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A simple proof of reflexivity and separability of $N^{1,p}$ Sobolev spaces

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Abstract. We present an elementary proof of a well-known theorem of Cheeger which states that if a metric-measure space X supports a p-Poincaré inequality, then the $N^{1,p}(X)$ Sobolev space is reflexive and separable whenever $p \in (1,\infty)$. We also prove separability of the space when $p = 1$. Our proof is based on a straightforward construction of an equivalent norm on $N^{1,p}(X)$, $p \in [1,\infty)$, that is uniformly convex when $p \in (1,\infty)$. Finally, we explicitly construct a functional that is pointwise comparable to the minimal p-weak upper gradient, when $p \in (1,\infty)$.

Yksinkertainen todistus Sobolevin avaruuksien $N^{1,p}$ refleksiivisyydelle ja separoituvuudelle

Tiivistelmä. Esitämme alkeellisen todistuksen tunnetulle Cheegerin lauseelle, jonka mukaan p-Poincarén epäyhtälön toteuttavan metrisen mitta-avaruuden X Sobolevin avaruudet $N^{1,p}(X)$ ovat refleksiivisiä ja separoituvia kaikilla $p \in (1,\infty)$. Osoitamme separoituvuuden myös kun $p = 1$. Todistuksemme perustuu kaikilla $p \in [1,\infty)$ suoraviivaiseen tapaan rakentaa avaruudelle $N^{1,p}(X)$ yhtäpitävä normi, joka on tasaisesti konveksi, kun $p \in (1,\infty)$. Lopuksi rakennamme eksplisiittisesti funktionaalin, joka on pisteittäin verrannollinen minimaaliseen p-heikkoon ylägradienttiin, kun $p \in (1,\infty)$.

1. Introduction

Sobolev spaces on metric-measure spaces $M^{1,p}$ have been introduced in [10], and soon after, many other definitions followed. Independently, Cheeger [3] and Shanmugalingam [17] introduced notions of Sobolev spaces on metric-measure spaces based on the upper gradient of Heinonen and Koskela [13]. Their spaces are denoted by $H_{1,p}$ and $N^{1,p}$, respectively. While their definitions are different, it was observed by Shanmugalingam [17, Theorem 4.10], that the spaces $H_{1,p}$ and $N^{1,p}$ are isometrically isomorphic when $p > 1$.

Throughout the paper we assume that (X, d, μ) is a metric-measure space with a Borel regular doubling measure. In this setting, we define $N^{1,p}(X)$, $p \in [1,\infty)$, as the space of functions $u \in L^p(X)$ that have an upper gradient in $L^p(X)$. $N^{1,p}(X)$ is a Banach space with respect to the norm

$$\|u\|_{N^{1,p}(X)} := \Big(\|u\|^p_{L^p(X)} + \inf_g \|g\|^p_{L^p(X)}\Big)^{1/p}.$$

Here, the infimum is taken over all upper gradients g of u. See Section 2 for additional details regarding our setting and the space $N^{1,p}(X)$.

If there are no rectifiable curves in X, then $g = 0$ is an upper gradient of any function, and hence, $N^{1,p}(X) = L^p(X)$ isometrically. Therefore, in order to have a

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rich theory, we need a large family of rectifiable curves in X, which is guaranteed when the space supports a p-Poincaré inequality. Recall that the space (X, d, μ) supports a p-Poincaré inequality, $p \in [1, \infty)$, if the measure μ is doubling and there are constants $c_{\mathrm{PI}} > 0$ and $\lambda \geq 1$ such that

$$\fint_B |u - u_B| \, d\mu \leq c_{\mathrm{PI}} \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}$$

for all balls $B \subseteq X$, for all Borel functions $u \in L^1_{\mathrm{loc}}(X)$, and all upper gradients g of u. Here, and in what follows, the barred integral stands for the integral average and $u_B := \fint_B u \, d\mu$ is the integral average of u over the ball B. Also, $\operatorname{diam}(B)$ denotes the diameter of B, and λB stands for a ball concentric with B and radius λ times that of B.

Cheeger [3] proved that if the space (X, d, μ) supports a p-Poincaré inequality for some $p \in (1, \infty)$, then the space $N^{1,p}(X)$ is reflexive. In fact, he proved in this setting that the space $N^{1,p}(X)$ can be equipped with an equivalent uniformly convex norm, from which reflexivity follows. His proof of reflexivity is, however, very difficult and based on the celebrated construction of a measurable differentiable structure. Later Keith [15] proved the existence of a measurable differentiable structure and hence, reflexivity of $N^{1,p}(X)$, $p \in (1, \infty)$, under the so-called Lip-lip condition. As demonstrated by Heinonen [12, Section 12.5], for general metric-measure spaces, $N^{1,p}(X)$, $p \in (1, \infty)$, need not be reflexive.

A different approach to reflexivity was provided by Ambrosio, Colombo and Di Marino [1]. They proved reflexivity of $N^{1,p}(X)$, $p \in (1, \infty)$, under the assumptions that the metric space X is metric-doubling, complete, and the measure μ is finite on balls. They did not, however, assume that the space supports a p-Poincaré inequality. In fact, they proved reflexivity of a Sobolev type space $W^{1,p}(X)$ whose definition is based on a notion of p-relaxed slope, and they proved that the space is equivalent to $N^{1,p}(X)$ under the given assumptions. Their proof is actually quite difficult since it involves methods of mass-transportation, gradient flows, Γ-convergence, and Christ dyadic cubes, just to name a few. A simplification of this proof of reflexivity in the case when the space supports a p-Poincaré inequality was obtained by Durand-Cartagena and Shanmugalingam [5]; their proof follows arguments from [1] and, in particular, they use Γ-convergence and Christ dyadic cubes to construct an equivalent norm on $N^{1,p}(X)$ that is uniformly convex.

Recently, Eriksson-Bique and Soultanis [7], proved reflexivity of $N^{1,p}(X)$, $p \in (1, \infty)$, under the assumption that the space has finite Hausdorff dimension. Their proof is quite difficult too.

The purpose of this paper is to provide a further simplification of the proof of reflexivity of $N^{1,p}(X)$ when $p \in (1, \infty)$ and the space supports a p-Poincaré inequality. In fact, we provide an explicit construction of an equivalent norm on $N^{1,p}(X)$, $p \in [1, \infty)$, which is uniformly convex when $p \in (1, \infty)$. Our arguments are based on ideas from [1] and also [5], but our construction of the uniformly convex norm is direct and it does not require Γ-convergence nor Christ cubes.

A brief outline of our construction is below. All details can be found in Section 3.

For each $k \in \mathbb{Z}$, we select a covering of X by balls $\{B_i^k\}_i$, of radii 2^{-k}, such that the balls in the family $\left\{\frac{1}{5} B_i^k\right\}_i$ are pairwise disjoint. We say that balls B_i^k and B_j^k are neighbors if $\operatorname{dist}(B_i^k, B_j^k) < 2^{-k}$, and we denote neighbors by $B_i^k \sim B_j^k$. It follows from the doubling condition that the number of neighbors of a given ball B_i^k is bounded by some constant $N \in \mathbb{N}$ that is independent of k.

For each $x \in X$, there is a smallest index i such that $x \in B_i^k$, and we write $B^k[x] := B_i^k$. Then for $p \in [1, \infty)$ and $u \in L^1_{\mathrm{loc}}(X)$ we define

$$|T_k u(x)|_p := 2^k \left(\sum_{j: B_j^k \sim B^k[x]} |u_{B^k[x]} - u_{B_j^k}|^p \right)^{1/p},$$

where the sum is taken over all neighbors of $B^k[x] = B_i^k$, and we set $|T_k u(x)| := |T_k u(x)|_1$. Finally, we equip $N^{1,p}(X)$, $p \in [1, \infty)$ with a new norm,

$$\|u\|_{1,p}^* := \left(\|u\|_{L^p(X)}^p + \limsup_{k \to \infty} \big\| |T_k u|_p \big\|_{L^p(X)}^p \right)^{1/p}.$$

The main result of the paper reads as follows.

Theorem 1.1. *Suppose that the space (X, d, μ) supports a p-Poincaré inequality for some $p \in [1, \infty)$. Then $\| \cdot \|_{1,p}^*$ is an equivalent norm on $N^{1,p}(X)$. Moreover, if $p \in (1, \infty)$, then the space $N^{1,p}(X)$ with the equivalent norm $\| \cdot \|_{1,p}^*$ is uniformly convex and hence, the space $N^{1,p}(X)$ is reflexive.*

The notion of uniform convexity is recalled in Section 2.5.

The construction of the norm $\| \cdot \|_{1,p}^*$ is different from, but related to, the constructions given in [1, 5]. Recall that their constructions were less direct, as they required Γ-convergence and Christ cubes. The equivalence of the norms when $p = 1$ is, however, new. As a corollary, we also prove

Theorem 1.2. *Suppose that the space (X, d, μ) supports a p-Poincaré inequality for some $p \in [1, \infty)$. Then the space $N^{1,p}(X)$ is separable.*

It is well known that separability can be deduced from reflexivity when $p \in (1, \infty)$, see [3], but separability in the case $p = 1$ seems to be new.

It follows from the proof of Theorem 1.1 (more specifically, Proposition 3.7) that if $p \in [1, \infty)$, then there is $C \geq 1$ such that

$$C^{-1} \|g_u\|_{L^p(X)} \leq \limsup_{k \to \infty} \big\| |T_k u|_p \big\|_{L^p(X)} \leq C \|g_u\|_{L^p(X)},$$

where g_u is the minimal p-weak upper gradient of u. The next result shows not only a comparison of norms, but a pointwise comparison under the additional assumptions that X is complete and $p > 1$.

