k of characteristic zero is given by a symmetric bilinear form B over k such that $Q(x) = B(x, x)$ for any $x \in k^n$. We say that Q is *nondegenerate* in k^n if the rank of the matrix associated to B has maximal rank. For each $p \in S$, a quadratic form $Q_p(x)$ is isotropic over $\mathbb{Q}_p$ if there exists a nonzero vector x such that $Q_p(x) = 0$. Over S-adic products, a quadratic $Q = (Q_p)_{p\in S}$ over $\mathbb{Q}_S$ is said to be nondegenerate (resp. isotropic) if and only if Q_p is nondegenerate (resp. isotropic) over $\mathbb{Q}_p$ for each $p \in S$. The special orthogonal group of a quadratic form $(Q_p)_{p\in S}$ is the product of S of orthogonal groups $\mathrm{SO}(Q_p)$, the latter is a Lie group which is semisimple as soon as Q_p is nondegenerate. The orthogonal group $\mathrm{SO}(Q_p)$ is said to be isotropic over $\mathbb{Q}_p$ if it has a nontrivial split subtorus over $\mathbb{Q}_p$ [3, §20.1]. The group $\mathrm{SO}(Q_p)$ is isotropic over $\mathbb{Q}_p$ if and only if Q_p is isotropic [3, §23.4]. It is well-known that over local fields, $\mathrm{SO}(Q_p)$ is isotropic if and if it is has no compact factors. If $H = \prod_{p\in S} H_p$ is a product of p-adic Lie groups and $S_1 \subseteq S$ be a finite subset of places, then H is said to be isotropic over S_1 if for every $p \in S_1$, H_p is isotropic over $\mathbb{Q}_p$. A quadratic form $(Q_p)_{p\in S}$ is said to be (globally) *rational* if there exists a form Q_0 with rational coefficients such that $Q = \lambda Q_0$ for some nonzero $\lambda \in \mathbb{Q}_S$, and *irrational* otherwise. Note that it can happen that a quadratic form $Q = (Q_p)_{p\in S}$ is irrational while being rational at some place $p \in S$. More precisely, the fact that $Q = (Q_p)_{p\in S}$ is irrational over $\mathbb{Q}_S$ does not prevent Q_p to be proportional to a form with rational coefficients for some $p \in S$.

Any vector space V over k equipped with a quadratic form Q, can be decomposed in virtue of the Witt's decomposition theorem as follows

$$V = \mathrm{rad}(Q) \perp V_{an} \perp P_1 \perp \ldots \perp P_r$$

where the restriction Q to V_{an} is anisotropic over k, $\mathrm{rad}(Q)$ is the radical of Q, which is equal to zero if Q is nondegenerate and $P_i (1 \leqslant i \leqslant r)$ are hyperbolic planes such that the restriction of Q to each P_i is isotropic (see e.g. [18, §4, I]). In particular, $P_1 \oplus \ldots \oplus P_r$ is the maximal isotropic subspace, $2r$ is called the isotropy index and r is called the Witt index denoted $i(Q)$, remark that Q is isotropic over k if and only if $i(Q) \geq 1$.

Algebraic projective varieties. If we consider an algebraic variety defined over $\mathbb{Q}$ by a prime ideal $I = \langle f_1, \ldots, f_r\rangle$ for which closed points of given by

$$X = \left\{x \in \overline{\mathbb{Q}}^n \mid f_i(x) = 0,\ 1 \leq i \leq r\right\}$$

where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. The polynomials defining the prime ideal I are supposed to be *homogeneous* in n variables with rational coefficients. For each $1 \leq i \leq r$, if we denote $d_i = \deg f_i$ thus we must have that for any $\lambda \in \overline{\mathbb{Q}}$, $f_i(\lambda x) = \lambda^{d_i} f_i(x)$, in particular the zero locus $X = V(I)$ can be seen an *algebraic projective* *variety.* Neverthless for practical reasons we need our varieties to be embedded in vectors spaces, therefore we will be stuck with the affine point view.

Let us consider a variety $X/\mathbb{Q}$ embedded diagonally in a finite product of completions relative to a finite set of places S in $\mathbb{Q}$ containing the archimedean one. Using this embedding we can consider the family of polynomials $(f_{i,p})_{p\in S}$ over $\mathbb{Q}_S$ and for each $p \in S$ we define X_p to be the zero set of the $(f_{i,p})$ $(1 \leq i \leq r)$, this defines an affine variety over $\mathbb{Q}_p$. Therefore X_S is the direct product of the completions of X_p $(p \in S)$ in $\prod_{s\in S} \overline{\mathbb{Q}}_p^n$ where $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$.

Algebraic groups and their actions. Let G be the special linear algebraic group

$$\mathrm{SL}_{n|\mathbb{Q}} := \{(g_{ij}) \in \mathcal{M}_n(\mathbb{Q}) \mid \det(g) - 1 = 0\}.$$

It is defined over $\mathbb{Q}$, in the sense that the coefficients of $\det(g)$ viewed as a polynomial in the n^2 variables $g_{11}, g_{12}, \ldots, g_{nn}$ has rational coefficients. Here we can simply write SL_n instead of $\mathrm{SL}_{n|\mathbb{Q}}$ since the context is clear. For any ring A, the set of A-points of SL_n denoted $\mathrm{SL}_n(A)$ is just the set $SL_n \cap \mathcal{M}_n(A)$.

Let us consider the left action of SL_n on the $\mathbb{Q}$-vector space $V = \mathbb{Q}[x_1, \ldots, x_n]$ which is given for each $g \in G$ and $f \in V$ by

$$g.f(x) = f(g^{-1}x) \quad \text{for all } x \in \mathbb{Q}^n.$$

We would like to define a subgroup of G which leaves globally invariant the variety $X = V(I)$ but also every element of the ideal of definition I. For this purpose, we introduce the subgroup H of G defined over $\mathbb{Q}$ given by

$$H = \{h \in G \mid h \cdot f_i = f_i,\ 1 \leq i \leq r\}.$$

From its very definition H is an algebraic group which acts linearly on the variety X. The *automorphism group* of X under the action of G (PSL_n in the projective setting) is defined as

$$\mathrm{Aut}_G(X) = \{x \in X \mid g \cdot x = x\}.$$

It is immediate to see that we have the following inclusions

$$\mathrm{Aut}_G(X) \subseteq H \subseteq G.$$

In particular, one can note that X can have a large group H while having a small or maybe even finite a automorphsim group. For this reason, we prefer to consider the action of H rather than $\mathrm{Aut}_G(X)$ which in the ideal case would be a large enough Lie group in order to apply the strong approximation property.

Near vectors to a variety. At some point, we will have to consider the set of $\mathbb{Q}_p$-points of X_p which is given by $X_p(\mathbb{Q}_p) = X_p \cap \mathbb{Q}_p^n$, it is equipped with the p-adic topology induced by the base field $\mathbb{Q}_p$. For each $p \in S$ the set $X_p(\mathbb{Q}_p)$ of $\mathbb{Q}_p$-points of X_p can endowed with a structure of analytic variety over $\mathbb{Q}_p$. Using some analogy with analytic complex geometry, given any real $\varepsilon > 0$ we define the *ε-tubular neighborhood* of X_S for the S-adic topology by

$$X_S^\varepsilon = \{x \in \mathbb{Q}_S^n : |f_{j,p}(x)|_p \leq \varepsilon \text{ for every } 1 \leq j \leq r \text{ and every } p \in S\}.$$

The elements of $\mathbb{Q}_S^n$ which are in X_S^ε are called ε-near vectors to X_S. The fact that we have chosen the f_i's to be homogeneous implies that for any $\varepsilon > 0$, the intersection of X_S^ε with the lattice $\mathbb{Z}_S^n$ contains at least the null vector. Thus for any $\varepsilon > 0$, showing that $X_S^\varepsilon \cap \mathbb{Z}_S^n \neq \{0\}$ amounts to find a nonzero $x \in \mathbb{Z}_S^n$ such that for every $p \in S$ and $1 \leq i \leq r$ we have,

$$|f_{i,p}(x)|_p \leq \epsilon.$$

If we consider the polynomial map $\phi\colon \mathbb{Q}_S^n \longrightarrow \mathbb{Q}_S^r$ associated to X_S, given by

$$\phi(x) = (f_1(x), \ldots, f_r(x)),$$

then $X_S^\varepsilon \cap \mathbb{Z}_S^n \neq \{0\}$ if and only if the origin in $\mathbb{Q}_S^r$ is an accumulation point in $\phi(\mathbb{Z}_S^n)$ for the S-adic topology. A more ambitious question regarding our initial problem in (1) is to ask the density of $\phi(\mathbb{Z}_S^n)$ in $\mathbb{Q}_S^r$, note that for real places (1), i.e. non-discreteness at the origin, is equivalent to density.

