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$\mathcal{A}$-harmonic equation and cavitation

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Abstract. Suppose that f is a homeomorphism from the punctured unit disk $D\setminus\{0\}$ onto the annulus $A(r')=\{r'<|z|<1\}$, $r'\geq 0$, and f is quasiconformal in every $A(r)$, $r>0$, but not in D. If $r'>0$ then f has cavitation at 0 and no cavitation if $r'=0$. The singular factorization problem is to find harmonic functions h in $A(r')$ such that $h\circ f$ satisfies the elliptic PDE associated with f with a singularity at 0. Sufficient conditions in terms of the dilatation $K_{f^{-1}}(z)$ together with the properties of h are given to the factorization problem, to the continuation of $h\circ f$ to 0 and to the regularity of $h\circ f$. We also give sufficient conditions for cavitation and non-cavitation in terms of the complex dilatation of f and demonstrate both cases with several examples.

$\mathcal{A}$-harmoninen yhtälö ja kavitaatio

Tiivistelmä. Julkaisussa tarkastellaan homeomorfismia $f\colon D\setminus\{0\}\to A(r')$, missä D on tason yksikköympyrä ja $A(r')=\{r'<|z|<1\}$, $r'\in[0,1)$, sekä f on kvasikonforminen jokaisessa $A(r)$:ssä, $r>0$, mutta ei ympyrässä D. Jos $r'>0$, niin f:llä on kavitaatio 0:ssa ja tapauksessa $r'=0$ ei kavitaatiota. Kummallekin tapaukselle annetaan riittäviä ehtoja f:n kompleksidilataation avulla. Kuitenkin osoitetaan esimerkillä, että pelkästään dilataation kasvu ei ratkaise ongelmaa. Tilanteeseen liittyy myös ns. singulaarinen faktorisointiprobleema, jolla tarkoitetaan niiden harmonisten funktioiden h löytämistä $A(r')$:ssa, joilla $h\circ f$ toteuttaa f:ään liittyvän singulaarisen elliptisen osittaisdifferentiaaliyhtälön. Faktorisointiprobleemalle, funktion $h\circ f$ säännöllisyydelle ja jatkamiselle 0:aan annetaan riittäviä ehtoja kuvauksen f^{-1} dilataation ja h:n ominaisuuksien avulla. Useat esimerkit valaisevat ehtojen tarkkuutta.

1. Introduction

Let D be an open unit disk in the complex plane $\mathbb{C}$. We study a local behavior of distributional solutions of the $\mathcal{A}$-harmonic equation in two variables

$$\operatorname{div}[\mathcal{A}(z)\nabla u]=0,\quad z\in D\setminus\{0\},\ z=x+iy, \tag{1.1}$$

at a neighborhood of the point $z=0$ provided that $z=0$ is the unique singular point of (1.1), i.e. the point where the uniform ellipticity condition is violated. Here $\mathcal{A}(z)=\{a_{ij}\}$ is a measurable 2×2 symmetric elliptic matrix function with $\det\mathcal{A}(z)=1$ a.e. in D.

For the matrix function $\mathcal{A}(z)$ there is a real valued function $1\leq K_f(z)<\infty$, finite for almost all $z\in D$, such that for almost every $z\in D$

$$\mathcal{A}(z)\xi\cdot\xi\geq\frac{1}{K_f(z)}|\xi|^2 \tag{1.2}$$

and

$$|\mathcal{A}(z)\xi|\leq K_f(z)|\xi| \tag{1.3}$$

for all $\xi\in\mathbb{C}$.

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In the non-singular case the equation (1.1) is intimately connected to quasiconformal mappings and harmonic functions. If $f\colon D \to D$ is a quasiconformal mapping satisfying the linear Beltrami equation

$$f_{\bar z} = \mu(z) f_z, \quad z \in D, \tag{1.4}$$

with complex dilatation μ of the form

$$\mu(z) = \frac{1}{\det(I + \mathcal{A})}(a_{22} - a_{11} - 2ia_{12}). \tag{1.5}$$

and

$$\mathcal{A}(z) = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}, \tag{1.6}$$

then every solution $u \in W^{1,2}_{\mathrm{loc}}(D)$ to the equation (1.1) can be factorized as $u = h \circ f$ where h is harmonic in D, see [4, Theorem 16.2.1] and [2].