Theorem 1.3. *Suppose that the space (X, d, μ) is complete and supports a p-Poincaré inequality for some $p \in (1, \infty)$. Then there exists a constant $C \geq 1$ such that for every $u \in N^{1,p}(X)$,*

$$C^{-1} g_u(x) \leq \limsup_{k \to \infty} |T_k u(x)| \leq C g_u(x) \quad \text{for } \mu\text{-a.e. } x \in X,$$

where $g_u \in L^p(X)$ denotes the minimal p-weak upper gradient of u.

Remark 1.4. Note that we could replace $|T_k u(x)|$ in Theorem 1.3 by $|T_k u(x)|_p$, because the number of neighbors is bounded by N and all norms in $\mathbb{R}^N$ are equivalent. However, in Theorem 1.1 we have to work with $|T_k u|_p$ in order to guarantee uniform convexity of the norm.

The paper is structured as follows. In Section 2 we fix notation used in the paper, recall basic definitions, and state known results that will be used in the subsequent sections. In Section 3 we carefully explain the statement of the main result, Theorem 1.1, and we prove it. In Section 4 we prove Theorem 1.2 and finally, in Section 5 we prove Theorem 1.3.

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2. Preliminaries

2.1. Notational conventions. Let $\mathbb{Z}$ denote all integers and $\mathbb{N}$ all (strictly) positive integers. By C we denote a generic constant whose actual value may change from line to line. For nonnegative quantities, $L, R \geq 0$, the notation $L \lesssim R$ will be used to express that there exists a constant $C > 0$, perhaps dependent on other constants within the context, such that $L \leq CR$. If $L \lesssim R$ and simultaneously $R \lesssim L$, then we will simply write $L \approx R$ and say that the quantities L and R are *equivalent* (or *comparable*).

The characteristic function of a set E will be denoted by χ_E.

We assume that all function spaces are linear spaces over the field of real numbers.

We use a convention that the names "Theorem" and "Proposition" are reserved for new results, while well-known results and results of technical character are called "Lemma" or "Corollary".

2.2. Metric-measure spaces. A metric-measure space is a triplet (X, d, μ) where (X, d) is a metric space and μ is a Borel measure such that $0 < \mu(B) < \infty$ for every ball $B \subseteq X$. We will assume that μ is *Borel regular*, in the sense that every μ-measurable set is contained in a Borel set of equal measure. We will also assume that μ is *doubling*, i.e., there is a constant $C_d \geq 1$ such that $\mu(2B) \leq C_d\mu(B)$ for every ball $B \subseteq X$.

We will need the following version of the Lebesgue differentiation theorem.

Lemma 2.1. *Assume that μ is a Borel regular doubling measure on X and $u \in L^1_{\rm loc}(X)$. Then for μ-a.e. $x \in X$ the following is true. If $\{B_i\}_i$ is a sequence of balls such that $x \in B_i$ for all i and $\operatorname{diam}(B_i) \to 0$ as $i \to \infty$, then*

$$\lim_{i\to\infty} \fint_{B_i} u\, d\mu = u(x). \tag{1}$$

Equality (1) is satisfied whenever x is a Lebesgue point of u. This result is well known if μ is the Lebesgue measure in $\mathbb{R}^n$, but the standard proofs easily generalize to the case of metric-measure spaces equipped with a Borel regular doubling measure.

2.3. Integrating along curves in metric spaces and modulus of the path family. By a *curve* in X, we mean a continuous mapping $\gamma\colon [a, b] \to X$. Given a curve γ, the *image of* γ is denoted by $|\gamma| := \gamma([a, b])$ and $\ell(\gamma)$ stands for the *length of* γ. We will say that γ is *rectifiable* if $\ell(\gamma) < \infty$ and the family of all non-constant rectifiable curves in X will be denoted by $\Gamma(X)$. Every $\gamma \in \Gamma(X)$ admits a unique (orientation preserving) *arc-length parameterization* $\widetilde{\gamma}\colon [0, \ell(\gamma)] \to X$, and the arc-length parameterization is 1-Lipschitz; see, e.g., [11, Theorem 3.2]. Given a curve $\gamma \in \Gamma(X)$ and a Borel measurable function $\varrho\colon |\gamma| \to [0, \infty]$, we define

$$\int_\gamma \varrho\, ds := \int_0^{\ell(\gamma)} \varrho(\widetilde{\gamma}(t))\, dt.$$

We can naturally define the integral over a curve for a general function by considering the positive and negative parts of the function.

Let $\Gamma \subseteq \Gamma(X)$ and consider the collection $F(\Gamma)$ of all Borel functions $\varrho\colon X \to [0,\infty]$ satisfying

$$\int_\gamma \varrho\, ds \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

Then, for each $p \in [1,\infty)$, the *p-modulus of the family* Γ is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf_{\varrho\in F(\Gamma)} \int_X \varrho^p\, d\mu.$$

Note that Mod_p is an outer-measure on $\Gamma(X)$ and, in particular, it is countably subadditive; see, e.g., [11, Theorem 5.2]. A family of curves $\Gamma \subseteq \Gamma(X)$ is called *p-exceptional* if $\mathrm{Mod}_p(\Gamma) = 0$ and a statement is said to hold for Mod_p-*a.e.* curve $\gamma \in \Gamma(X)$ if the family of curves in $\Gamma(X)$ for which this statement does not hold is p-exceptional.

For the next result, see [2, Proposition 2.45]. It follows from Hölder's inequality and [2, Proposition 1.37(c)].

Lemma 2.2. *If a family of curves is p-exceptional for some $p \in (1,\infty)$, then it is q-exceptional for every $q \in [1,p]$.*

We will also need the following important result; see, e.g., [11, Theorem 5.7] and [2, Lemma 2.1].

Lemma 2.3. (Fuglede's lemma) *Let $p \in [1,\infty)$ and assume that $\{g_k\}_{k=1}^\infty$ is a sequence of Borel functions that converges in $L^p(X)$ to a Borel function $g \in L^p(X)$. Then, there is a subsequence $\{g_{k_i}\}_{i=1}^\infty$, such that for Mod_p-a.e. curve $\gamma \in \Gamma(X)$, one has*

$$\int_\gamma g_{k_i}\, ds \to \int_\gamma g\, ds \quad \text{and} \quad \int_\gamma |g_{k_i} - g|\, ds \to 0 \quad \text{as } i \to \infty,$$

where all of the integrals are well defined and finite.

2.4. Sobolev spaces in metric-measure spaces. A Borel measurable function $g\colon X \to [0,\infty]$ is called an *upper gradient* of a Borel measurable function $u\colon X \to [-\infty,\infty]$ if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds, \tag{2}$$

for every rectifiable curve $\gamma\colon [a,b] \to X$, with the convention that $|(\pm\infty) - (\pm\infty)| = \infty$. The function g shall be referred to as a *p-weak upper gradient* of u, $p \in [1,\infty)$, if (2) holds true for Mod_p-a.e. curve $\gamma \in \Gamma(X)$.

The next result shows that p-weak upper gradients can be approximated by upper gradients in the L^p norm; see e.g. [11, Lemma 6.3]

Lemma 2.4. *If g is a p-weak upper gradient of u which is finite μ-a.e., then for every $\varepsilon \in (0,\infty)$ there is an upper gradient g_ε of u such that*

$$g_\varepsilon \geq g \text{ pointwise everywhere in } X \quad \text{and} \quad \|g_\varepsilon - g\|_{L^p(X)} < \varepsilon.$$

For $p \in [1,\infty)$ we define $\widetilde{N}^{1,p}(X)$, to be the space of all Borel measurable functions $u : X \to [-\infty,\infty]$ for which

$$\|u\|_{N^{1,p}(X)} := \left(\|u\|^p_{L^p(X)} + \inf_g \|g\|^p_{L^p(X)}\right)^{1/p} < \infty, \tag{3}$$

where the infimum is taken over all upper gradients g of u. Equivalently, we can take the infimum over all p-weak upper gradients in (3) since every p-weak upper gradient can be approximated in L^p by upper gradients (Lemma 2.4).

The functional $\|\cdot\|_{N^{1,p}(X)}$ is a seminorm on $\widetilde{N}^{1,p}$ and a norm on $N^{1,p}(X) := \widetilde{N}^{1,p}(X)/{\sim}$, where the equivalence relation $u \sim v$ is given by $\|u - v\|_{N^{1,p}(X)} = 0$. Furthermore, the space $N^{1,p}(X)$ is complete and thus a Banach space, see [17, Theorem 3.7].

For the next result see, e.g., [11, Corollary 7.7].

Lemma 2.5. *If $u, v \in \widetilde{N}^{1,p}(X)$ and $u = v$ pointwise μ-a.e. in X, then $u \sim v$ i.e., the two functions define the same element in $N^{1,p}(X)$.*

For $p \in [1,\infty)$, every $u \in N^{1,p}(X)$ has a *minimal p-weak upper gradient* $g_u \in L^p(X)$ in the sense that if $g \in L^p(X)$ is another p-weak upper gradient of u, then $g \geq g_u$ pointwise μ-a.e. in X, see, e.g., [11, Theorem 7.16]. Hence, the infimum in (3) is attained with g_u, which is given uniquely up to pointwise a.e. equality.

Recall that the *pointwise lower Lipschitz-constant* of a function $\eta\colon X \to \mathbb{R}$ is given by

$$\operatorname{lip}\eta(x) := \liminf_{r\to 0^+} \sup_{y\in B(x,r)} \frac{|\eta(x)-\eta(y)|}{r}, \quad x \in X. \tag{4}$$

For the next lemma, see, e.g., [11, Lemma 6.7] or [14, Lemma 6.2.6].