For each $p \in S$, we define the p-adic Lie subgroup

$$H_p = \{h \in \mathrm{SL}_n(\mathbb{Q}_p) \mid h \cdot f_{i,p} = f_{i,p},\ 1 \leq i \leq r\}.$$

and let denote the S-product by $H_S = \prod_{p\in S} H_p$. Clearly the action of H on X induces an equivariant action of H_S on X_S with respect to the diagonal embedding. For each $p \in S$, we say that H_p is rational over $\mathbb{Q}$ when the ideal I_p of defintion of H consists in polynomials with rational coefficients. This definition applies to any other algebraic group.

Notations.

- If A is a subset of $\mathbb{Q}$, we denote by $\overline{A}^{(p)}$ its p-adic closure in $\mathbb{Q}_p$ and for any set of places S in $\Sigma_\mathbb{Q}$, $\overline{A}^{(S)} = \prod_{p\in S} \overline{A}^{(p)}$.
- If A and B are two sets, we denote $A + B$ by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

  More generally if $A_1, \ldots, A_l$ are sets, their Minkowski sum is denoted

$$A_1 + \ldots + A_l = \sum_{1\leq i\leq l}^{(M)} A_i = \{a_1 + \ldots + a_n \mid a_i \in A_i \text{ for } 1 \leq i \leq l\}.$$

  In particular for every integer $n > 0$, the n-times sum $A + \ldots + A$ is denoted by $n * A$.
- For each $p \in S$, we denote by $\mathrm{Sym}^2(\mathbb{Q}_p^n)$ the set of bilinear symmetric forms with coefficients in $\mathbb{Q}_p$ with n variables. We identify this set with the set of all quadratic forms in n variables with coefficients in $\mathbb{Q}_p$ since we are in characteristic zero.

1.2. Main results. Our main result, namely, Theorem 1.1 gives sufficient conditions in order to ensure that the system (1) has nontrivial solutions. These conditions are realized if we can find a *rational triple* (H_0, X_0, f_0) where H_0 acts on $X_0 = V(f_0)$ and where H_0 satisfies some assumptions prior to the application of the strong approximation property. As an application, we provide an answer to (1) for systems involving one quadratic form and one/several linear form(s) which are well-understood in the real case, i.e. $S = \{\infty\}$. The three dimensional case was treated by Dani and Margulis in [8] where solutions to (1) was provided for pairs $(f_1, f_2) = (Q, L)$. In higher dimensions, a similar result has been proved by Gorodnik [13]. The case of values of quadratic forms restricted to affine subspaces defined linear forms has been treated by Dani [7]. Very recently for the case concerning the values of linear forms on a quadric hypersurfaces, Sargent was able to prove an Oppenheim type density result in the real case [35]. All these results relies on deep results from the ergodic theory of unipotent flows on homogeneous spaces. The most powerful tool to prove such density results is Ratner's orbit closure theorem. The Oppenheim conjecture [29] is a direct consequence of the Raghunathan conjecture, but this conjecture was not yet proved at the time when Margulis [21] and then

Borel–Prasad [5] published their results. The Raghunathan conjecture was proved by Ratner in full generality [33], and also for p-adic Lie groups [34]. The p-adic version of the Raghunathan conjecture has been proved by different methods by Margulis and Tomanov [23].

The main results of the work of Borel–Prasad [5] and Borel [4] can be summarized as follows: given any isotropic nondegenerate quadratic form $F = (F_s)_{s\in S}$ with coefficients in $\mathbb{Q}_S$ in $n \geq 3$ variables such that F is not proportional to a quadratic form with coefficients in $\mathbb{Q}$ then $F(\mathbb{Z}_S^n)$ is dense in $\mathbb{Q}_S$.

Our theorem 1.1 is a generalization of the result of Borel and Prasad given above in the most general setting so that we can ensure solutions of (1) for homogeneous polynomials of arbitrary degrees. To give some credit to the point of view made by Borel and Prasad, we apply our main theorem to find sufficient conditions so that (1) holds when the polynomials involded are quadratic forms and linear forms together. In our appplications, we only focus in low degree polynomials (i.e. less than 3) since there exists better methods to treat such problems in higher dimension, among those the Hardy–Littlewood circle method is better suited when for cubics for instance.

Theorem 1.1. *Let S be a finite set of places in $\mathbb{Q}$ containing the archimedean one. For each $s \in S$, we are given a projective algebraic variety X_s over $\mathbb{Q}_s$ defined by a homogeneous prime ideal I_s of $\mathbb{Q}_s[x_1, \dots, x_n]$ and let H_s be the algebraic $\mathbb{Q}_s$-subgroup of $\mathrm{SL_n}(\mathbb{Q}_s)$ leaving invariant every generator of I_s. Assume that the following subset of places $S_1 \subset S_f$ is nonempty,*

$$S_1 = \{p \in S_f \mid H_p \text{ is rational over } \mathbb{Q}\}.$$

If there exists a connected algebraic subgroup H_0 of G rational over $\mathbb{Q}$ and a hypersurface $X_0 = V(f_0)$ defined over $\mathbb{Q}$ such that

(1) *H_0 is a semisimple absolutely almost simple algebraic $\mathbb{Q}$-group.*
(2) *For every prime $p \in S_1$, $X_p = X_0$ and $H_p = H_0$.*
(3) *Every $\mathbb{Q}$-simple factor of H_0 is isotropic over S_1.*

Then for any $\epsilon > 0$, there exists a nonzero S-integral vector lying ϵ-near X_S i.e.

$$X_S^{\varepsilon} \cap \mathbb{Z}_S^n \neq \{0\}.$$

As an application we use this theorem in order to prove the existence of near integral vectors to a variety of the form $X_S = \{Q = L = 0\}$ which is seen as the nondegenerate quadric $Q = 0$ cutted out by the hyperplane of equation $\{L = 0\}$. In the following result we extend the validity of a previous result of the author [20, Corollary 2.2] proved under the condition that any nontrivial linear combinaison $\alpha_s Q_s + \beta_s L_s^2$ should be irrational at all places $s \in S$. Indeed we are able to prove that the same result holds if we only assume that $\alpha Q + \beta L^2$ is (globally) irrational over $\mathbb{Q}_S$ allowing $\alpha Q + \beta L^2$ to be rational at some nonarchimedean place.

Corollary 1.2. *Assume S is as before and let $Q = (Q_s)_{s\in S}$ be a quadratic form and $L = (L_s)_{s\in S}$ be a linear form on $\mathbb{Q}_S^n$ with $n \geq 4$ and $L_s \neq 0$ for all $s \in S$. Suppose that the pair (Q, L) satisfies the following conditions,*

(1) *Q is nondegenerate.*
(2) *$Q_{|L=0}$ is nondegenerate and isotropic.*
(3) *For any choice α, β in $\mathbb{Q}_S$ with $(\alpha, \beta) \neq (0, 0)$, the form $\alpha Q + \beta L^2$ is irrational with at least one place $v \in S_f$ where $\alpha_v Q + \beta_v L^2$ is proportional to a rational form.*

Then for any $\varepsilon > 0$, there exists a nonzero $x \in \mathbb{Z}_S^n$ such that

$$|Q_s(x)|_s < \varepsilon \quad \text{and} \quad |L_s(x)|_s < \varepsilon \quad \text{for each } s \in S.$$

In the same vein, we are also able to treat the case when we consider several linear forms instead of one. This result is a variation of a recent result due to O. Sargent [35], which gives a S-arithmetic version of using Theorem 1.1.

Corollary 1.3. *Assume S is a finite set of places in $\mathbb{Q}$ containing the archimedean one and suppose that Q is an isotropic nondegenerate quadratic form in $n \geq 3$ variables over $\mathbb{Q}_S$ with rational coefficients. Let $M = (L_1, \ldots, L_r)$ be a linear map $M\colon \mathbb{Q}_S^n \to \mathbb{Q}_S^r$ where $(L_{i,s})_{s\in S}$ $(1 \leqslant i \leqslant r)$ are linear forms of rank r over $\mathbb{Q}_S^n$ in $n \geq 3$ variables which satisfies the following conditions:*

(1) $n > \max_{s\in S}(\dim \ker M_s) + 2$.
(2) rank $(Q_{s|\mathrm{Ker}M_s}) = r$ *and* $Q_{|\mathrm{Ker}M_s}$ *is isotropic and , for every* $s \in S$. .
(3) *For each choice of $\alpha_1, \ldots, \alpha_r$ in $\mathbb{Q}_S$ with $(\alpha_1, \ldots, \alpha_r) \neq (0, \ldots, 0)$, the linear form $\alpha_1 L_1 + \ldots + \alpha_r L_r$ is irrational and at some place $v \in S_f$, $\alpha_{1,v}L_{1,v} + \ldots + \alpha_{r,v}L_{r,v}$ is proportional to a rational form.*

Then for any $\varepsilon > 0$, there exists a nonzero $x \in \mathbb{Z}_S^n$ such that

$$Q_s(x) = 0 \quad \text{and} \quad |L_{i,s}(x)|_s < \varepsilon \quad \text{for each } s \in S \text{ and } 1 \leqslant i \leqslant r.$$

Remarks. (1) The method used in Theorem 1.1 is an adaptation of the work of Borel–Prasad in the case when S contains at least one place where the form is rational (see [5, §4]).