Conversely, the real part of any quasiconformal solution $f \in W^{1,2}_{\mathrm{loc}}(D)$ to the equation (1.4) with a given measurable dilatation coefficient μ solves the $\mathcal{A}(z)$-harmonic equation (1.1) in D whose ellipticity condition for $\mathcal{A}(z)$ now follows from

$$|\mu(z)| \le \frac{K_f(z) - 1}{K_f(z) + 1} \quad \text{a.e. in } D \tag{1.7}$$

where $K_f(z)$ is bounded.

In the singular case we consider the situation where μ satisfies (1.7) and $K_f(z) \ge 1$ is not bounded in D but still satisfies for each $r \in (0,1)$

$$\operatorname*{ess\,sup}_{r<|z|<1} K_f(z) \le \beta(r) < \infty. \tag{1.8}$$

In other words, the $\mathcal{A}$-harmonic equation has in $\overline{D}$ unique singular point at $z = 0$ and equation (1.1) is referred to as non-uniformly $\mathcal{A}$-harmonic, or a degenerate $\mathcal{A}$-harmonic equation in D.

In this case, finding some conditions on $K_f(z)$, which guarantee the existence of continuous solutions seems to be far from settled, see e.g. [10, 19, 22] and the references therein. Some sufficient conditions in terms of $K_f(z)$, that guarantee the existence of continuous solutions $u \in W^{1,1}_{\mathrm{loc}}(D)$ to the degenerate equation (1.1) can be deduced from the corresponding existence theorems for the degenerate Beltrami equations, see e.g. [4, 14, 15] and the references therein.

We first consider the behaviour of the locally quasiconformal mapping f in the singular case. Let us fix some notations. Set $A(r_1, r_2) = \{r_1 < |z| < r_2\}$ where $0 \le r_1 < r_2$, $A(r) = \{r < |z| < 1\}$ and $B(0,r) = \{|z| < r\}$ where $0 \le r < 1$. Note that $A(0)$ stands for the punctured disk $D \setminus \{0\}$.

Theorem 1.1. *Let μ be a measurable function in D satisfying (1.7) and (1.8) for each $r \in (0,1)$. Then there exists a homeomorphism $f\colon A(0) \cup \partial D \to A(r') \cup \partial D$, $0 \le r' < 1$, such that $f|_{A(r)}$ is $\beta(r)$-quasiconformal for each $r \in (0,1)$ and satisfies the Beltrami equation with the dilatation μ in each $A(r)$.*

The proof of this theorem is based on the measurable Riemann mapping theorem and can be found, for example, in [5, Theorem 9], see also [6, 7] and also [23, Th. 2.9].

The following notion will play an important role in our further considerations. We say that f has *cavitation* at 0 if $r' > 0$. Note that r' depends only on μ. If f has no cavitation, then f extends to a homeomorphism from $\overline{D}$ onto itself. However, f

need not be a quasiconformal mapping in D due to the singularity of μ at 0. If there is cavitation, then f still extends to a homeomorphism of $\overline{D}\setminus\{0\}$ onto $\overline{D}\setminus\{|z|\le r'\}$.

In Section 2 we prove a harmonic factorization theorem for the distributional solutions to the equation (1.1) in the case of the doubly connected domain $D\setminus\{0\}$. The result holds both in the cavitation and non-cavitation case. Contrary to the proof of Theorem 16.2.1 in [4] we use purely real analytic methods combined with the theory of quasiconformal mappings, see Theorems 2.1 and 2.2. This leads to the same equation but allows more harmonic functions and avoids the difficulties associated with multivalued mappings that arise in the theory of vector fields in multiply connected domains.

Section 3 is devoted to the study of distributional solutions u defined in $D\setminus\{0\}$ which can be extended by continuity to 0. Using the factorization results, we show in Theorem 3.1 that, regardless of the behavior of $K_f(z)$ in a neighborhood of the point $z=0$, there always exist solutions u of (1.1) in $D\setminus\{0\}$ with continuous extension to $z=0$. Theorem 3.1 also solves the Dirichlet problem for the equation (1.1). In Theorem 3.2 we give conditions in terms of $K_{f^{-1}}(z)$ and of a harmonic function h that allow to extend the solution $u=h\circ f$ of (1.1) in $D\setminus\{0\}$ either to a very weak solution $u\in C(D)\cap W^{1,1}_{\mathrm{loc}}(D)$ or to an ordinary solution in $C(D)\cap W^{1,2}_{\mathrm{loc}}(D)$ of (1.1) in the whole disk D.