Lemma 2.6. *$\operatorname{lip}\eta$ is an upper gradient of any Lipschitz continuous function η on a metric space.*

Lemma 2.7. *Fix $p \in [1,\infty)$ and suppose that $u \in N^{1,p}(X)$ and $\eta\colon X \to \mathbb{R}$ is a bounded Lipschitz function. Then $\eta u \in N^{1,p}(X)$ and the function $h := |\eta| g_u + |u| \operatorname{lip}\eta$ is a p-weak upper gradient for ηu, where $g_u \in L^p(X)$ is the minimal p-weak upper gradient of u.*

Proof. The Leibniz rule for p-weak upper gradients, [14, Lemma 6.3.28], the fact that functions in $N^{1,p}(X)$ are absolutely continuous on Mod_p-a.e. curve, [11, Lemma 7.6], and Lemma 2.6 imply that the function $h := |\eta| g_u + |u| \operatorname{lip}\eta$ is a p-weak upper gradient for ηu. Since $\eta u \in L^p(X)$ and $h \in L^p(X)$, it follows that $\eta u \in N^{1,p}(X)$. □

We say that (X, d, μ) supports a *p-Poincaré inequality*, $p \in [1,\infty)$, if there exist constants $c_{\mathrm{PI}} > 0$ and $\lambda \geq 1$ such that

$$\fint_B |u - u_B|\, d\mu \leq c_{\mathrm{PI}} \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p} \tag{5}$$

for all balls $B \subseteq X$, all Borel functions $u \in L^1_{\mathrm{loc}}(X)$, and all upper gradients g of u. Recall that we always assume that μ is a Borel regular doubling measure in this setting. In this situation we say that the space supports a p-Poincaré inequality with constants c_{PI} and λ.

The following lemma is an immediate consequence of Lemma 2.4.

Lemma 2.8. *Suppose that X supports a p-Poincaré inequality for some $p \in [1,\infty)$. If $u \in L^1_{\mathrm{loc}}(X)$ is Borel, then*

$$\fint_B |u - u_B|\, d\mu \leq c_{\mathrm{PI}} \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, d\mu \right)^{1/p}, \tag{6}$$

for all balls $B \subseteq X$ and all p-weak upper gradients g of u that are finite μ-a.e.

2.5. Uniformly convex spaces. We begin with a definition due to Clarkson [4]. We say that a normed space $(Z, \|\cdot\|)$ is *uniformly convex* if for every $\varepsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ with the property that $\|x+y\| \le 2(1-\delta)$ whenever $x, y \in Z$ satisfy $\|x\| = \|y\| = 1$, and $\|x-y\| > \varepsilon$.

From a geometric point of view, uniform convexity implies that the boundary of the unit ball does not contain any segments and that the unit ball is, in a sense, uniformly "round".

The next result is well known, but it is not easy to find a proof in the literature.

Lemma 2.9. *A normed space $(Z, \|\cdot\|)$ is uniformly convex if and only if for every $\varepsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ with the property that $\|x+y\| \le 2(1-\delta)$ whenever $x, y \in Z$ satisfy $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| > \varepsilon$.*

Proof. One direction is clear. To see the other, fix $\varepsilon \in (0,\infty)$ and suppose that $x, y \in Z$ satisfy $\|x\|, \|y\| \le 1$ and $\|x-y\| > \varepsilon$. Since Z is uniformly convex, there is $\tilde{\delta} \in (0,\infty)$ associated to the choice of $\varepsilon/3$. Let $\delta := \min\{\varepsilon/6, \tilde{\delta}/3\}$. If either $\|x\| \le 1-2\delta$ or $\|y\| \le 1-2\delta$, then $\|x+y\| \le 2(1-\delta)$. If $\|x\|, \|y\| > 1-2\delta$, then $\tilde{x} := x/\|x\|$ and $\tilde{y} := y/\|y\|$ satisfy $\|x-\tilde{x}\|, \|y-\tilde{y}\| < 2\delta$, and hence, $\|\tilde{x}-\tilde{y}\| > \varepsilon - 4\delta \ge \varepsilon/3$. Since $\|\tilde{x}\| = \|\tilde{y}\| = 1$, uniform convexity yields $\|\tilde{x}+\tilde{y}\| \le 2(1-\tilde{\delta})$ and hence, $\|x+y\| \le \|x-\tilde{x}\| + \|\tilde{x}+\tilde{y}\| + \|y-\tilde{y}\| \le 2(1-\delta)$. □

A clever proof of the next result that avoids the use of Clarkson's inequalities can be found in [14, Proposition 2.4.19].

Lemma 2.10. (Clarkson) *$L^p(X)$ is uniformly convex for $p \in (1,\infty)$.*

For a proof of the following theorem, see, e.g., [14, Theorem 2.4.9].

Lemma 2.11. (Milman–Pettis' theorem) *Every uniformly convex Banach space is reflexive.*

By ℓ^p_M we will denote $\mathbb{R}^M$ with the norm $|x|_p := \big(\sum_{j=1}^M |x_j|^p\big)^{1/p}$, where $x = (x_1, \dots, x_M)$, and so $L^p(X, \ell^p_M)$ is a Banach space equipped with the norm

$$\Phi(f) := \left(\sum_{j=1}^M \|f_j\|^p_{L^p(X)}\right)^{1/p}, \quad \text{where} \quad f = (f_1, \dots, f_M). \tag{7}$$

Corollary 2.12. *If $p \in (1,\infty)$ and $M \in \mathbb{N}$, then $L^p(X, \ell^p_M)$ is uniformly convex.*

Proof. Since $L^p(X, \ell^p_M)$ is isometric to $L^p(X_M)$, where

$$X_M = X \sqcup X \sqcup \ldots \sqcup X = X \times \{1, 2, \dots, M\}$$

is the disjoint union of M copies of the measure space (X, μ), the result follows from Lemma 2.10. □

2.6. Dunford–Pettis theorem. Recall that a family of μ-measurable functions $\mathcal{F}$ is said to be *equi-integrable* if for every $\varepsilon \in (0,\infty)$ there exists $\delta \in (0,\infty)$ such that for every μ-measurable set $S \subseteq X$ with $\mu(S) < \delta$ we have

$$\sup_{f \in \mathcal{F}} \int_S |f| \, d\mu < \varepsilon. \tag{8}$$

The proof for the following version of the Dunford–Pettis theorem can be found in, e.g., [8, Theorem 2.54].

Lemma 2.13. (Dunford–Pettis' theorem) *Let $\mathcal{F} \subseteq L^1(X)$. Then every sequence in $\mathcal{F}$ has a subsequence that is weakly convergent in $L^1(X)$ if and only if the following two conditions are satisfied:*

(a) *$\mathcal{F}$ is bounded in $L^1(X)$ and equi-integrable;*
(b) *for every $\varepsilon \in (0, \infty)$ one can find a μ-measurable set $E \subseteq X$ such that $\mu(E) < \infty$ and*

$$\sup_{f\in\mathcal{F}} \int_{X\setminus E} |f|\, d\mu < \varepsilon. \tag{9}$$

Remark 2.14. Observe that whenever $\mu(X) < \infty$, condition (b) is trivially satisfied by setting $E = X$, which is why it is omitted in literature that discusses the Dunford–Pettis theorem over spaces of finite measure.

3. The main result

In this section we will carefully record all of the notation and technical lemmata used in the poof of the main result, Theorem 1.1, and then we will prove it. The reader is reminded that we are always assuming (X, d, μ) is a metric-measure space, where μ is a Borel regular doubling measure. However, unless explicitly stated, we do not assume that the space supports a p-Poincaré inequality.

3.1. Notation. For each $k \in \mathbb{Z}$, let $\{B_i\}_{i=1}^{M(k)}$, $M(k) \in \mathbb{N} \cup \{\infty\}$, be a covering of X by balls of radius 2^{-k} such that the balls in the family $\{\frac{1}{5}B_i\}_{i=1}^{M(k)}$ are pairwise disjoint. The existence of such coverings follows from the familiar $5r$-covering lemma.

The doubling property of the measure μ implies that for each fixed $\theta \in [1, \infty)$, the family $\{\theta B_i\}_{i=1}^{M(k)}$ of enlarged balls has bounded overlapping, in the sense that there exists a constant $C_0 \in [1, \infty)$ such that $\sum_i \chi_{\theta B_i}(x) \le C_0$ for every $x \in X$. Note that C_0 depends only on θ and C_d (the doubling constant of μ). In particular, C_0 is independent of k.

For each $k \in \mathbb{Z}$, we have a different family of balls (referred to as *balls of generation k*) and we will write $B_i^k := B_i$ if we wish to stress for which $k \in \mathbb{Z}$ the family was constructed.

We say that balls B_i^k and B_j^k are *neighbors* if $\operatorname{dist}(B_i^k, B_j^k) < 2^{-k}$, and we will write $B_i^k \sim B_j^k$ in this case. Note that there exists $N \in \mathbb{N}$, such that each ball has at most N neighbors, where N depends only on the doubling constant of μ and, in particular, is independent of k.

If $B_{i,1}, \dots, B_{i,n_i}$, $n_i < N$ are *all* of the neighbors of B_i then we set

$$B_{i,n_i+1}, \dots, B_{i,N} := B_i.$$

That is, we set the last $N - n_i$ balls in the sequence $\{B_{i,j}\}_{j=1}^{N}$ to be identical copies of B_i. While this construction is somewhat formal, for reasons that will be clear later, we need to have the same number of balls "around" each of the B_i's.

Let $A_1 := B_1$ and $A_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1})$ for each $i \ge 2$. Then $X = \bigcup_{i=1}^{M(k)} A_i$, and so, in particular, $\{A_i\}_{i=1}^{M(k)}$ is a partition of X into pairwise disjoint sets. For each $x \in X$, there is a unique i such that $x \in A_i$. In other words, i is the smallest index such that $x \in B_i$. As such, we define $B[x] := B_i$ and set $B[x, j] := B_{i,j}$ for $j \in [1, N]$. In particular, $B[x, j] = B[x]$ if $j \in (n_i, N]$.