(2) The proof of Theorem 1.1 is based on a strenghtening of the strong approximation theorem which apply to Zariski dense subgroups of reductive groups proved by Matthews, Vaserstein and Weisfeiller [24, 37] and later by Nori [28]. In the meanwhile, Venkataramana [36, Proposition (5.3)] proved that there exists Zariski dense subgroup of integral points of SL_n which contains no unipotent elements and thus such subgroup might be eligible for strong approximation even if it is far from being unipotent or generated by unipotents elements. For more general details about strong approximation in algebraic groups and more particularly this version, we advice the reader the recent survey of Rapinchuk [32].

(3) A nice feature of the proof of Corollaries 1.2 and 1.3 is that we do not have to reduce to lower dimension. Indeed, the reduction process was prerequisted for proceeding to the classification of intermediate subgroups arising from the application Ratner's theorem.

2. Proof of Theorem 1.1

Let us consider as in the assumptions of the theorem and denote by S_2 the set of places of S which are disjoint from S_1, in particular S_1 contains only nonarchimedean places with let us say $S_1 = \{p_1, \ldots, p_s\}$ and $S_2 = \{q_1, \ldots, q_{l-1}, \infty\}$. Let us introduce Λ the stabilizer of the standard lattice $\mathbb{Z}_S^n$ under the action of $H_0(\mathbb{Q})$ i.e.

$$\Lambda = \{h \in H_0(\mathbb{Q}) \mid h(\mathbb{Z}_S^n) = \mathbb{Z}_S^n\}.$$

Under the diagonal embedding, Λ can be seen as an S-arithmetic subgroup which is discrete in $H_0(\mathbb{Q}_S)$ for the product topology. Consider the universal $\mathbb{Q}$-isogeny $\pi\colon \widetilde{H_0} \to H_0$, here $\widetilde{H_0}$ is a semisimple simply connected group defined over $\mathbb{Q}$ such isogeny always exists in this case [31, Theorem 2.6]. Let us choose an arbitrary S-arithmetic subgroup $\widetilde{\Lambda}$ of $\widetilde{H_0}(\mathbb{Q})$ embedded as a discrete subgroup in $\widetilde{H_0}(\mathbb{Q}_S)$ for the product topology. Since $\widetilde{H_0}$ is absolutely almost simple over $\mathbb{Q}$ and every simple factor H_i of $\widetilde{H_0}$ is noncompact at every place in S_1 by definition of S_1, in particular $\sum_{s\in S}\mathrm{rank}_{\mathbb{Q}_s}H_i \geq \sum_{s\in S_1}\mathrm{rank}_{\mathbb{Q}_s}H_i > 0$. Borel's density theorem (see e.g. [22, Proposition 3.2.10] applied to $\tilde{H}_0$ with $K = k = \mathbb{Q}$ gives us that the discrete subgroup $\widetilde{\Lambda}$ is Zariski-dense in $\widetilde{H_0}$. A special instance of the strong approximation for Zariski-dense subgroups (see e.g. [31, Theorem 7.14] applied in $\widetilde{H_0}$ for the set of primes $S_1 = \{p_1, \ldots, p_s\}$ implies that the closure $\overline{\widetilde{\Lambda}}$ is open in $\widetilde{H_0}(\mathbb{A}_{S_1})$. The latter adelic group is described by the following product

$$\prod_{p\in S_1}\widetilde{H}_0(\mathbb{Q}_p) \times \prod_{q\in S_2}\widetilde{H}_0(\mathbb{Z}_q) \times \prod_{q\notin S_1\cup S_2}\widetilde{H}_0(\mathbb{Z}_q).$$

Thus for every $q \in S_2$, the projection $\overline{\widetilde{\Lambda}}^{(q)}$ is open in $\tilde{H}_0(\mathbb{Z}_q)$. The latter fact can also be deduced from the fact that $\widetilde{H_0}(\mathbb{Z}_p)$ is a virtually pro p-group for every $p \in S_2$ (see [32, Lemma 2.7 and §3A]).

The universal $\mathbb{Q}$-isogeny π transforms $\tilde{\Lambda}$ into an arithmetic subgroup $\pi(\tilde{\Lambda})$ in $H_0(\mathbb{Q})$ such that $\overline{\pi(\tilde{\Lambda})}^{(S_2)}$ is open in $H_0(\mathbb{Q}_{S_2})$. The two arithmetic subgroups $\pi(\tilde{\Lambda})$ and Λ are commensurable in H_0, thus $\pi(\tilde{\Lambda})\cap\Lambda$ is a finite index subgroup in $\pi(\tilde{\Lambda})$ (see e.g. [22, Cor. 3.2.9]). For each $p \in S$, we define U_p to be the projection of $\pi(\tilde{\Lambda}) \cap \Lambda$ onto its p-component, that is,

$$U_p := (\pi(\tilde{\Lambda}) \cap \Lambda)_p. \tag{2}$$

As we have seen above, the p-adic closure $\overline{U_p}^{(p)}$ lies in the open subgroup $\overline{\pi(\tilde{\Lambda})}^{(p)}$ of $H_0(\mathbb{Q}_p)$ for each $p \in S_2$ and U_{S_2} is contained in $\overline{\Lambda}^{(S_2)}$ by definition.

Now let us introduce the following subset of vectors in $\mathbb{Q}_p^n$ defined for each $p \in S_2$, by

$$\mathfrak{X}_p = U_p^{-1}X_p(\mathbb{Q}_p) - X_0(\mathbb{Q}_p)$$

where $U_p^{-1} = \{a_p^{-1} | a_p \in U_p\}$ is an open subset of invertible matrices in $H_0(\mathbb{Q}_p)$ where U_p was defined in (2). We claim that $\mathfrak{X}_p$ is a nonempty open cone in $\mathbb{Q}_p^n$ for $p \in S_2$ which does not contains the null vector i.e. $0 \notin \mathfrak{X}_p$. Indeed, since X_p is not rational over $\mathbb{Q}$ for each $p \in S_2$ while X_0 is, this forces $X_p \neq X_0$ for $p \in S_2$. The Zariski density of $X_p(\mathbb{Q}_p)$ (resp. $X_0(\mathbb{Q}_p)$) in X_p (resp. X_0) implies that $X_p(\mathbb{Q}_p) \neq X_0(\mathbb{Q}_p)$ for each $p \in S_2$. In particular for each $p \in S_2$, there exists an $x \in X_p(\mathbb{Q}_p) - X_0(\mathbb{Q}_p)$, thus taking $a_p = I_n$ we get that $x \in \mathfrak{X}_p$. Now let us fix $x \in \mathfrak{X}_p$ and consider $y \in \mathbb{Q}_p^n$ sufficiently close to x, so that we can find a $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ such that $y = gx$ where g is close to I_n in $\mathrm{SL}_n(\mathbb{Q}_p)$. Since $I_n \in U_p$ then we can assume that g is arbitrarily close to I_n in the open set U_p. We have $f_{i,p}(g_p x) = 0$ for some $g_p \in U_p$ close to I_n and since $y = gx$ we get $f_{i,p}((g_p g^{-1})y) = f_{i,p}(g_p x) = 0$ with $g_p g^{-1} \in U_p$ this implies that $y \in \mathfrak{X}_p - \{0\}$. By definition $\mathfrak{X}_p$ does not contains the origin. For any $\lambda \in \mathbb{Q}_p^*$ and $x \in \mathfrak{X}_p$, $\lambda x \in \mathfrak{X}_p$ by homogeneity of the f_i's. The claim is proved.