Theorems 4.2 and 4.4 in Section 4 give sufficient and necessary conditions in terms of $\mu(z)$ for a mapping f to have cavitation and no cavitation at $z=0$. These results use the concepts of angular and radial dilatation of f. We also demonstrate these conditions by examples. In particular, Examples 4.1 and 4.2 show that the growth properties of $K_f(z)$ alone cannot solve the cavitation problem.

2. Harmonic factorization

Next, we turn to the study of the following problem. Suppose that h is a harmonic function in $A(r')$ and a mapping f is from Theorem 1.1. Which equation does $h\circ f$ satisfy in $A(0)$?

We first write the Beltrami equation (1.4) in term of the matrices

$$\mathcal{A}(z)f'(z)^* = J(z,f)f^{-1\,\prime}(f(z)). \tag{2.1}$$

Define a mapping $\mathbb{A}\colon D\times\mathbb{C}\to\mathbb{C}$ so that $\mathbb{A}(z,\xi)$ is defined for a.e. $z\in D$ for every ξ in $\mathbb{C}$ as

$$\mathbb{A}(z,\xi) = J(z,f)\,f^{-1\,\prime}(f(z))f^{-1\,\prime}(f(z))^*\xi.$$

Here $J(z,f)$ is the Jacobian of f. Note that there are two linear mappings from $\mathbb{C}$ into $\mathbb{C}$ and their composition is evaluated at $\xi\in\mathbb{C}$ and L^* stands for the transpose of a linear mapping $L\colon\mathbb{C}\to\mathbb{C}$.

From the equation (2.1) it follows that $\mathbb{A}(z,\xi)=\mathcal{A}(z)\xi$.

A function $u\in W^{1,1}_{\mathrm{loc}}(A(0))$ is a distributionl solution of $\operatorname{div}[\mathbb{A}(z,\nabla u)]=0$ in $A(0)$ if for all $\varphi\in C_0^\infty(A(0))$

$$\int_{A(0)} \mathbb{A}(z,\nabla u(z))\cdot\nabla\varphi(z)\,dm_z = 0. \tag{2.2}$$

Note that we do not require that (2.2) holds for all $\varphi\in C_0^\infty(D)$. In order that the left hand side of (2.2) has a meaning we need the following lemma. The lemma also shows that operator $\mathbb{A}$ creates a good linear elliptic operator in each $A(r)$ with 0 as an only singular point. For the importance of the inequalities (2.3)–(2.5) below see Chapter 3 in [16].

Lemma 2.1. *The operator $\mathbb{A}$ has in each $A(r)$, $0 < r < 1$, the following properties for every $\xi \in \mathbb{C}$ and a.e. $z \in A(r)$:*

$$\mathbb{A}(z,\xi)\cdot\xi \geq \frac{|\xi|^2}{\beta(r)}, \tag{2.3}$$

$$|\mathbb{A}(z,\xi)| \leq \beta(r)|\xi|, \tag{2.4}$$

and for ξ_1, $\xi_2 \in \mathbb{C}$, $\xi_1 \neq \xi_2$

$$(\mathbb{A}(z,\xi_1) - \mathbb{A}(z,\xi_2))\cdot(\xi_1 - \xi_2) > 0, \tag{2.5}$$

$$\det[J(z,f)\, f^{-1\,\prime}(f(z)) f^{-1\,\prime}(f(z))^*] = 1. \tag{2.6}$$

Moreover, the integral in (2.2) is well defined for each $\varphi \in C_0^\infty(A(0))$.

Proof. To prove (2.3) and (2.4) we use the $\beta(r)$-quasiconformality of f and f^{-1} in $A(r)$ and in $f(A(r))$, respectively, to obtain

$$\begin{aligned}\mathbb{A}(z,\xi)\cdot\xi &= J(z,f) f^{-1\,\prime}(f(z))^*\xi \cdot f^{-1\,\prime}(f(z))^*\xi \\ &= J(z,f)|f^{-1\,\prime}(f(z))^*\xi|^2 \geq J(z,f)\,\mathcal{L}(f^{-1\,\prime}(f(z))^2\,|\xi|^2 \\ &\geq \frac{J(z,f)|f^{-1\,\prime}(f(z)|^2|\xi|^2}{\beta(r)} \geq \frac{J(z,f)\,J(f(z),f^{-1})\,|\xi|^2}{\beta(r)} = \frac{|\xi|^2}{\beta(r)}\end{aligned}$$

and for (2.4)

$$|\mathbb{A}(z,\xi)| \leq J(z,f)|f^{-1\,\prime}(f(z))|^2|\xi| \leq J(z,f)\beta(r)J(f(z),f^{-1})|\xi| = \beta(r)|\xi|.$$