For $u \in L^1_{\mathrm{loc}}(X)$ and $k \in \mathbb{Z}$, we define

$$S_k u := \sum_{i=1}^{M(k)} u_{B_i}\, \chi_{A_i}, \tag{10}$$

and note that $S_k u(x) = u_{B[x]}$ for each $x \in X$. According to Lebesgue's differentiation theorem (Lemma 2.1), $S_k u \to u$ pointwise μ-a.e. in X as $k \to \infty$.

For $u \in L^1_{\mathrm{loc}}(X)$ and $k \in \mathbb{Z}$, we also define

$$T_k u(x) := 2^k \big[u_{B[x]} - u_{B[x,1]}, \ldots, u_{B[x]} - u_{B[x,N]}\big] \in \mathbb{R}^N,$$

for $x \in X$, or equivalently,

$$T_k u := 2^k \sum_{i=1}^{M(k)} \big[u_{B_i} - u_{B_{i,1}}, \ldots, u_{B_i} - u_{B_{i,N}}\big]\chi_{A_i}.$$

Observe that if $x \in A_i$ and $n_i < N$ (the actual number of neighbors of $B[x] = B_i$), then the vector $T_k u(x) \in \mathbb{R}^N$ has zeros in the last $N - n_i$ components, i.e.,

$$T_k u(x) := 2^k \big[u_{B_i} - u_{B_{i,1}}, \ldots, u_{B_i} - u_{B_{i,n_i}}, 0, \ldots, 0\big].$$

Equipping $\mathbb{R}^N$ with the ℓ^p_N norm, $p \in [1, \infty)$, we have

$$\begin{aligned} |T_k u(x)|_p &= 2^k \left(\sum_{j=1}^{N} |u_{B[x]} - u_{B[x,j]}|^p\right)^{1/p} \\ &= 2^k \sum_{i=1}^{M(k)} \left(\sum_{j=1}^{N} |u_{B_i} - u_{B_{i,j}}|^p\right)^{1/p} \chi_{A_i}(x). \end{aligned} \tag{11}$$

In particular,

$$|T_k u(x)| := |T_k u(x)|_1 = 2^k \sum_{j=1}^{N} |u_{B[x]} - u_{B[x,j]}| = 2^k \sum_{i=1}^{M(k)} \sum_{j=1}^{N} |u_{B_i} - u_{B_{i,j}}|\, \chi_{A_i}(x).$$

Clearly the norms $|\cdot|$ and $|\cdot|_p$ are equivalent on $\mathbb{R}^N$, but we will have to work with the norm $|\cdot|_p$ in order to prove uniform convexity of $N^{1,p}$. Note that when $\mathbb{R}^N$ is equipped with $|\cdot|_p$, the $L^p(X, \ell^p_N)$ norm of $T_k u$ is

$$\big\||T_k u|_p\big\|_{L^p(X)} = 2^k \left(\sum_{i=1}^{M(k)} \sum_{j=1}^{N} \big(|u_{B_i} - u_{B_{i,j}}|\mu(A_i)^{1/p}\big)^p\right)^{1/p},$$

where we have used the fact that the A_i's are pairwise disjoint.

Fix $p \in [1, \infty)$ and define

$$\|u\|^*_{1,p} := \left(\|u\|^p_{L^p(X)} + \limsup_{k\to\infty} \big\||T_k u|_p\big\|^p_{L^p(X)}\right)^{1/p} \quad \text{for each } u \in N^{1,p}(X). \tag{12}$$

Observe that components of $T_k u(x)$ are averaged difference quotients of u in all possible directions, i.e., over all balls that are neighbours of $B[x]$. As we shall see, in some sense $|T_k u|$ (or $|T_k u|_p$) is an approximation of the minimal p-weak upper gradient of u (see Lemma 3.4, Proposition 3.7, and Theorem 5.1).

3.2. Auxiliary results. In this subsection we will prove technical lemmata which will be needed in the proof of the main result (Theorem 1.1).

Lemma 3.1. *Suppose that the space supports a p-Poincaré inequality* (5) *for some $p \in [1, \infty)$. Then there exists a constant $C = C(p, C_d, c_{\mathrm{PI}}) > 0$ such that if $u \in L^1_{\mathrm{loc}}(X)$ is Borel measurable and g is a p-weak upper gradient of u that is finite μ-a.e., then for each $k \in \mathbb{Z}$, we have*

$$|T_k u(x)|_p \le C \left(\fint_{5\lambda B[x]} g^p \, d\mu \right)^{1/p} \quad \text{for all } x \in X. \tag{13}$$

Therefore, there is a constant $C' = C'(p, C_d, C_{\mathrm{PI}}, \lambda) > 0$ such that

$$\big\| |T_k u|_p \big\|_{L^p(X)} \le C' \|g\|_{L^p(X)}. \tag{14}$$

Consequently, if $u \in N^{1,p}(X)$, then the sequence $\{T_k u\}_{k \in \mathbb{Z}}$ is bounded in $L^p(X, \ell^p_N)$.

Proof. Fix $k \in \mathbb{Z}$ and $x \in X$, along with $j \in [1, N]$. Since $B[x]$ and $B[x, j]$ are neighbors (by definition) we have that $B[x, j] \subseteq 5B[x] \subseteq 10B[x, j]$. Therefore, the doubling condition of μ implies that $\mu(B[x, j]) \approx \mu(5B[x])$. Applying Lemma 2.8 to the pair (u, g) we can estimate

$$\begin{aligned} |u_{B[x]} - u_{B[x,j]}| &\le |u_{B[x]} - u_{5B[x]}| + |u_{5B[x]} - u_{B[x,j]}| \\ &\lesssim \fint_{5B[x]} |u - u_{5B[x]}| \, d\mu \le C 2^{-k} \left(\fint_{5\lambda B[x]} g^p \, d\mu \right)^{1/p}, \end{aligned}$$

where $C \in (0, \infty)$ depends only on C_d and c_{PI}. From the formula for $|T_k u|_p$, we have

$$|T_k u(x)|_p \le 2^k \left(\sum_{j=1}^{N} C^p 2^{-kp} \fint_{5\lambda B[x]} g^p \, d\mu \right)^{1/p} = C \cdot N^{1/p} \left(\fint_{5\lambda B[x]} g^p \, d\mu \right)^{1/p}, \tag{15}$$

where C and N only depend on C_d and c_{PI}. This proves (13).

Turning our attention to proving (14), observe that estimate (15) is equivalent to

$$|T_k u(x)|_p \le C \cdot N^{1/p} \left(\sum_{i=1}^{M(k)} \chi_{A_i}(x) \fint_{5\lambda B_i} g^p \, d\mu \right)^{1/p}, \tag{16}$$

because the right hand sides of (15) and (16) are equal. The bounded overlapping of the family of enlarged balls $\{5\lambda B_i\}_{i=1}^{M(k)}$ and the fact that $A_i \subseteq B_i$ together yield

$$\big\| |T_k u|_p \big\|^p_{L^p(X)} \le C^p N \sum_{i=1}^{M(k)} \mu(A_i) \left(\fint_{5\lambda B_i} g^p \, d\mu \right) \le C^p N \sum_{i=1}^{M(k)} \int_{5\lambda B_i} g^p \, d\mu \le C' \|g\|^p_{L^p(X)},$$

where $C' = C'(p, C_d, C_{\mathrm{PI}}, \lambda)$.

Finally, if $u \in N^{1,p}(X)$ then (14) applied with $g = g_u \in L^p(X)$ proves boundedness of $\{T_k u\}_{k \in \mathbb{Z}}$ in $L^p(X, \ell^p_N)$. □

The next result follows immediately from the definition of $\|\cdot\|^*_{1,p}$ in (12), Lemma 2.5, and Lemma 3.1.

Corollary 3.2. *Suppose that the space supports a p-Poincaré inequality for some $p \in [1, \infty)$. Then $\|\cdot\|^*_{1,p} \colon N^{1,p}(X) \to [0, \infty)$ as in* (12) *is a well-defined norm on $N^{1,p}(X)$ and there exists $C \in (0, \infty)$ satisfying*

$$\|u\|^*_{1,p} \le C \|u\|_{N^{1,p}(X)} \quad \text{for all } u \in N^{1,p}(X).$$

The reader is reminded of the definition of $S_k u$ in (10).

Lemma 3.3. *Let $u \in N^{1,p}(X)$ with $p \in [1,\infty)$ and assume that $\{h_k\}_{k=1}^{\infty}$ is a sequence of nonnegative Borel functions in $L^p(X)$ such that*

$$|S_k u(x) - S_k u(y)| \le \int_\gamma h_k \, ds, \tag{17}$$

whenever $k \in \mathbb{N}$, $x, y \in X$ satisfy $d(x,y) \ge 2^{-k}$, and γ is a rectifiable curve connecting x and y. If $\{h_k\}_{k=1}^{\infty}$ contains a subsequence that converges weakly in $L^p(X)$ to some nonnegative Borel function $h \in L^p(X)$, then h is a p-weak upper gradient of u and hence, $h \ge g_u$ pointwise μ-a.e. in X, where $g_u \in L^p(X)$ is the minimal p-weak upper gradient of u.