For each $p \in S_1$, denote by $\Delta_p(r)$ the hypercube centered at 0 with radius r in $\mathbb{Q}_p^n$, i.e.

$$\Delta_p(r) = \left\{x \in \mathbb{Q}_{S_1}^n \mid |x_i|_p \leq r\right\}$$

and denote $\Delta_{S_1}(r) = \prod_{p\in S_1}\Delta_p(r)$. Our aim is to show that $\bigcap_{\varepsilon>0} X_S^{\varepsilon} \cap \mathbb{Z}_S^n \neq \{0\}$, a first step consists to show the existence of nonzero lattice point for the domain $(\Delta_{S_1}(\delta) \times \mathfrak{X}_{S_2})$ which can be seen as a sort of approximation of X_S^{ε}.

Lemma 2.1. *For every $\delta > 0$, we have*

$$(\Delta_{S_1}(\delta) \times \mathfrak{X}_{S_2}) \cap \mathbb{Z}_S^n \neq \{0\}.$$

Proof of the lemma. Let us fix a place $p \in S_2$ whether it is archimedean or not, and set $l = |S_2|$. Let $v_{1,p} \in \mathbb{Q}_p^n$ be a nonzero in vector $\mathfrak{X}_p$ and complete it to a basis $\{v_{1,p}, v_{2,p}, \dots, v_{p,n}\}$ of $\mathbb{Q}_p^n$. For each real $a > 0$ we introduce the hypercubes $V_p(a)$ in $\mathbb{Q}_p^n$ and $W_p(a)$ in $\mathbb{Q}_p^{n-1}$ defined as

$$V_p(a) = \left\{ \sum_{i=1}^{n} \alpha_i v_{i,p} \mid |\alpha_i|_p \le a \right\} \quad \text{and} \quad W_p(a) = \left\{ \sum_{i=2}^{n} \alpha_i v_{i,p} \mid |\alpha_i|_p \le a \right\}.$$

Since we know that $\mathfrak{X}_p$ is open, we can find an infinitesimal hypercube $v_{1,p} \oplus W_p(\alpha)$ for some small enough real $\alpha > 0$ so that it is contained in $\mathfrak{X}_p$. However we have the following fact: the resulting infinitesimal hypercube $v_{1,p} \oplus W_p(\alpha)$ remains in the cone $\mathfrak{X}_p$ if we perform a translation in the direction of $v_{1,p}$ away from $\mathbb{Z}_p$ [1]. In other words, we can find an real positive α small enough so that for each $p \in S_2$ and for any given arbitrary $\eta \in \mathbb{Q}_p$ with $|\eta|_p > 1$ (i.e. $\eta \notin \mathbb{Z}_p$) we have simultaneously

$$\eta v_{1,p} \oplus W_p(\alpha) \subset \mathfrak{X}_p \quad \text{and} \quad V_q(\alpha) \subset \mathfrak{X}_p. \tag{3}$$

Indeed, let us set $u = \eta v_{1,p} + \sum_{i=2}^{n} \alpha_i v_{i,p} \in \eta v_{1,p} \oplus W_p(\alpha)$ for $\eta \in \mathbb{Q}_p^* - \mathbb{Z}_p$. Thus u can be written as

$$u = \eta \left(v_{1,p} + \sum_{i=2}^{n} \eta^{-1} \alpha_i v_{i,p} \right).$$

It is clear that for every $2 \le i \le n$, $|\eta^{-1}\alpha_i|_p < \alpha$, thus

$$v_{1,p} + \sum_{i=2}^{n} \eta^{-1} \alpha_i v_{i,p} \in v_{1,p} \oplus W_p(\alpha).$$

Therefore using the cone invariance for $\mathfrak{X}_p$, we infer that $u \in \mathfrak{X}_p$, which proves the claim (3). For this choice of α and for each reals $\delta, t > 0$ we introduce the S-adelic domain

$$\mathcal{C}_p(\delta, t) = \Delta_{S_1}(\delta/2l) \times [0, t] v_{1,p} \oplus W_p(\alpha/2) \times \prod_{q \in S_2 \setminus \{p\}} V_q(\alpha/2l) \times \prod_{s \notin S} \mathbb{Z}_p \subset \mathbb{A}_S^n.$$

Let us define $C_p(\delta)$ to be $2 * \mathcal{C}_p(\delta, 1)$. It is a compact subset of $\mathbb{A}_S^n$ and thus it meets the discrete subgroup $\mathbb{Z}_S^n$ in finitely many points at a number of $k = |C_p(\delta) \cap \mathbb{Z}_S^n|$. The set of S-integral vectors $\mathbb{Z}_S^n$ is a cocompact lattice in $\mathbb{A}_S^n$ i.e. $\mathbb{A}_S^n / \mathbb{Z}_S^n$ is a compact space of finite volume for the measure μ induced by $\mathrm{vol}_{\mathbb{A}_S}$ on the quotient space. Let us denote by $\mathcal{D}$ a fundamental domain for the quotient $\mathbb{A}_S^n / \mathbb{Z}_S^n$ and by $\mu(\mathcal{D})$ its volume, in particular we have

$$\mathbb{A}_S^n = \bigcup_{y \in \mathcal{D}} y + \mathbb{Z}_S^n.$$

We can find some τ large enough so that exists $k+2$ disctinct points $y_0, \dots, y_{k+1}$ in $\mathcal{C}_p(\delta, \tau/2)$ such that $y_i - y_j \in \mathbb{Z}_S^n$ for any $1 \leqslant i < j \leqslant k+1$. Indeed, for this it

[1] Note that if p is nonarchimedean, $\mathbb{Q}_p$ is not an ordered field neither even partially, so one has to be careful with the meaning of this assertion. The real meaning is arithmetical rather than geometrical as it can be seen just below.

suffices to remark that the function $t \mapsto \mathrm{vol}_{\mathbb{A}_S}(\mathcal{C}_p(\delta, t))$ in increasing, so for τ large enough we have the inequality

$$\mathrm{vol}_{\mathbb{A}_S}(\mathcal{C}_p(\delta, \tau/2)) > (k+1)\mu(\mathcal{D}).$$

Now applying the adelic Blichfeldt's principle in $\mathbb{A}_S^n$ (see e.g. [25, Lemma 4, §5.2]) we obtain the required y_i's. Let us put $K = \mathbb{A}_S^n - \mathcal{C}_p(\delta)$ and $x_i = y_0 - y_i$ for $1 \leq i \leq k+1$. The previous assertion tells us that $x_1, \ldots, x_{k+1}$ are nonzero elements of $\mathbb{Z}_S^n$, namely we have a set of $k+1$ integral vectors. But there are only k elements in $\mathcal{C}_p(\delta) \cap \mathbb{Z}_S^n$, thus exactly one of them must lie in the complement $K_p \cap \mathbb{Z}_S^n$, call this unique element $x(p)$. By definition $x(p)$ is the difference of two elements in $\mathcal{C}_p(\delta, \tau/2)$, thus $x(p) \in 2 * \mathcal{C}_p(\delta, \tau/2)$.

Hence for each $p \in S_2$, we assigns a unique nonzero element $x(p)$ of $\mathbb{Z}_S^n$ which lies in $K_p \cap 2 * \mathcal{C}_p(\delta, \tau/2)$, moreover the latter subset satisfies the following inclusion

$$K_p \cap 2 * \mathcal{C}_p(\delta, \tau/2) \subset \Delta_{S_1}(\delta/l) \times (1, \tau]v_{1,p} \oplus W_p(\alpha) \times \prod_{q \in S_2 \setminus \{p\}} V_q(\alpha/l) \times \prod_{s \notin S} 2 * \mathbb{Z}_p.$$

Finally let us set $x = \sum_{p \in S_2} \pi_S(x(p))$ where π_S is the projection to the S-factor. It is already clear that x is a nonvector vector in $\mathbb{Z}_S^n$. It remains to verify that $x \in \Delta_{S_1}(\delta) \times \mathfrak{X}_{S_2}$. For the S_1-components, we have

$$\pi_{S_1}(x) = \sum_{p \in S_2} \pi_{S_1}(x(p)) \in \sum_{p \in S_2}^{(M)} \pi_{S_1}\left(K_p \cap 2 * \mathcal{C}_p(\delta, \tau/2)\right) \subset l * \Delta_{S_1}(\delta/l) \subset \Delta_{S_1}(\delta).$$

On S_2 side, we isolate the diagonal component in order to obtain

$$\pi_{S_2}(x) = x(p)_p + \sum_{q \in S_2 \setminus \{p\}} x(p)_q \in (1, \tau]v_{1,p} \oplus W_p(\alpha) \times \prod_{q \in S_2 \setminus \{p\}} \sum_{p' \in S_2 \setminus \{p\}}^{(M)} V_q(\alpha/l)_{p'}.$$

Remembering the choice of $\alpha > 0$ made in (3), we infer that

$$\pi_{S_2}(x) \in \mathfrak{X}_p \times \prod_{q \in S_2 \setminus \{p\}} V_q(\alpha) \subset \mathfrak{X}_{S_2}.$$