Above we have also used the following properties of linear maps $L\colon \mathbb{C}\to\mathbb{C}$:

$$\begin{gathered} L\xi_1\cdot\xi_2 = \xi_1\cdot L^*\xi_2,\ \xi_1,\xi_2\in\mathbb{C}, \\ |L| = |L^*|,\ |L| = \sup_{|\xi|=1}|L(\xi)| \text{ is the operator norm of } L, \\ \mathcal{L}(L) = \mathcal{L}(L^*),\ \mathcal{L}(L) = \inf_{|\xi|=1}|L(\xi)| \text{ is the minimal operator norm of } L. \end{gathered}$$

Inequality (2.5) follows from

$$(\mathbb{A}(z,\xi_1) - \mathbb{A}(z,\xi_2))\cdot(\xi_1-\xi_2) = J(z,f)\,|f^{-1\,\prime}(f(z))^*\,(\xi_1-\xi_2)|^2 > 0$$

because for a.e. $z \in D$, $J(z,f) > 0$ and the linear mapping $f^{-1\,\prime}(f(z))^*$ is injective. Finally for $u \in W^{1,1}_{\mathrm{loc}}(D)$ and $\varphi \in C_0^\infty(A(0))$ the integral in (2.2) is well defined by (2.4) because the support of φ lies in some $A(r)$, $r > 0$. $\square$

Remark 2.1. Let f be a mapping from Theorem 1.1. If the distortion function $K_f(z) \in L^1(D)$, then partial derivatives f_z and $f_{\bar z}$ are in $L^1(D)$. Indeed, since $f|_{A(r)}$ is $K_f(z)$–quasiconformal for each $r \in (0,1)$, we see that

$$(|f_z| + |f_{\bar z}|)^2 \leq K_f(z)J(z,f) \quad \text{a.e. in } A(r)$$

and hence the inequality

$$\iint_{A(r)} (|f_z| + |f_{\bar z}|)\,dm_z \leq \iint_{A(r)} \sqrt{K_f(z)}\sqrt{J(z,f)}\,dm_z$$

holds for every $r \in (0,1)$. Making use of Schwarz's inequality, we get

$$\begin{aligned}\left(\iint_{A(r)} (|f_z| + |f_{\bar z}|)\,dm_z\right)^2 &\leq \iint_{A(r)} K_f(z)\,dm_z \cdot \iint_{A(r)} J(z,f)\,dm_z \\ &\leq \pi \iint_D K_f(z)\,dm_z < \infty.\end{aligned}$$

Passing to the limit as $r \to 0$, we arrive at the required result.

Theorem 2.1. *If h is a harmonic function in $A(r')$, $r' \in [0,1)$, then $u = h \circ f$ is a solution of* $\operatorname{div}[\mathbb{A}(z, \nabla u)] = 0$ *in* $A(0)$.

Proof. Let $\varphi \in C_0^\infty(A(0))$. Then $\varphi \circ f^{-1} \in C(A(r')) \cap W^{1,2}(A(r'))$ with compact support in $A(r')$ and

$$\nabla(\varphi \circ f^{-1})(w) = f^{-1\,\prime}(w)^* \nabla \varphi(w)$$

for a.e. w in $A(r')$. Now $\nabla(h \circ f)(z) = f'(z)^* \nabla h(f(z))$ for a.e. $z \in A(0)$ and thus

$$\begin{aligned}
&\int_{A(0)} \mathbb{A}(z, \nabla (h \circ f)(z)) \cdot \nabla \varphi(z)\, dm_z \\
&= \int_{A(0)} J(z,f)\, f^{-1\,\prime}(f(z)) f^{-1\,\prime}(f(z))^* f'(z)^* \nabla h(f(z)) \cdot \nabla \varphi(z)\, dm_z \\
&= \int_{A(0)} J(z,f)\, f^{-1\,\prime}(f(z)) \nabla h(f(z)) \cdot \nabla \varphi(f^{-1}(f(z))\, dm_z \\
&= \int_{A(r')} f^{-1\,\prime}(w) \nabla h(w) \cdot \nabla \varphi(f^{-1}(w))\, dm_w \\
&= \int_{A(r')} \nabla h(w) \cdot f^{-1\,\prime}(w)^* \nabla \varphi(f^{-1}(w))\, dm_w \\
&= \int_{A(r')} \nabla h(w) \cdot \nabla(\varphi \circ f^{-1})(w)\, dm_w = 0
\end{aligned}$$