Proof. Without loss of generality, we can assume that $\{h_k\}_{k=1}^{\infty}$ converges weakly in $L^p(X)$ to some Borel function $h \in L^p(X)$. Then by Mazur's lemma (see, e.g., [14, p. 19]), there exists a sequence

$$g_k := \sum_{\ell=k}^{L(k)} \alpha_{k,\ell} h_\ell \to h \quad \text{in } L^p(X) \text{ as } k \to \infty,$$

where $\alpha_{k,\ell} \ge 0$ and $\sum_{\ell=k}^{L(k)} \alpha_{k,\ell} = 1$ for each $k \in \mathbb{N}$ (with $L(k) \in \mathbb{N}$). By further passing to a subsequence, if necessary, we can assume that $g_k \to h$ pointwise μ-a.e. in X as $k \to \infty$. Consider the corresponding family of convex combinations of $S_k u$, $v_k := \sum_{\ell=k}^{L(k)} \alpha_{k,\ell} S_\ell u$. Since $S_k u \to u$ pointwise μ-a.e. in X by Lebesgue's differentiation theorem (Lemma 2.1), we have that $v_k \to u$ pointwise μ-a.e. in X, as well.

Clearly, if $d(x,y) \ge 2^{-k}$ for some $k \in \mathbb{N}$ and $\gamma \in \Gamma(X)$ connects x and y, then by (17) we have

$$|v_k(x) - v_k(y)| \le \int_\gamma g_k \, ds. \tag{18}$$

Define $\tilde{u}\colon X \to [-\infty,\infty]$ by setting $\tilde{u}(x) := \limsup_{k\to\infty} v_k(x)$ for every $x \in X$ and note that $\tilde{u} = u$ pointwise μ-a.e. in X. We will prove that $\tilde{u}$ is finite everywhere on the image $|\gamma|$ for Mod_p-a.e. curve $\gamma \in \Gamma(X)$. To this end, by Fuglede's lemma (Lemma 2.3), there is a set $\Gamma_1 \subseteq \Gamma(X)$ with $\operatorname{Mod}_p(\Gamma_1) = 0$ and a subsequence of $\{g_k\}_{k=1}^{\infty}$ (also denoted by $\{g_k\}_{k=1}^{\infty}$) such that

$$\int_\gamma g_k \, ds \to \int_\gamma h \, ds \in \mathbb{R} \quad \text{as } k \to \infty, \tag{19}$$

for every curve $\gamma \in \Gamma(X) \setminus \Gamma_1$. Next, let E be the set of all $x \in X$ for which the convergence $v_k(x) \to u(x) \in \mathbb{R}$ does not hold, and set

$$\Gamma_2 := \big\{\gamma \in \Gamma(X) \colon |\gamma| \subseteq E\big\}.$$

Note that $E \subseteq X$ is μ-measurable and $\mu(E) = 0$, which implies that $\|\infty \cdot \chi_E\|_{L^p(X)} = 0$. This, together with the observation that $\infty \cdot \chi_E \in F(\Gamma_2)$, immediately gives $\operatorname{Mod}_p(\Gamma_2) = 0$ and hence, $\operatorname{Mod}_p(\Gamma_1 \cup \Gamma_2) = 0$.

Now fix a curve $\gamma \in \Gamma(X) \setminus (\Gamma_1 \cup \Gamma_2)$ and let $\widetilde{\gamma}$ be the arc-length parameterization of γ. We claim that the sequence $\{v_k(\widetilde{\gamma}(s))\}_{k=1}^{\infty}$ of real numbers is bounded for every $s \in [0, \ell(\gamma)]$. Let $s \in [0, \ell(\gamma)]$ and note that since $\gamma \notin \Gamma_2$, there is a point $t \in [0, \ell(\gamma)]$ such that $\widetilde{\gamma}(t) \notin E$. By definition of the set E, we have that $v_k(\widetilde{\gamma}(t)) \to u(\widetilde{\gamma}(t)) \in \mathbb{R}$. In particular, $\{v_k(\widetilde{\gamma}(t))\}_{k=1}^{\infty}$ is a bounded sequence. To proceed, it is enough to consider the scenario when $s \le t$ as the other case is handled similarly. If $\widetilde{\gamma}(s) = \widetilde{\gamma}(t)$ then $\{v_k(\widetilde{\gamma}(s))\}_{k=1}^{\infty}$ is bounded by the choice of t. If, on the other hand, $\widetilde{\gamma}(s) \ne \widetilde{\gamma}(t)$

then we have $d(\widetilde{\gamma}(t),\widetilde{\gamma}(s)) \geq 2^{-k}$ for all sufficiently large $k \in \mathbb{N}$ and so, by appealing to (18) we can write

$$\begin{aligned}\big|v_k(\widetilde{\gamma}(s))\big| &\leq \big|v_k(\widetilde{\gamma}(s)) - v_k(\widetilde{\gamma}(t))\big| + \big|v_k(\widetilde{\gamma}(t))\big| \\ &\leq \int_s^t g_k(\widetilde{\gamma}(\tau))d\tau + \big|v_k(\widetilde{\gamma}(t))\big| \leq \int_\gamma g_k\, ds + \big|v_k(\widetilde{\gamma}(t))\big|.\end{aligned} \tag{20}$$

Since $\gamma \notin \Gamma_1$, we have that $\int_\gamma g_k\, ds$ converges to the finite number (19), as $k \to \infty$ and hence, is bounded. Therefore, the right-hand side of (20) is bounded by a finite constant that is independent of k, and it follows that $\{v_k(\widetilde{\gamma}(s))\}_{k=1}^\infty$ is a bounded sequence for each fixed $s \in [0, \ell(\gamma)]$. Consequently, $\tilde{u}$ is finite on the image $|\gamma|$ whenever $\gamma \in \Gamma(X) \setminus (\Gamma_1 \cup \Gamma_2)$.

Moving on, we claim next that h is a p-weak upper gradient of $\tilde{u}$. Fix $\gamma \in \Gamma(X) \setminus (\Gamma_1 \cup \Gamma_2)$ and let $x, y \in X$ be the end-points of γ. If $x = y$, then the inequality $|\tilde{u}(x) - \tilde{u}(y)| \leq \int_\gamma h\, ds$ is trivially satisfied, since $\tilde{u}(x) = \tilde{u}(y) \in \mathbb{R}$. If $x \neq y$, then $d(x, y) \geq 2^{-k}$ for all $k \in \mathbb{N}$, large enough, and so (18) is satisfied. Since $\gamma \notin \Gamma_1 \cup \Gamma_2$, we have that (19) holds and $\tilde{u}(x), \tilde{u}(y) \in \mathbb{R}$. As such, we can estimate

$$|\tilde{u}(x) - \tilde{u}(y)| \leq \limsup_{k\to\infty} |v_k(x) - v_k(y)| \leq \limsup_{k\to\infty} \int_\gamma g_k\, ds = \int_\gamma h\, ds\,.$$

Therefore, $h \in L^p(X)$ is a p-weak upper gradient of $\tilde{u}$, and hence $\tilde{u} \in N^{1,p}(X)$. Since $u = \tilde{u}$ pointwise μ-a.e. in X and both u and $\tilde{u}$ belong to $N^{1,p}(X)$, the function h is also a p-weak upper gradient of u by Lemma 2.5. Therefore, $h \geq g_u$ pointwise μ-a.e. in X by the definition of a minimal p-weak upper gradient. This completes the proof of Lemma 3.3. □

We will show that the sequence $h_k := 4|T_k u|_p$ satisfies the hypotheses of Lemma 3.3. We first verify estimate (17).

Lemma 3.4. *Let $u \in L^1_{\mathrm{loc}}(X)$ and suppose that γ is a rectifiable curve in X with endpoints x and y. If $d(x, y) \geq 2^{-k}$ for some $k \in \mathbb{Z}$, then*

$$|S_k u(x) - S_k u(y)| \leq 4 \int_\gamma |T_k u|_p\, ds,$$

where $S_k u$ is as in (10).

Proof. We can assume that $\gamma\colon [0, L] \to X$ is parametrized by arc-length and $x = \gamma(0)$, $y = \gamma(L)$. Consider a partition

$$0 = t_0 < t_1 < \cdots < t_m = L,$$

of $[0, L]$ such that the length of each subinterval $[t_{i-1}, t_i]$ satisfies

$$2^{-(k+1)} \leq t_i - t_{i-1} < 2^{-k}.$$

This is possible because $L = \ell(\gamma) \geq d(x, y) \geq 2^{-k}$. Since γ is parametrized by the arc-length, it follows that

$$\ell(\gamma|_{[t_{i-1},t_i]}) = t_i - t_{i-1} < 2^{-k}. \tag{21}$$

Observe that

$$\begin{aligned}|S_k u(x) - S_k u(y)| &\le \sum_{i=1}^{m} \big|S_k u(\gamma(t_i)) - S_k u(\gamma(t_{i-1}))\big| \\ &= \sum_{i=1}^{m} \big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t_{i-1})]}\big|.\end{aligned} \tag{22}$$

On the other hand, if $i \in \{1, \dots, m\}$ is fixed then for any $t \in [t_{i-1}, t_i]$ we have

$$\big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t_{i-1})]}\big| \le \big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t)]}\big| + \big|u_{B[\gamma(t)]} - u_{B[\gamma(t_{i-1})]}\big|. \tag{23}$$

It follows from (21) that the distance between $\gamma(t)$ and any of the points $\gamma(t_{i-1})$ and $\gamma(t_i)$ is less than 2^{-k} and hence, each of the balls $B[\gamma(t_{i-1})]$ and $B[\gamma(t_i)]$ is a neighbor of $B[\gamma(t)]$. Combining this with (23) and the formula for $|T_k u(\gamma(t))|_p$ in (11) yields

$$\big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t_{i-1})]}\big| \le 2 \cdot 2^{-k} |T_k u(\gamma(t))|_p.$$

Therefore, integration with respect to $t \in [t_{i-1}, t_i]$ gives

$$(t_i - t_{i-1})\big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t_{i-1})]}\big| \le 2 \cdot 2^{-k} \int_{t_{i-1}}^{t_i} |T_k u(\gamma(t))|_p \, dt.$$

Since $t_i - t_{i-1} \ge 2^{-(k+1)}$ we have

$$\big|u_{B[\gamma(t_i)]} - u_{B[\gamma(t_{i-1})]}\big| \le 4 \int_{t_{i-1}}^{t_i} |T_k u(\gamma(t))|_p \, dt.$$

Given that $i \in \{1, \dots, m\}$ was arbitrary, we can add the inequalities in (22) to obtain

$$|S_k u(x) - S_k u(y)| \le 4 \int_0^L |T_k u(\gamma(t))|_p \, dt = 4 \int_\gamma |T_k u|_p \, ds. \qquad \square$$

Next we show that if the space supports a p-Poincaré inequality for some $p \in [1, \infty)$, and $u \in N^{1,p}(X)$, then we can always extract a subsequence of $\{|T_k u|_p\}_{k=1}^\infty$ that converges weakly in $L^p(X)$. In the case when $p > 1$, we can rely on the reflexivity of L^p and Lemma 3.1. However, the case of $p = 1$ is more delicate; it relies on the Dunford–Pettis theorem (Lemma 2.13) and some ideas from [9].