Hence for any $\delta > 0$, we can always find a nonzero vector x lying in $(\Delta_{S_1}(\delta) \times \mathfrak{X}_{S_2}) \cap \mathbb{Z}_S^n$ and this achieves the proof of the lemma. $\square$

We are now ready to prove the theorem, for this let us fix an $\varepsilon > 0$. The fact that the f_i's are homogenous polynomials, in particular with $f_i(0) = 0$, implies the existence of a real number $\delta(\varepsilon) > 0$ small enough so that $\Delta_{S_1}(\delta(\varepsilon)) \subset X_{S_1}^{\varepsilon}$. Using Lemma 2.1 with $\delta = \delta(\varepsilon)$, one obtains a nonzero S-integral vector $x \in \mathbb{Z}_S^n$ such that $\pi_{S_1}(x) \in X_{S_1}^{\varepsilon}$ and $\pi_{S_2}(x) \in \mathfrak{X}_{S_2}$. The latter condition means that for each $p \in S_2$ and corresponding $x_p \in \mathfrak{X}_p$, there exists some $u_p \in U_p$ such that $f_{i,p}(u_p x_p) = 0$. Since U_p is open there exists $g_p \in U_p$ such that

$$0 < |f_{i,p}(g_p x_p)|_p \leq \varepsilon/2.$$

With the help of the strong approximation theorem, we have seen earlier that U_{S_2} is contained in $\overline{\Lambda}^{(S_2)}$, and incidentally we find $(\gamma_p)_{p \in S_2} \in \Lambda_{S_2}$ such that for every $p \in S_2$ and $1 \leq i \leq r$, one has

$$0 < |f_{i,p}(\gamma_p x_p)|_p \leq \varepsilon. \tag{4}$$

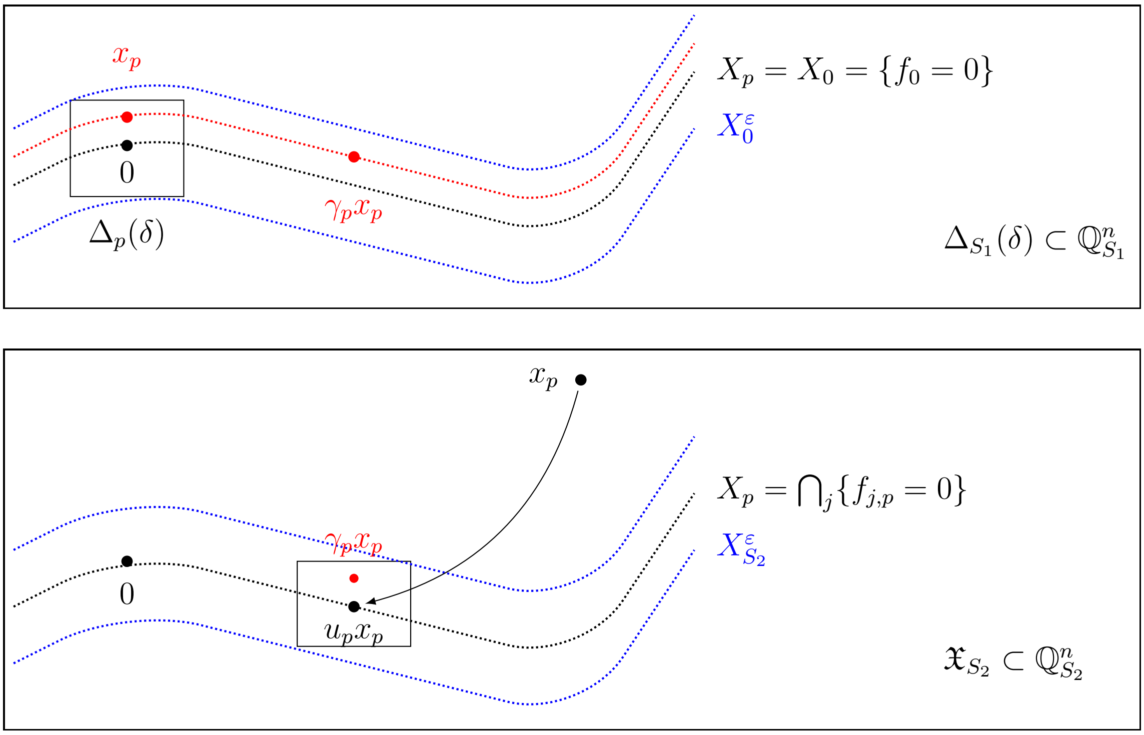

Figure 1. The space between the blue dotted lines is the tubular neighborhood X_S^ε.

Consider the projection $\Lambda_S \twoheadrightarrow \Lambda_{S_2}$ and let $(\tilde{\gamma})_{p\in S}$ be a lift of $(\gamma_p)_{p\in S_2}$, that is, $\tilde{\gamma}_p = \gamma_p$ for every $p \in S_2$. We claim that $y = \tilde{\gamma}x$ is the solution of our problem. Indeed, on the one hand the inequalities in (4) imply that $\pi_{S_2}(y) \in X_{S_2}^\varepsilon$.

On the other hand since $\pi_{S_1}(x) \in X_{S_1}^\varepsilon$ and X_p^ε is H_0-invariant with $\Lambda_p \subseteq H_0$ for every $p \in S_1$, we deduce that $y_p = (\tilde{\gamma}_p x_p) \in X_p^\varepsilon$ for every $p \in S_1$, i.e. $\pi_{S_1}(y) \in X_{S_1}^\varepsilon$. This allows us to conclude the existence of a nonzero vector $y \in X_S^\varepsilon \cap \mathbb{Z}_S^n$ and this finishes the proof of the theorem.

3. Proof of Corollary 1.2

Let us consider a pair $(Q_s, L_s)_{s\in S}$ over $\mathbb{Q}_S$ satisfiying all the assumptions of Corollary 1.2. For each $s \in S$, we set X_s to be the algebraic (projective) variety given by $\{Q_s = L_s = 0\}$, geometrically this can be seen as the cone $Q_s = 0$ cutted out by the hyperplane of equation $L_s = 0$ in $\mathbb{Q}_s^n$. It is more suitable here to think X_s as the quadric of equation $\{Q_{s|_{L_s=0}} = 0\}$, in that way the assumptions (1) and (2) amounts to say that this quadric $\{Q_{s|_{L_s=0}} = 0\}$ is nondegenerate and contains at least one nonzero vector in $\mathbb{Q}_s^n$ for every $s \in S$. We need to introduce the following map associated to the pair (Q, L) over $\mathbb{Q}_S$

$$\begin{array}{rrcl} \psi: & \mathbb{P}^1(\mathbb{Q}_S) & \to & \mathrm{Sym}^2(\mathbb{Q}_S^n), \\ & (\alpha:\beta) & \mapsto & \alpha Q + \beta L^2. \end{array}$$

This induces at each place $s \in S$, a (local) map $\psi_s(\alpha_s : \beta_s) = \alpha_s Q_s + \beta_s L_s^2$.

The assumption (3) says that the range of ψ is in the subspace $\mathrm{Sym}^2_{ir.}(\mathbb{Q}_S^n)$ consisting of quadratic forms in $\mathrm{Sym}^2(\mathbb{Q}_S^n)$ which are not proportional to rational form over $\mathbb{Q}_S$. The complement consisting of quadratic forms over $\mathbb{Q}_S^n$ (resp. $\mathbb{Q}_s^n$) which are proportional to a rational form is denoted $\mathrm{Sym}^2_{rat.}(\mathbb{Q}_S^n)$ (resp. $\mathrm{Sym}^2_{rat.}(\mathbb{Q}_s^n)$).

Case 1. If we are in the case where ψ_s assumes its values in $\mathrm{Sym}^2_{ir.}(\mathbb{Q}_s^n)$ for every $s \in S$, there is nothing to prove. Indeed, Corollary 2.2. in [20] already gives the required result.

Now we treat the case of interest here.