because $\varphi \circ f^{-1} \in C_0(A(r')) \cap W^{1,2}(A(r'))$ and hence can be used as a test function for the Laplace equation $\operatorname{div}[\nabla h] = 0$ in the distributional sense in $A(0)$. Note that in the above equalities all the computations are performed in compact sets of $A(r)$ and $A(r')$ respectively. We also used the change of variables using a quasiconformal map f and the fact that $L z_1 \cdot z_2 = z_1 \cdot L^* z_2$ where L is a linear map. □

Theorem 2.1 has also a converse.

Theorem 2.2. *If v is a function in $A(r')$ and $u = v \circ f \in W^{1,2}_{\mathrm{loc}}(A(0))$ is a solution of* $\operatorname{div}[\mathbb{A}(z, \nabla(u)(z)))] = 0$ *in $A(0)$, then v is harmonic in $A(r')$.*

Proof. Note first that $v = u \circ f^{-1}$ belongs to $W^{1,2}_{\mathrm{loc}}(A(r'))$. Next let $\psi \in C_0^\infty(A(r'))$. Then $\varphi = \psi \circ f \in C_0(A(0)) \cap W^{1,2}_{\mathrm{loc}}(A(0))$ and can be used as a test function in (2.2), see Lemma 3.11 in [16]. Now for a.e. $z \in A(0)$

$$\begin{aligned}
\mathbb{A}(z, \nabla(v \circ f)(z)) \cdot \nabla\varphi(z) &= J(z,f)\, f^{-1\,\prime}(f(z)) f^{-1\,\prime}(f(z))^* \nabla(v \circ f)(z) \cdot \nabla\varphi(z) \\
&= J(z,f)\, \nabla v(f(z)) \cdot \nabla\psi(f(z))
\end{aligned}$$

and hence

$$\begin{aligned}
0 &= \int_{A(0)} \mathbb{A}(z, \nabla(v \circ f)(z)) \cdot \nabla\varphi(z)\, dm_z \\
&= \int_{A(0)} J(z,f)\, \nabla v(f(z)) \cdot \nabla\psi(f(z))\, dm_z \\
&= \int_{A(r')} \nabla v(w) \cdot \nabla\psi(w)\, dm_w = \int_{A(r')} v\, \Delta\psi\, dm_w.
\end{aligned}$$

Since this holds for all $\psi \in C_0^\infty(A(r'))$, the function v is harmonic in $A(r')$ by the Weyl lemma [24]. □

3. Continuous extension and regularity of solutions

In Theorem 2.1 the function $u = h \circ f$ need not belong to $C(\overline{D})$. The following corollary is a direct consequence of Theorem 2.1 and the mean value property of harmonic functions.

Corollary 3.1. *Suppose that $h \in C(\overline{A(r')})$ is harmonic in $A(r')$ with $h(z) = c = \text{const.}$ for $|z| = r' > 0$ and*

$$h(0) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{i\theta})\,d\theta,$$

if $r' = 0$. Then $h \circ f$ belongs to $C(\overline{D})$ and is a solution of (1.1) in $A(0)$.

A typical example in the cavitation case above is the harmonic function $h(z) = \log|z|$. Note that $h \circ f$ cannot be continuous in $\overline{D}$ if h is not constant on $\partial B(0, r')$. The following theorem solves the Dirichlet problem in the cavitation and the non-cavitation case.