For the next result, see also [9, Lemma 6]. We will only need it for $p = 1$.

Lemma 3.5. *Suppose that the space supports a p-Poincaré inequality for some $p \in [1, \infty)$. If $u \in N^{1,p}(X)$, then every subsequence of $\{|T_k u|_p^p\}_{k=1}^\infty$ has a further subsequence that is weakly convergent in $L^1(X)$.*

Proof. Fix $u \in N^{1,p}(X)$. We will prove that $\{|T_k u|_p^p\}_{k=1}^\infty$ satisfies (a) and (b) in Lemma 2.13.

To verify (b), fix $\varepsilon \in (0, \infty)$ and $k \in \mathbb{N}$. Since inequality (13) is satisfied with $g = g_u \in L^p(X)$, we have

$$|T_k u(x)|_p^p \lesssim \fint_{5\lambda B[x]} g_u^p \, d\mu = \sum_{i=1}^{M(k)} \chi_{A_i}(x) \fint_{5\lambda B_i} g_u^p \, d\mu \quad \text{for every } x \in X.$$

Consequently, since $A_i \subseteq B_i$, for every measurable set $S \subseteq X$, we have

$$\int_S |T_k u|_p^p \, d\mu \lesssim \sum_{i=1}^{M(k)} \frac{\mu(S \cap B_i)}{\mu(5\lambda B_i)} \int_{5\lambda B_i} g_u^p \, d\mu \le \sum_{i:\, S \cap B_i \ne \varnothing} \int_{5\lambda B_i} g_u^p \, d\mu. \tag{24}$$

Fix $x_o \in X$, $R > 6\lambda$, and let $S_R := X \setminus B(x_o, 2R)$. Each of the balls B_i has radius $2^{-k} < 1 \le \lambda$. Thus, if $S_R \cap B_i \neq \varnothing$, then $5\lambda B_i \cap B(x_o, R) = \varnothing$, by a simple application of the triangle inequality, and hence (24) and the bounded overlapping of the balls $\{5\lambda B_i\}_i$ yield

$$\int_{X\setminus B(x_o,2R)} |T_k u|_p^p \, d\mu \lesssim \int_{X\setminus B(x_o,R)} g_u^p \, d\mu < \varepsilon,$$

provided R is sufficiently large. This proves condition (b) in Lemma 2.13 with $E := B(x_o, 2R)$.

Next, we prove that condition (a) holds. Note that Lemma 3.1 implies that $\{|T_k u|_p^p\}_{k=1}^\infty$ is bounded in $L^1(X)$. Thus it remains to prove that the family is equi-integrable.

Fix $\varepsilon \in (0, \infty)$ and $k \in \mathbb{N}$, and let $\sigma \in (0, \infty)$ be any number. The value of σ will be fixed later.

Given a μ-measurable set $S \subseteq X$, we define $\mathcal{G}$ to be the collection of all integers $i \in [1, M(k)]$ satisfying $\mu(S \cap B_i) \le \sigma\mu(5\lambda B_i)$, and we let $\mathcal{B}$ consist of all integers $i \in [1, M(k)] \setminus \mathcal{G}$. Note that $\mu(5\lambda B_i) < \mu(S \cap B_i)/\sigma$ for all $i \in \mathcal{B}$. Thus $\mathcal{G}$ and $\mathcal{B}$ partition the set of integers in $[1, M(k)]$ and (24) yields

$$\int_S |T_k u|_p^p \, d\mu \le C_1 \left(\sigma \sum_{i\in\mathcal{G}} \int_{5\lambda B_i} g_u^p \, d\mu + \sum_{i\in\mathcal{B}} \int_{5\lambda B_i} g_u^p \, d\mu \right), \tag{25}$$

where the constant C_1 does not depend on σ or k.

Assume that the overlapping constant of the balls $\{5\lambda B_i\}_i$ is bounded by C_2. Now we fix $\sigma \in (0, \infty)$ such that

$$C_2 \sigma \|g_u\|_p^p < \frac{\varepsilon}{2C_1}.$$

Then the first sum in (25) can be estimated by

$$\sigma \sum_{i\in\mathcal{G}} \int_{5\lambda B_i} g_u^p \, d\mu \le C_2 \sigma \int_X g_u^p \, d\mu < \frac{\varepsilon}{2C_1}. \tag{26}$$

Regarding the second sum in (25), we have

$$\sum_{i\in\mathcal{B}} \int_{5\lambda B_i} g_u^p \, d\mu \le C_2 \int_G g_u^p \, d\mu \quad \text{where} \quad G := \bigcup_{i\in\mathcal{B}} 5\lambda B_i. \tag{27}$$

Note that

$$\mu(G) \le \sum_{i\in\mathcal{B}} \mu(5\lambda B_i) \le \sum_{i\in\mathcal{B}} \frac{\mu(S \cap B_i)}{\sigma} \le \frac{C_2\, \mu(S)}{\sigma}. \tag{28}$$

Absolute continuity of the integral yields $\tilde{\delta} \in (0, \infty)$ such that

$$\int_G g_u^p \, d\mu < \frac{\varepsilon}{2C_1 C_2}, \tag{29}$$

provided $\mu(G) < \tilde{\delta}$.

Let $\delta := \sigma\tilde{\delta}/C_2$. If $\mu(S) < \delta$, then $\mu(G) < \tilde{\delta}$ by (28) and hence (29) is satisfied. This, in concert with (25), (26), and (27) yield

$$\int_S |T_k u|_p^p \, d\mu < C_1 \Big(\frac{\varepsilon}{2C_1} + C_2 \frac{\varepsilon}{2C_1 C_2} \Big) = \varepsilon,$$

and that completes the proof of the equi-integrability and the proof of Lemma 3.5. □

Corollary 3.6. *Suppose the space supports a p-Poincaré inequality for some $p \in [1,\infty)$. If $u \in N^{1,p}(X)$ then, every subsequence of $\{|T_k u|_p\}_{k=1}^\infty$ has a further subsequence that converges weakly in $L^p(X)$.*

Proof. If $p > 1$, then the sequence $\{|T_k u|_p\}_{k=1}^\infty$ is bounded in $L^p(X)$ by Lemma 3.1, and the result follows from the reflexivity of $L^p(X)$. If $p = 1$, then the existence of a weakly convergent subsequence is guaranteed by Lemma 3.5. □

3.3. Proof of the main result.

Proof of Theorem 1.1. We need to prove that:

- $\|\cdot\|_{1,p}^* \approx \|\cdot\|_{N^{1,p}(X)}$ on $N^{1,p}(X)$ when $p \in [1,\infty)$;
- the norm $\|\cdot\|_{1,p}^*$ is uniformly convex on $N^{1,p}(X)$ when $p \in (1,\infty)$.

Then reflexivity of $N^{1,p}(X)$, $p \in (1,\infty)$, will follow directly from the Milman–Pettis theorem, Lemma 2.11.

Therefore, the proof of Theorem 1.1 is contained in Proposition 3.7 and Proposition 3.9 below.

Proposition 3.7. *Suppose the space supports a p-Poincaré inequality for some $p \in [1,\infty)$. Then there exists $C = C(p, C_d, c_{\rm PI}, \lambda) \in (0,\infty)$ such that*

$$4^{-1}\|g_u\|_{L^p(X)} \le \limsup_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} \le C\|g_u\|_{L^p(X)},$$

for all $u \in N^{1,p}(X)$. Consequently, $\|u\|_{1,p}^* \approx \|u\|_{N^{1,p}(X)}$ for all $u \in N^{1,p}(X)$.

Proof. Fix $u \in N^{1,p}(X)$ and let $g_u \in L^p(X)$ denote the minimal p-weak upper gradient of u. In view of Lemma 3.1, we immediately have that

$$\limsup_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} \le C\|g_u\|_{L^p(X)} \tag{30}$$

for some $C = C(p, C_d, c_{\rm PI}, \lambda) \in (0,\infty)$.