Case 2. Suppose there exists a place $v \in S_f$ such that the range of ψ_v is in $\mathrm{Sym}^2_{rat.}(\mathbb{Q}_v^n)$. This means concretely that for any not both zero couple of constants $(\alpha_v, \beta_v) \in \mathbb{Q}_v^2$ we have that $\alpha_v Q_v + \beta_v L_v^2$ is proportional to a quadratic form Q_0 with rational coefficients, thus $\alpha_v Q_v + \beta_v L_v^2 = \lambda_v Q_0$ for some $\lambda_v \in \mathbb{Q}_v^*$, in particular $\psi_v(1:0) = Q_v$ and $\psi_v(0:1) = L_v$ are proportional to a rational form. Thus we can write $L_v = \mu_v L_0$ where L_0 is a rational linear form and $\mu_v \in \mathbb{Q}_v^*$ and from the definition of Q_0 we have that $\alpha_v Q_{v|L_v=0} = \lambda_v Q_{0|L_0=0}$ and in particular $SO(Q_{v|L_v=0}) = SO(Q_{0|L_0=0})$. Now for each $s \in S$ we denote by H_s the stabilizer of the pair (Q_s, L_s) to be the subgroup of $\mathrm{SL}_n(\mathbb{Q}_s)$ leaving both invariant Q_s and L_s. Following [20, Lemma 4.1], since Q_v and L_v are both proportional to rational forms then we can arrange a basis $\{w_1, \ldots, w_{n-1}, u\}$ consisting of rational vectors with the condition that $\{L_v = 0\} = \langle w_1, \ldots, w_{n-1}\rangle$ and $L_v(u) = u$. Hence we can find a $g \in \mathrm{SL}_n(\mathbb{Q})$ such that $H_v = g^{-1} \left[\begin{array}{c|c} SO(Q_{v|L_v=0}) & 0 \\ \hline 0 & 1 \end{array}\right] g$.

Let us define the hypersurface defined over $\mathbb{Q}$ by $X_0 = \{Q_0 = L_0 = 0\} = \{Q_{0|L_0=0} = 0\}$ and set

$$H_0 := g^{-1} \left[\begin{array}{c|c} SO(Q_{0|L_0=0}) & 0 \\ \hline 0 & 1 \end{array}\right] g.$$

Clearly we have $X_v = X_0$ and $H_v = H_0$ where v is as chosen above. The form $Q_{0|L_0=0}$ is nondegenerate thus H_0 is semisimple, it is also isotropic which implies that H_0 is noncompact and isotropic at v. Moreover, from the rationality of $Q_{0|L_0=0}$ we infer that H_0 is defined over $\mathbb{Q}$. Therefore the triple $(X_0, Q_{0|L_0=0}, H_0)$ satisfies all the conditions of Theorem 1.1 unless the connectedness for H_0 which is not ensured at all. To remedy to this situation we can consider the connected component of the identity of H_0 which we denote H_0^+. The point is that H_0^+ is now connected but it is not obvious that the other properties (isotropy and rationality) are preserved by performing this operation. The key fact is that H_0^+ has finite index in H_0 thus since H_0^+ is still noncompact and therefore isotropic, in addition H_0^+ is defined over $\mathbb{Q}$ since H_0 is (see e.g. [3, Prop. 1.2(b)] and also [22, 2.3.2 and Remark 2 just after]. A last remark concerns the conservation by the connected component under the central isogeny. This point is quite crucial since at some stage in the proof of the Theorem 1.1 we need to pass to the universal covering. The fact that $\pi\colon \widetilde{H_0} \to H_0$ is an isogeny defined over $\mathbb{Q}$, we have that $H_0^+ = \pi(\widetilde{H_0})^+ = \pi(\widetilde{H_0}^+)$ (see e.g. [3, Cor. 1.4 (b)]). Hence π induces an central isogeny $\pi^+\colon \widetilde{H_0}^+ \to H_0^+$ where $\widetilde{H_0}^+$ is simply connected. In fact, the latter subgroup is explicitely given by

$$\widetilde{H_0}^+ = g^{-1} \left[\begin{array}{c|c} \mathrm{Spin}(Q_{0|L_0=0}) & 0 \\ \hline 0 & 1 \end{array}\right] g.$$

To sum up, given an arbitrary $\varepsilon > 0$, the triple $(X_0, Q_{0|L_0=0}, H_0^+)$ satisfies all the conditions of Theorem 1.1 for $S_1 = \{v\}$ thus there exists a nonzero $x \in \mathbb{Z}_S^n$ such that $x \in X_S^\varepsilon$ i.e.

$$|Q_s(x)|_s < \varepsilon \quad \text{and} \quad |L_s(x)|_s < \varepsilon \quad \text{for each } s \in S.$$

4. Proof of Corollary 1.3

We proceed in the same way as the previous corollary. Let us consider a quadratic form Q and a linear map $M = (L_1, \ldots, L_r)$ satisfying all the assumptions of the corollary 1.3. For each $s \in S$, we define the following variety $X_s = V(Q_s, L_{1,s}, \ldots, L_{r,s})$ defined over $\mathbb{Q}_s^n$, in fact, for pratical reasons it is more suitable to see this variety as $X_s = V(Q_{s|M_s=0})$. Geometrically the locus of X_s is defined by a quadratic $\{Q_s = 0\}$ cutted out by the intersection of the hyperplanes of equations $\{L_{i,s} = 0\}$ $(1 \leqslant i \leqslant r)$, the later intersection is just the kernel of M_s i.e. $X_s = V(Q_{s|\ker M_s})$. It is assumed that $\operatorname{rank} M_s = r$ for every $s \in S$, thus $\dim \ker M_s = n - r$, that is to say, $L_{1,s}, \ldots, L_{r,s}$ are linearly independent over $\mathbb{Q}_s$. If we denote by B_s the symmetric bilinear form associated to Q_s, we can consider the following orthogonal decompositon of the quadratic space $(\mathbb{Q}_s^n, Q_s)$ with respect to B:

$$(\mathbb{Q}_s^n, Q_s) = (\ker M_s, Q_{s|\ker M_s}) \bigoplus ((\ker M_s)^\perp, Q_{s|(\ker M_s)^\perp})$$

and let $\{e_1, \ldots, e_n\}$ a basis of $\mathbb{Q}_s^n$ adapted to this decompostion above, that is, $M_s = \langle e_1, \ldots, e_{n-r}\rangle$ and $M_s^\perp = \langle e_{n-r+1}, \ldots, e_n\rangle$ such that $B(e_k, e_l) = 0$ for all $1 \leqslant k \leqslant n-r$ and $n-r+1 \leqslant k \leqslant n$. Now for any $s \in S$, we set the following subgroup of $\mathrm{SL}_n(\mathbb{Q}_s)$ given in the adapted basis by

$$H_s = \left[\begin{array}{c|c} SO(Q_{s|\ker M_s}) & 0 \\ \hline 0 & I_r \end{array}\right].$$

We claim that H_s leaves invariant both Q_s and M_s for each $s \in S$. Indeed let $h \in H_s$, and A an element of $SO(Q_{s|\ker M_s})$ such that

$$h = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & I_r \end{array}\right].$$

Note that, as given, the range of A is necessarily within the subspace $\ker M_s$. Let x be a vector in $\mathbb{Q}_s^n$ which decomposes into $x = x_1 + x_2$ with $x_1 \in \ker M_s$ and $x_2 \in (\ker M_s)^\perp$. Therefore $L_s(hx) = L_s(h(x_1, x_2)^t) = L_s(Ax_1 \oplus x_2) = L_s(Ax_1) + L_s(x_2) = 0 + L_s(x_2) = L_s(x_1) + L_s(x_2) = L_s(x)$, thus L_s is H_s-invariant. In the other hand, using the same decomposition for x we get

$$Q_s(hx) = Q(Ax_1 \oplus x_2) = Q_{s|\ker M_s}(Ax_1) + Q_s(x_2) = Q_s(x_1) + Q_s(x_2) = Q(x).$$

In particular, the claim shows that H_s acts linearly on $X_s = V(Q_{s|\ker M_s})$. Now given any constants $\alpha_1, \ldots, \alpha_r$ in $\mathbb{Q}_S$ not all zero, the form $\alpha_{1,s}L_{1,s} + \ldots + \alpha_{r,s}L_{r,s}$ is irrational for every place $s \in S\backslash\{v\}$ and proportional to a rational form for $s = v$. Let us set $f_0 := Q_v + \alpha_{1,v}L_{1,v} + \ldots + \alpha_{1,v}L_{r,v}$, it is clear that f_0 is proportional to a rational form and a suitable choice of constants allows us to assume that f_0 has rational coefficients. The crucial fact is that $f_{0|\ker M_v} = Q_{v|\ker M_v}$ is also a quadratic form with rational coefficients since $\ker M_v$ is a $\mathbb{Q}$-subspace. Thus if we put

$$H_0 = \left[\begin{array}{c|c} SO(f_{0|\ker M_v}) & 0 \\ \hline 0 & I_r \end{array}\right],$$

then $H_0 = H_v$ is a algebraic subgroup of $\mathrm{SL}_n(\mathbb{Q}_v)$ which is defined over $\mathbb{Q}$ and which acts on $X_0 = X_v$. Moreover, for the same reasons as in the previous corollary, H_0^+ is a connected semisimple algebraic subgroup which is isotropic at v since $Q_{v|\ker M_v}$ is isotropic, in particular H_0^+ has no compact factors. We obtain a rational triple by taking $(X_0, f_{0|\ker M_v}, H_0^+)$, since it satisfies all the conditions of Theorem 1.1 for $S_1 = \{v\}$, we infer that for any $\varepsilon > 0$, we can find a nonzero $x \in \mathbb{Z}_S^n$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_{i,s}(x)|_s < \varepsilon \text{ for each } s \in S \text{ and } 1 \leqslant i \leqslant r.$$