Theorem 3.1. (a) *Suppose that f has cavitation at 0 and $\psi \in C(\partial A(r'))$ such that $\psi|\partial B(0, r') = c = \text{const.}$ Then there is $h \in C(\overline{A(r')})$ which is harmonic in $A(r')$, $h|\partial A(r') = \psi$ and the function $u = h \circ f \in C(\overline{D}) \cap W^{1,2}_{\text{loc}}(A(0))$ is a solution of $\operatorname{div}[\mathbb{A}(z, \nabla u)] = 0$ in $A(0)$, $u|\partial D = \psi \circ f|\partial D$ and $u(0) = c$.*

(b) *Suppose that f has no cavitation at 0 and $\psi \in C(\partial D)$. Then there is $h \in C(\overline{D})$ which is harmonic in D and $h|\partial D = \psi \in C(\partial D)$. Now the function $u = h \circ f \in C(\overline{D}) \cap W^{1,2}_{\text{loc}}(A(0))$ is a solution of $\operatorname{div}[\mathbb{A}(z, \nabla u)] = 0$ in $A(0)$, $u|\partial D = \psi \circ f|\partial D$ and*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} \psi(e^{i\theta})\,d\theta. \tag{3.1}$$

Proof. Assume that f has cavitation at 0. Since $A(r')$ is a Dirichlet regular domain the required harmonic function h exists. Since $f|\partial D$ is a homeomorphism onto ∂D and $|f(z)| \to r'$ as $z \to 0$, the function u is continuous in $\overline{D} = \overline{A(0)}$ with the required boundary values. That u is a solution of (2.2) follows from Theorem 2.1.

If f has no cavitation at 0, then there exists a function $h \in C(\overline{D})$ such that h is harmonic in D with prescribed boundary values ψ on ∂D. Proceeding as above we obtain the required function u since $u(0) = h(f(0)) = h(0)$ satisfies (3.1) by the mean value property of harmonic functions. □

The above functions $h \circ f$ need not be solutions to $\operatorname{div}[\mathbb{A}(z, h \circ f)] = 0$ in D. However, in the non-cavitation case the following result holds.

Theorem 3.2. *Suppose that f has no cavitation at 0 and h is a harmonic function in D. If*

$$\int_{B(0,r_0)} K_{f^{-1}}(w)\,|\nabla h(w)|^2\,dm_w < \infty \tag{3.2}$$

for some $r_0 > 0$, then $h \circ f \in C(D) \cap W^{1,1}_{\text{loc}}(D)$ and $h \circ f$ is a very weak solution of $\operatorname{div}[\mathbb{A}(z, \nabla(h \circ f))] = 0$ in D. Moreover, if

$$\int_{B(0,r_0)} K_{f^{-1}}(w)^4\,|\nabla h(w)|^4\,dm_w < \infty \tag{3.3}$$

for some $r_0 > 0$, then $h \circ f$ is an ordinary $C(D) \cap W^{1,2}_{\text{loc}}(D)$-solution in D.

Proof. Choose $r_1 > 0$ so that $f(B(0,r_1)) \subset B(0,r_0)$. We first show that

$$\int_{B(0,r_1)} |\mathbb{A}(z, \nabla(h\circ f)(z))|\, dm_z < \infty. \tag{3.4}$$

By the definition of $\mathbb{A}$ and using the Hölder inequality we obtain

$$\begin{aligned}
&\int_{B(0,r_1)} |\mathbb{A}(z, \nabla(h\circ f)(z))|\, dm_z = \int_{B(0,r_1)} J(z,f)|f^{-1\,\prime}(f(z))\nabla h(f(z))|\, dm_z \\
&= \int_{B(0,r_0)} |f^{-1\,\prime}(w)\nabla h(w)|\, dm_w \le \int_{B(0,r_0)} K_{f^{-1}}(w)^{1/2}\, J(w,f^{-1})^{1/2}\, |\nabla h(w)|\, dm_w \\
&\le \Big(\int_{B(0,r_0)} J(w,f^{-1})\, dm_w\Big)^{1/2} \Big(\int_{B(0,r_0)} K_{f^{-1}}(w)|\nabla h(w)|^2\, dm_w\Big)^{1/2} \\
&\le \pi^{1/2} \Big(\int_{B(0,r_0)} K_{f^{-1}}(w))|\nabla h(w)|^2\, dm_w\Big)^{1/2} < \infty
\end{aligned}$$

as required.

It is clear that $h\circ f \in C(D)$ and as above

$$\begin{aligned}
\int_{B(0,r_1)} |\nabla\, (h\circ f)(z)|\, dm_z &\le \int_{B(0,r_1)} K_f(z)^{1/2} J(z,f)^{1/2}|\nabla h(f(z))|\, dm_z \\
&\le \pi^{1/2}\Big(\int_{B(0,r_0)} K_{f^{-1}}(w))|\nabla h(w)|^2\, dm_w\Big)^{1/2} < \infty
\end{aligned}$$

since $K_f(z) = K_{f^{-1}}(f(z))$ for a.e. $z \in D$ and thus $h\circ f \in W^{1,1}_{\mathrm{loc}}(D)$.