To see the opposite inequality, take a subsequence $\{|T_{k_j} u|_p\}_{j=1}^\infty$ of $\{|T_k u|_p\}_{k=1}^\infty$ such that

$$\lim_{j\to\infty} \big\||T_{k_j} u|_p\big\|_{L^p(X)} = \liminf_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)}. \tag{31}$$

In light of Corollary 3.6, by passing to a further subsequence, we can assume $\{|T_{k_j} u|_p\}_{j=1}^\infty$ converges weakly in $L^p(X)$. Let $|T|(u) \in L^p(X)$ be a Borel representative of the weak limit of $\{|T_{k_j} u|_p\}_{j=1}^\infty$ and set $h_k := 4|T_k u|_p$ and $h := 4|T|(u)$. Note that h and each h_k are nonnegative Borel functions. Since $\{h_{k_j}\}_{j=1}^\infty$ converges weakly to h in $L^p(X)$, by appealing to Lemma 3.4, we can conclude that the pair $\big(\{h_k\}_{k=1}^\infty, h\big)$ satisfies the hypotheses of Lemma 3.3. Therefore, we have that h is a p-weak upper gradient of u and hence, $h \ge g_u$ pointwise μ-a.e. in X. Combining this fact with (31) and the lower semicontinuity of the L^p-norm (with respect to the weak convergence), we can estimate

$$\begin{aligned}\limsup_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} &\ge \liminf_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} = 4^{-1} \lim_{j\to\infty} \|h_{k_j}\|_{L^p(X)} \\ &\ge 4^{-1}\|h\|_{L^p(X)} \ge 4^{-1}\|g_u\|_{L^p(X)}.\end{aligned} \tag{32}$$

The proof of Proposition 3.7 is now complete. □

Remark 3.8. Combining (30) and (32) we can conclude that for $p \in [1,\infty)$, there is a finite constant $\xi = \xi(p, C_d, c_{\rm PI}, \lambda) \ge 1$ satisfying

$$\liminf_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} \le \limsup_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)} \le \xi \liminf_{k\to\infty} \big\||T_k u|_p\big\|_{L^p(X)},$$

for every $u \in N^{1,p}(X)$.

We will now proceed to showing that $\|\cdot\|_{1,p}^*$ is uniformly convex on $N^{1,p}$ when $p \in (1,\infty)$.

Proposition 3.9. *Suppose the space supports a p-Poincaré inequality for some $p \in (1,\infty)$. Then the norm $\|\cdot\|_{1,p}^*$ is uniformly convex on $N^{1,p}(X)$. In particular, the Banach space $(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)})$ is reflexive.*

Proof. Fix $\varepsilon \in (0,\infty)$. We will first prove that there exists $\delta \in (0,\infty)$ such that $\|u+v\|_{1,p}^* \le 2(1-\delta)$ whenever $u, v \in N^{1,p}(X)$ satisfy $\|u\|_{1,p}^* < 1$, $\|v\|_{1,p}^* < 1$, and $\|u-v\|_{1,p}^* > \varepsilon$. Fix $u, v \in N^{1,p}(X)$ as above. Then by definition of $\|\cdot\|_{1,p}^*$, we have that

$$\left(\|u\|_{L^p(X)}^p + \big\||T_k u|_p\big\|_{L^p(X)}^p\right)^{1/p} < 1 \quad \text{and} \quad \left(\|v\|_{L^p(X)}^p + \big\||T_k v|_p\big\|_{L^p(X)}^p\right)^{1/p} < 1 \tag{33}$$

for all sufficiently large $k \in \mathbb{N}$. In light of Remark 3.8, we can estimate

$$\begin{aligned}\varepsilon < \|u-v\|_{1,p}^* &\le \left(\|u-v\|_{L^p(X)}^p + \xi^p \liminf_{k\to\infty} \big\||T_k(u-v)|_p\big\|_{L^p(X)}^p\right)^{1/p} \\ &\le \xi \left(\|u-v\|_{L^p(X)}^p + \liminf_{k\to\infty} \big\||T_k u - T_k v|_p\big\|_{L^p(X)}^p\right)^{1/p},\end{aligned}$$

where we have used the fact that $\xi \ge 1$ and T_k is linear in obtaining the last inequality. Consequently, (33) and

$$\left(\|u-v\|_{L^p(X)}^p + \big\||T_k u - T_k v|_p\big\|_{L^p(X)}^p\right)^{1/p} > \varepsilon/\xi, \tag{34}$$

hold true for all sufficiently large $k \in \mathbb{N}$. Fix such a k. Since $T_k u$ and $T_k v$ are vectors in $\mathbb{R}^N$, we can write

$$\big\||T_k u|_p\big\|_{L^p(X)}^p = \sum_{\ell=1}^N \|T_k^\ell u\|_{L^p(X)}^p \quad \text{and} \quad \big\||T_k v|_p\big\|_{L^p(X)}^p = \sum_{\ell=1}^N \|T_k^\ell v\|_{L^p(X)}^p,$$

where $T_k u = (T_k^1 u, \dots, T_k^N u)$ and $T_k v = (T_k^1 v, \dots, T_k^N v)$. Therefore, if we let

$$f := (u, T_k^1 u, \dots, T_k^N u) \quad \text{and} \quad g := (v, T_k^1 v, \dots, T_k^N v),$$

then $f, g \in L^p(X, \ell_{N+1}^p)$ and, with Φ defined as in (7), a rewriting of (33) and (34) yields

$$\begin{aligned}\Phi(f) &= \left(\|u\|_{L^p(X)}^p + \big\||T_k u|_p\big\|_{L^p(X)}^p\right)^{1/p} < 1, \\ \Phi(g) &= \left(\|v\|_{L^p(X)}^p + \big\||T_k v|_p\big\|_{L^p(X)}^p\right)^{1/p} < 1\end{aligned}$$

and

$$\Phi(f-g) = \left(\|u-v\|_{L^p(X)}^p + \big\||T_k u - T_k v|_p\big\|_{L^p(X)}^p\right)^{1/p} > \varepsilon/\xi.$$

By Corollary 2.12, $L^p(X, \ell_{N+1}^p)$ is uniformly convex and so (keeping in mind Lemma 2.9) there exists $\delta \in (0,\infty)$, which depends on ε and ξ, but is independent of f and g (in particular, δ is independent of u, v, and k), such that

$$\left(\|u+v\|_{L^p(X)}^p + \big\||T_k(u+v)|_p\big\|_{L^p(X)}^p\right)^{1/p} = \Phi(f+g) \le 2(1-\delta). \tag{35}$$

Note that we have used the linearity of T_k in obtaining the equality in (35). Given that (35) holds for all sufficiently large $k \in \mathbb{N}$, it follows that $\|u+v\|_{1,p}^* \le 2(1-\delta)$.

To complete the proof of the proposition, suppose that $u, v \in N^{1,p}(X)$ are such that $\|u\|^*_{1,p} = \|v\|^*_{1,p} = 1$, and $\|u - v\|^*_{1,p} > \varepsilon$. Then, for all $\theta \in (0,1)$ sufficiently close to 1, we have that $\theta u, \theta v \in N^{1,p}(X)$ satisfy $\|\theta u\|^*_{1,p}, \|\theta v\|^*_{1,p} < 1$ and $\|\theta u - \theta v\|^*_{1,p} > \varepsilon$. As such, we have $\|\theta u + \theta v\|^*_{1,p} \le 2(1-\delta)$ by what has been established above. Since δ is independent of θ, passing to the limit as $\theta \to 1^-$ yields $\|u + v\|^*_{1,p} \le 2(1-\delta)$. Given that $\varepsilon \in (0,\infty)$ was arbitrary, it follows that $\|\cdot\|^*_{1,p}$ is a uniformly convex norm on $N^{1,p}(X)$.

Finally, the assertion that $(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)})$ is reflexive follows as an immediate consequence of the Milman–Pettis theorem (see Lemma 2.11) and the fact that a reflexive space remains reflexive for an equivalent norm. The proof of Proposition 3.9 is now complete. □

This completes the proof of Theorem 1.1. □

4. Separability from reflexivity

In this section we will prove separability of $N^{1,p}(X)$ for $p \in [1,\infty)$ (Theorem 1.2). In its proof we will employ a general result that provides a mechanism for using reflexivity to establish separability, see Proposition 4.1. Recall that we always assume that the measure on X is doubling and Borel regular.

Throughout this section, all vector spaces are over the field of real numbers. Also, as a notational convention, if S is a set of vectors then we let $\operatorname{span}_{\mathbb{Q}} S$ and $\operatorname{span} S$ denote the set of all finite linear combinations of vectors in S with coefficients in $\mathbb{Q}$ and $\mathbb{R}$, respectively.

Proposition 4.1. *If $T\colon V \to W$ is a linear and bounded injective map of a reflexive Banach space V into a separable normed space W, then V is separable.*

Proof. It suffices to prove that the unit ball $B \subseteq V$ is separable. Given that W is separable, there is a set $\{v_k\colon k \in \mathbb{N}\} \subseteq B$ such that the set $\{T(v_k)\colon k \in \mathbb{N}\}$ is dense in $T(B)$. Since $\operatorname{span}_{\mathbb{Q}}\{v_k\colon k \in \mathbb{N}\} \subseteq \operatorname{span}\{v_k\colon k \in \mathbb{N}\}$ is countable and dense, we conclude that $\operatorname{span}\{v_k\colon k \in \mathbb{N}\}$ is separable and hence, it suffices to prove that

$$\operatorname{span}\{v_k\colon k \in \mathbb{N}\} \cap B \ \text{ is dense in } B. \tag{36}$$

Let $v \in B$. Then, there exists a sequence $\{v_{k_i}\}_i \subseteq B$ such that $T(v_{k_i}) \to T(v)$ in W as $i \to \infty$. Since $\{v_{k_i}\}_i$ is bounded in V and V is reflexive, by passing to a subsequence, we can assume that $\{v_{k_i}\}_i$ converges weakly in V to some $\tilde{v} \in V$. Then Mazur's lemma yields a sequence of convex combinations that converge to $\tilde{v}$ in the norm on V:

$$\operatorname{span}\{v_k\colon k \in \mathbb{N}\} \cap B \ni \sum_{j=i}^{L(i)} \alpha_{i,j} v_{k_j} \to \tilde{v} \quad \text{in } V \text{ as } i \to \infty, \tag{37}$$

where $\alpha_{i,j} \ge 0$ and $\sum_{j=i}^{L(i)} \alpha_{i,j} = 1$ with $L(i) \in \mathbb{N}$. Appealing to the boundedness and linearity of T, we have

$$T(\tilde{v}) = \lim_{i\to\infty} T\left(\sum_{j=i}^{L(i)} \alpha_{i,j} v_{k_j}\right) = \lim_{i\to\infty} \sum_{j=i}^{L(i)} \alpha_{i,j} T(v_{k_j}) = T(v).$$

Since T is injective, we conclude that $\tilde{v} = v$ and (36) now follows from (37) because $v \in B$ was chosen arbitrarily. □

Lemma 4.2. *Suppose that the space (X, d, μ) supports a p-Poincaré inequality for some $p \in [1, \infty)$. Then the space $\mathrm{Lip}_b(X)$ of Lipschitz functions with bounded support is a dense subset of $N^{1,p}(X)$.*

For a proof see [2, Corollary 5.15]. In fact, they proved density of compactly supported Lipschitz functions under the additional assumption that the space X is complete. Without assuming completeness of X, the same proof gives density of Lipschitz functions with bounded support. Completeness of X, since the space is equipped with a doubling measure, implies that bounded and closed sets are compact and hence Lipschitz functions with bounded support have compact support.