To conclude, one has to remark that since Q_s is rational and $x \in \mathbb{Z}_S^n = (\mathbb{A}_S \cap \mathbb{Q})^n$, $Q_s(x) \in \mathbb{Q}$ for every $s \in S$. Since $\mathbb{Q}$ is discrete in every completion, we deduce that if ε is small enough we can find $x \in \mathbb{Z}_S^n - \{0\}$ such that

$$Q_s(x) = 0 \text{ and } |L_{i,s}(x)|_s < \varepsilon \text{ for each } s \in S \text{ and } 1 \leqslant i \leqslant r. \qquad \square$$

5. Existence of rational triples (X_0, H_0, f_0) for general varieties

In this section we adress some remarks concerning the class of varieties X which falls into the conditions of the theorem. The varieties involved in the theorem are called *complete intersection* in the litterature and they have been subject to extensive research until now and still many problems remains open concerning those projective varieties. From our point of view, we are more concerned with the invariant theory of the space of homogeneous polynomials.

In order to apply the main theorem one has to find a rational triple (X_0, H_0, f_0) such that X_S (resp. H_S) can be splited in the form $X_S = V(f_0) \times X_{S_2}$ (resp. $H_S = H_0 \times H_{S_2}$) without loss in generality we assume that $S_1 = \{v\}$ for some $v \in S_f$ thus $S_2 = S \backslash \{v\}$. In particular H_0 acts rationally on the $\mathbb{Q}$-hypersurface $X_0 = V(f_0)$.

Bounds on the degrees $(d_1, \dots, d_r)$ of the generators of the ideal of X.

• A first constrain is the equality $X_v = X_0$, that is, in algebraic terms

$$V(f_{1,v}, \dots, f_{r,v}) = V(f_0).$$

Applying Hilbert's Nullstellensatz in an algebraic closure of $\mathbb{Q}_v$ yields

$$\sqrt{(f_0)} = \sqrt{(f_{1,v}, \dots, f_{r,v})}$$

where $\sqrt{J}$ is the radical of an ideal J in $\overline{\mathbb{Q}_v}[x_1, \dots, x_n]$, in particular there exists an integer $\rho > 0$ and (homogeneous) polynomials $P_1, \dots, P_r$ over $\mathbb{Q}_v$ such that

$$f_0^\rho = P_1 f_{1,v} + \ldots + P_r f_{r,v} \tag{5}$$

Let us denote by N_i (resp. d_i) the total homogeneous of P_i (resp. $f_{i,v}$) for each $1 \leqslant i \leqslant r$. Thus the previous equality reads in terms of degrees as

$$\rho \deg f_0 = \max_{1 \leqslant i \leqslant r} \{N_i + d_i\}. \tag{6}$$

Let us assume that the degrees are ordered as follows $N_1 \geqslant N_2 \geqslant \ldots \geq N_r$ and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_r$, then (6) reads

$$\rho \deg f_0 = N_1 + d_1. \tag{7}$$

When $d_i \neq 2$ $(1 \leqslant i \leqslant r)$, upper bounds for ρ can be effectively computed, the following sharp estimates for ρ are due to Kollár [17, Corollary 1.7].

$$\rho \leq \begin{cases} d_1 d_2 \ldots d_r & \text{if } r \leq n, \\ d_1 d_2 \ldots d_{n-1} d_r & \text{if } r > n, \end{cases} \tag{8}$$

and for each $1 \leqslant i \leqslant r$

$$N_i + d_i = \deg(P_i f_{i,v}) \leq \begin{cases} (1 + d_0) d_1 d_2 \ldots d_r & \text{if } r \leq n, \\ (1 + d_0) d_1 d_2 \ldots d_{n-1} d_r & \text{if } r > n, \end{cases} \tag{9}$$

where d_0 denotes $\deg f_0$. To sum up, saying that X_v equals the hypersurface $X_0 = V(f_0)$ amounts to find some polynomials $(P_j)_{1\leqslant i\leqslant r}$ such that

$$f_0^\rho = P_1 f_{1,v} + \ldots + P_r f_{r,v}$$

where ρ satisfies condition (8) and after (9) we get that for every $1 \leqslant i \leqslant r$,

$$(10) \qquad 0 \leq \deg(P_j) \leq \begin{cases} (1+d_0)d_1 d_2 \ldots d_r - d_j & \text{if } r \leq n, \\ (1+d_0)d_1 d_2 \ldots d_{n-1} d_r - d_j \text{if } r > n. \end{cases}$$

The equidimensional case: If we assume that $d_1 = \ldots = d_r = d \neq 2$, we obtain the following bounds for every $1 \leqslant i \leqslant r$,

$$(11) \qquad 0 \leq \deg(P_j) \leq \begin{cases} (1+d_0)d^r - d & \text{if } r \leq n, \\ (1+d_0)d^n - d & \text{if } r > n. \end{cases}$$

If we want that the right hand in (6), that is, f_0^ρ to have only degree d, then necessarily $\rho = 1$ and from (11) we get that the polynomials P_i should be constant polynomials α_i $(1 \leqslant i \leqslant r)$

$$(12) \qquad f_0 = \alpha_1 f_{1,v} + \ldots + \alpha_r f_{r,v}.$$

• Rationality conditions $X_v = X_0 = V(f_0)$ with f_0 rational. The relation (5) shows that if the polynomial f_0 has rational coefficients then some linear combination (over $\mathbb{Q}_v$) of the $f_{i,v}$'s must be rational. In particular, this and (12) explain why the condition (3) in both corollaries 1.2 and 1.3 is necessary.

From complete intersections towards invariant group of the ideal of definition. The main issue is that given a variety X defined over a field of characteristic zero K, say a complete intersection, to find the largest subgroup of $G = \mathrm{SL}_{n|K}$ which acts trivially on the ideal of definition I_X. For instance, let us be given a complete intersection $X = V(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r$ are homogeneous polynomial of degrees $d_1 \leq \ldots \leq d_r$. The ideal of definition of X is given by $I_X = \langle f_1, \ldots, f_r \rangle$. The central role is played by the stabilizer H of the ideal which is defined to be

$$H_s = \bigcap_{1\leqslant i\leqslant r} \{g \in G \mid g.f_i = f_i\}.$$

The ideal stabilizer H_s obviously does act on X_s for every $s \in S$, and it gives an action of H_S on X_S induces by the usual of $G = \mathrm{PSL_n}$ on the vector space of homogeneous polynomials in $\mathbb{Q}_s[x_1, \ldots, x_n]$. In particular H_s contains $\mathrm{Aut}_G(X)$ the group automorphism of X under G which is the pointwise stablizer of X_s under G. The ideal stabilizer is an algebraic subgroup of G given by the following equations in the variables $(g_{i,j})$

$$f_k(gx) = f_k(x) \quad \text{and} \quad \det(g_{i,j}) - 1 = 0.$$

Let us try to solve those equation with $g = (g_{i,j})_{i,j}$, for this let us explicit the coefficients of the f_k and assume that they are of form

$$f_k(x_1, \ldots, x_n) = \sum_{|\alpha|=d_k} a_\alpha^{(k)} x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

Therefore we have n equations which takes places in $K[x_1, \ldots, x_n]_{(d_k)}[(g_{i,j})_{i,j}]$

$$\sum_{|\alpha|=d_k} a_\alpha^{(k)} \left(\left(\sum_{j_1=1}^n g_{1j_1} x_{j_1} \right)^{\alpha_1} \cdots \left(\sum_{j_n=1}^n g_{nj_n} x_{j_n} \right)^{\alpha_n} - x_1^{\alpha_1} \ldots x_n^{\alpha_n} \right) = 0,$$

$$\sum_{|\alpha|=d_k} a_\alpha^{(k)} \left(\sum_{|\beta|=\alpha_1} \binom{\alpha_1}{\beta} (g_{11}x_1)^{\beta_1} \dots (g_{1n}x_n)^{\beta_n} \right) \dots \left(\sum_{|\beta|=\alpha_n} \binom{\alpha_n}{\beta} (g_{n1}x_1)^{\beta_1} \dots (g_{nn}x_n)^{\beta_n} \right)$$
$$= \sum_{|\alpha|=d_k} a_\alpha^{(k)} x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

We do not need to go further to observe that such compuations unless we are dealing with low degrees (i.e. $d = 2, 3$) leads to tremendous compuations and trying to obtain the ideal stabilizer in such a way is quite compromized. Using elimination when $d = 2, 3$ one could provide the required invariant groups where the solutions $g = (g_{i,j})$ are given by functions of the coefficients $a_\alpha^{(k)}$ of the f_k's. A last step would be to decide if the invariant group is compact/semisimple/isotropic which could ask some additional efforts, this task is crucial in order to apply our main theorem.

Final comments. The main problem is to determine the ideal stabilizer H of a given projective variety $X = V(f_1, \dots, f_r)$. This consists in finding a subgroup H such that

$$K[x_1, \dots, x_n]^H = K[f_1, \dots, f_r].$$

This question is dual to the *Invariant theory*, indeed in invariant theory we fix a group G and we try to understand the ring of invariants $k[X]^G$ of the variety X. This theory has reached a good level of maturity, notably with the rise of the *geometric invariant theory* (G.I.T. [26]) and more recently with the theory of *prehomogeneous spaces* for which we hope that we could derive an analog of the work of Yukie ([38]) using our main theorem. In terms of category, the invariant theory is an attempt to understand the image of the functor

$$\begin{array}{rccc} F(X): & \mathbf{Grps} & \to & \mathbf{Rings} \\ & G & \to & k[X]^G. \end{array}$$

The so called *Inverse Invariant theory* consists the dual situation, namely the image of the following functor given a fixed projective variety

$$\begin{array}{rccc} F(X)^*: & \mathbf{Rings} & \to & \mathbf{Grps} \\ & A & \to & G \end{array}$$

where we define the functor $F(X)^*$ as follows $F(X)^*(A) = G$ if $A = k[X]^G$. As far as we know, this theory has been only developped for finite groups and in particular for linear groups over *finite fields*. The latter has been studied in detail by Neusel using tools from algebraic topology such as Steenrod operations. It should be very interesting to have such a theory for linear groups over fields in null characteristic and to have a criterion which ensures that the group obtained is reductive or/and noncompact. If we could have such theory it would open a large range of applications and more particularly for solving the diophantine inequalities in (1).

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References

[1] Bombieri, E., and W. Gubler: Heights in Diophantine geometry. - Cambridge Univ. Press, 2006.

[2] Borel, A.: Introduction aux groupes arithmetiques. - Hermann, Paris, 1969.

[3] Borel, A.: Linear algebraic groups. Second edition. - Grad. Texts in Math. 126, Springer-Verlag New-York Inc., 1991.

[4] Borel, A.: Values of indefinite quadratic forms at integral points and flows on spaces of lattices. - Bull. Amer. Math. Soc. 32:2, 1995, 184–204.

[5] Borel, A., and G. Prasad: Values of isotropic quadratic forms at S-integral points. - Compositio Math. 83, 1992, 347–372.

[6] Borel, A., and J. Tits: Homomorphismes "abstraits" de groupes algébriques simples. - Ann. of Math. (2) 97, 1973, 499–571.

[7] Dani, S. G.: Simultaneous diophantine approximation one quadratic forms and linear forms. - J. Mod. Dynam. 2, 2008, 129–138.

[8] Dani, S. G., and G. A. Margulis: Orbit closures of generic unipotent flows on homogeneous spaces of $\mathrm{SL}(3,\mathbb{R})$. - Math. Ann. 286, 1990, 101–128.

[9] Ghosh, A., A. Gorodnik, and A. Nevo: Diophantine approximation and automorphic spectrum. - Int. Math. Res. Not. IMRN 2013:21, 2013, 5002–5058.

[10] Ghosh, A., A. Gorodnik, and A. Nevo: Metric Diophantine approximation on homogeneous varities. - Compositio Math. 150, 2014, 1435–1456.

[11] Ghosh, A., A. Gorodnik, and A. Nevo: Diophantine approximation exponents in homogeneous varieties. - In: Recent Trends in Ergodic Theory and Dynamical Systems (Conference in honor of S. G. Dani), Contemp. Math. 631, 2015, 181–200.

[12] Ghosh, A., A. Gorodnik, and A. Nevo: Optimal density for values of generic polynomial maps. - Preprint, arxiv.1801.01027, 2018.

[13] Gorodnik, A.: Values of pairs involving one quadratic form and one linear form at S-integral points. - Trans. Amer. Math. Soc. 187, 2004, 200–217.

[14] Kleinbock, D., N. Shah, and A. Starkov: Dynamics of subgroup actions on homogeneous spaces of Lie groups and application to number theory. - In: Handbook of dynamical systems 1A (edited by B. Hasselblatt and A. Katok), Elsevier Science, 2002, 1–109.

[15] Kneser, M.: Schwache Approximation in algebrischen gruppen. - Colloque sur la théorie des groupes algébriques, CBRM Brussels, 1962, 41–52.

[16] Kneser, M.: Strong approximation. - In: 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 187–196.

[17] Kollar, J.: Sharp effective Nullstellensatz. - J. Amer. Math. Soc. 1:4, 1988, 963–975.

[18] Lam, T. Y.: Introduction to quadratic forms over fields. - Grad. Stud. Math. 67, Amer. Math. Soc., 2005.

[19] Lang, S.: Algebraic number theory. Second edition. - Grad. Texts in Math. 110, Springer-Verlag, New-York, 1994.

[20] Lazar, Y.: Values of pairs involving one quadratic form and one linear form at S-integral points. - J. Number Theory 187, 2017, 200–217.

[21] Margulis, G. A.: Indefinite quadratic forms and unipotent flows on homogeneous spaces. - Dynamical Systems and Ergodic Theory 23, Banach Center Publ., PWN-Polish Scientific Publ., Warsaw, 1989, 399–409.

[22] Margulis, G. A.: Discrete subgroups of semisimple Lie groups. - Springer Verlag, 1991.

[23] Margulis, G. A., and G. Tomanov: Invariant measures for actions of unipotent flows over local fields on homogeneous spaces. - Invent. Math. 116:1-3, 1994, 347–392.

[24] Matthews, C. R., L. N. Vaserstein, and B. Weisfeiler: Congruence properties of Zariski-dense subgroups I. - Proc. London Math. Soc. 48:3, 1984, 514–532.

[25] McFeat, R. B.: Geometry of numbers in adele spaces. - Dissertationes Math. (Rozprawy Mat.) 88, 1971, 49 pp.

[26] Mumford, D., J. Fogarty, and F. Kirwan: Geometric invariant theory. Third edition. - Ergeb. Math. Grenzgeb. (3) 34, Springer-Verlag, Berlin Heidelberg, 1994.

[27] Neusel, M. D.: Inverse invariant theory and Steenrod operations. - Mem. Amer. Math. Soc. 146:692, 2000.

[28] Nori, M. V.: On subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$. - Invent. Math. 88:2, 1987, 257–275.

[29] Oppenheim, A.: The minima of indefinite quaternary quadratic forms. - Proc. Nat. Acad. Sci. USA 15, 1929, 724–727.

[30] Platonov, V. P.: The problem of strong approximation and the Kneser–Tits conjecture. - Izv. Akad. Nauk SSSR Ser. Mat. 3, 1969, 1139–1147; Addendum, ibid. 4, 1970, 784–786.

[31] Platonov, V. P., and A. S. Rapinchuk: Algebraic groups and number theory. - Pure and Applied Mathematics 139, Academic Press, Inc., Boston, MA, 1994.

[32] Rapinchuk, A. S.: Strong approximation for algebraic groups. - In: Thin groups and superstrong approximation, Math. Sci. Res. Inst. Publ. 61, Cambridge Univ. Press, Cambridge, 2014, 269–298.

[33] Ratner, M.: On measure rigidity of unipotent subgroups of semisimple groups. - Acta Math. 165, 1990, 229–309.

[34] Ratner, M.: Raghunathan's conjectures for p-adic Lie groups. - Int. Math. Res. Not. IMRN 1993:5, 1993, 141–146.

[35] Sargent, O.: Density of values of linear maps on quadratic surfaces. - J. Number Theory 143, 2014, 363–384.

[36] Venkataramana, B. N.: Zariski dense subgroups of arithmetic groups. - J. Algebra 108, 1987, 325–339.

[37] Weisfeiler, B.: Strong approximation for Zariski-dense subgroups of semisimple groups. - Ann. of Math. (2) 120, 1984, 271–315.

[38] Yukie, A.: Prehomogeneous spaces and ergodic theory I. - Duke Math. J. 90:1, 1997, 123–147.

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