Lemma 4.2 follows also from Theorem 8.2.1 and the proof of Proposition 7.1.35 in [14].

We are now ready to present the

Proof of Theorem 1.2. Suppose first that $p > 1$. Note that (X, d) is a separable metric space since μ is a doubling measure on X and so, $L^p(X)$ is separable by [14, Proposition 3.3.55]. Clearly, the identity mapping $\iota\colon N^{1,p}(X) \to L^p(X)$ is a linear and bounded injective map. Now, since the space $N^{1,p}(X)$ is reflexive by Proposition 3.9, Proposition 4.1 immediately implies that $N^{1,p}(X)$ is separable.

Suppose next that $p = 1$ and fix $q \in (1, \infty)$. It follows from Hölder's inequality that X supports a q-Poincaré inequality and by what we have already shown, $N^{1,q}(X)$ is separable so, there is a dense subset $\{\psi_i \colon i \in \mathbb{N}\}$ of $N^{1,q}(X)$.

Fix $x_o \in X$ and for each $k \in \mathbb{N}$ choose a Lipschitz function with bounded support $\eta_k \in \mathrm{Lip}_b(X)$ such that $\eta_k \equiv 1$ on $B(x_o, k)$. We will prove that $\mathcal{F} := \{\eta_k \psi_i \colon k, i \in \mathbb{N}\}$ is a dense subset of $N^{1,1}(X)$.

We first need to show that $\mathcal{F} \subseteq N^{1,1}(X)$. Fix $k, i \in \mathbb{N}$. It follows from Lemma 2.7 that $\eta_k \psi_i \in N^{1,q}(X)$ and $h_{k,i} := |\eta_k| g_{\psi_i} + |\psi_i| \operatorname{lip} \eta_k$ is a q-weak (hence, also 1-weak by Lemma 2.2) upper gradient for $\eta_k \psi_i$, where $g_{\psi_i} \in L^q(X)$ is the minimal q-weak upper gradient for $\psi_i \in L^q(X)$. Since $\eta_k \in \mathrm{Lip}_b(X)$, it follows from (4) that $\operatorname{lip} \eta_k$ is a bounded function with bounded support. Therefore, by Hölder's inequality we can conclude that $\eta_k \psi_i, h_{k,i} \in L^1(X)$ and hence, $\eta_k \psi_i \in N^{1,1}(X)$, as wanted.

In light of Lemma 4.2 it suffices to prove that any Lipschitz function with bounded support can be approximated in the $N^{1,1}$ norm by functions in $\mathcal{F}$. Fix $u \in \mathrm{Lip}_b(X)$ and let $k_o \in \mathbb{N}$ be such that $\operatorname{supp} u \subseteq B(x_o, k_o)$, so $\eta_{k_o} u = u$ pointwise in X. Since $u \in N^{1,q}(X)$, there is a sequence $\{\psi_{i_j}\}_j$ such that $\psi_{i_j} \to u$ in $N^{1,q}(X)$ as $j \to \infty$. Then it easily follows from Lemma 2.7 that

$$u - \eta_{k_o} \psi_{i_j} = \eta_{k_o}(u - \psi_{i_j}) \to 0 \quad \text{in } N^{1,1}(X) \text{ as } j \to \infty.$$

This completes the proof of Theorem 1.2. □

5. Pointwise estimates

The purpose of this section is to prove Theorem 1.3. In order to do so, it suffices to prove the following theorem.

Theorem 5.1. *Fix $p \in (1, \infty)$ and suppose that X supports a q-Poincaré inequality for some $q \in [1, p)$. Then there exists a constant $C \geq 1$ such that for all $u \in N^{1,p}(X)$,*

$$C^{-1} g_u(x) \leq \limsup_{k \to \infty} |T_k u(x)| \leq C g_u(x) \quad \text{for } \mu\text{-a.e. } x \in X, \tag{38}$$

where $g_u \in L^p(X)$ denotes the minimal p-weak upper gradient of u.

Indeed, Theorem 5.1 and the following deep result due to Keith and Zhong [16] (see also [6], [14, Theorem 12.3.9]) immediately yield Theorem 1.3.

Lemma 5.2. (Keith and Zhong) *Let (X, d, μ) be a complete metric-measure space that supports a p-Poincaré inequality for some $p \in (1, \infty)$. Then there exists $q \in [1, p)$ such that X supports a q-Poincaré inequality.*

In the proof of Theorem 5.1 we will make use of the *Hardy–Littlewood maximal operator* of a function $g \in L^1_{\mathrm{loc}}(X)$ which is defined by

$$\big(\mathcal{M}g\big)(x) := \sup_{r>0} \fint_{B(x,r)} |g|\, d\mu \quad \text{for all } x \in X.$$

We will use the boundedness of the maximal function in $L^p(X)$, [14, Theorem 3.5.6]:

Lemma 5.3. *If μ is a doubling measure on a metric space X and $p \in (1, \infty]$, then there is a constant C depending on p and the doubling constant of the measure only, such that $\|\mathcal{M}g\|_{L^p(X)} \le C\|g\|_{L^p(X)}$ for all $g \in L^p(X)$.*

Proof of Theorem 5.1. Assume that the q-Poincaré inequality holds with constants c'_{PI} and λ'. Since all norms in $\mathbb{R}^N$ are equivalent, it suffices to prove that there exists a constant $C \ge 1$ such that for all $u \in N^{1,p}(X)$,

$$C^{-1} g_u(x) \le \limsup_{k\to\infty} |T_k u(x)|_p \le C g_u(x) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{39}$$

Fix $u \in N^{1,p}(X)$. The second inequality in (39) follows from (13) and the Lebesgue differentiation theorem (Lemma 2.1) whenever x is a Lebesgue point of g_u^p. Indeed, it is immediate from Hölder's inequality that X supports a p-Poincaré inequality and so, Lemma 2.8 implies the pair (u, g_u) satisfies the p-Poincaré inequality (5).

There remains to prove the first inequality in (39). Our plan in this regard is to apply Lemma 3.3 with $h_k := 4\sup_{j\ge k} |T_j u|_p$ and $h := 4\limsup_{k\to\infty} |T_k u|_p$ in order to conclude that h is a p-weak upper gradient for u. To this end, first observe that clearly each h_k and h are nonnegative Borel functions. Moreover, Lemma 3.4 implies that if $d(x, y) \ge 2^{-k}$ for some $x, y \in X$ and $k \in \mathbb{N}$, and γ is a rectifiable curve connecting x and y, then

$$|S_k u(x) - S_k u(y)| \le \int_\gamma 4|T_k u|_p\, ds \le \int_\gamma h_k\, ds,$$

where $S_k u$ is as in (10). Hence, (17) in Lemma 3.3 holds.

Next, we claim that $\{h_k\}_{k\in\mathbb{N}}$ converges to h in $L^p(X)$.

Since g_u is a p-weak upper gradient of u, it is also a q-weak upper gradient by Lemma 2.2. Since g_u is finite μ-a.e., (13) in Lemma 3.1 (used here with q in place of p) yields

$$|T_k u(x)|_p \lesssim |T_k u(x)|_q \lesssim \left(\fint_{5\lambda' B[x]} g_u^q\, d\mu\right)^{1/q} \lesssim \big(\mathcal{M}g_u^q\big)^{1/q}(x) \quad \text{for every } x \in X, \tag{40}$$

where the implicit constant is independent of k. Note that in (40), the first inequality is a consequence of the fact that all norms on $\mathbb{R}^N$ are equivalent, and the last inequality follows from doubling condition and the definition of $\mathcal{M}$. Therefore, we have that $h_k \lesssim \big(\mathcal{M}g_u^q\big)^{1/q}$ pointwise on X for every $k \in \mathbb{N}$. On the other hand, since $g_u^q \in L^{p/q}(X)$ and $p/q > 1$, the boundedness of $\mathcal{M}$ on $L^{p/q}(X)$ (Lemma 5.3) implies that $h_k \lesssim (\mathcal{M}g_u^q)^{1/q} \in L^p(X)$. Clearly, $\{h_k\}_{k\in\mathbb{N}}$ converges pointwise to h and so, by Lebesgue's dominated convergence theorem, we have that $h_k \to h$ in $L^p(X)$. In particular, $\{h_k\}_{k\in\mathbb{N}}$ converges weakly to h in $L^p(X)$. Therefore, $\{h_k\}_{k\in\mathbb{N}}$ and h satisfy the hypotheses of Lemma 3.3, and it follows that h is a p-weak upper gradient for u which, in turn, implies that $h \geq g_u$ pointwise μ-a.e. in X by the definition of a minimal p-weak upper gradient. This completes the proof of the first inequality in (39) and, in turn, the proof of Theorem 5.1. □

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