To complete the proof we need to show that

$$\int_D \mathbb{A}(z, \nabla(h\circ f)(z)) \cdot \nabla\varphi(z)\, dm_z = 0, \tag{3.5}$$

for every $\varphi \in C_0^\infty(D)$. Let $r > 0$ be so small that $f(B(0,r)) \subset B(0,r_0)$ and choose a function $\psi_r \in C^\infty(\mathbb{C})$, $0 \le \psi \le 0$, such that $\psi_r = 1$ in $\mathbb{C} \setminus f(B(0,r))$ and $\psi_r = 0$ in $B(0,\tilde r)$ where $\overline{B(0,\tilde r)} \subset f(B(0,r))$. Moreover, by choosing $\tilde r$ small enough we have

$$\int_{f(B(0,r))} |\nabla\, \psi_r(w)|^2\, dm_w < \varepsilon \tag{3.6}$$

for arbitrary $\varepsilon > 0$ because the conformal (or 2)-capacity of the condenser $E = (\overline{B(0,\tilde r)}, f(B(0,r))$ satisfies $\mathrm{Cap}_2(E) \to 0$ as $\tilde r \to 0$, see Section 2.11 in [16]. Set $\varphi_r(z) = \varphi(z)\psi_r(f(z))$. By the properties of f and the functions φ and ψ_r the function φ_r belongs to $C_0(A(0)) \cap W^{1,2}(A(0))$ and

$$\nabla\varphi_r(z) = \psi_r(f(z))\nabla\varphi(z) + \varphi(z)f'(z)^*\nabla\psi_r(f(z)).$$

Now φ_r can be used as a test function for the solution $h\circ f$ of the equation $\operatorname{div}\mathbb{A}(z, \nabla(h\circ f)) = 0$ in $A(0)$ to obtain

$$\begin{aligned}
0 &= \int_D \mathbb{A}(z, \nabla(h\circ f)(z)) \cdot \nabla\varphi_r(z)\, dm_z \\
&= \int_{D\setminus B(0,r)} \mathbb{A}(z, \nabla(h\circ f)) \cdot \nabla\varphi\, dm_z + \int_{B(0,r)} \psi_r(f(z))\mathbb{A}(z, \nabla(h\circ f)) \cdot \nabla\varphi\, dm_z \\
&\quad + \int_{B(0,r)} \mathbb{A}(z, \nabla(h\circ f)(z)) \cdot \varphi(z)f'(z)^*\nabla\psi_r(f(z))\, dm_z = I + II + III
\end{aligned} \tag{3.7}$$

and we consider the above terms separately.

Since φ has compact support in D, (3.4) implies

$$I = \int_{D\setminus B(0,r)} \mathbb{A}(z, \nabla(h\circ f))\cdot\nabla\varphi\, dm_z \to \int_D \mathbb{A}(z,\nabla(h\circ f))\cdot\nabla\varphi\, dm_z \tag{3.8}$$

as $r \to 0$.

Set $C = \sup\{|\nabla\varphi(z)|\colon z\in D\}$. The term II can be handled as above:

$$|II| \le C\left|\int_{B(0,r)} |\mathbb{A}(z,\nabla(h\circ f))\right| dm_z| \to 0 \quad \text{as } r\to 0. \tag{3.9}$$

Setting $C_1 = \sup\{|\varphi(z)|\colon z\in D\} < \infty$ and using the properties of the operator $\mathbb{A}$, a change of variables and the Hölder inequality we can estimate the term III as follows:

$$\begin{aligned}
|III| &= \left|\int_{B(0,r)} \mathbb{A}(z,\nabla(h\circ f)(z))\cdot\varphi(z)f'(z)^*\nabla\psi_r(f(z))\, dm_z\right| \\
&= \left|\int_{B(0,r)} J(z,f)\,\nabla h(f(z))\cdot(\varphi(z)\,\nabla\psi_r(f(z)))\, dm_z\right| \\
&\le C_1\left(\int_{B(0,r)} J(z,f)|\nabla h(f(z))|^2\, dm_z\right)^{1/2}\left(\int_{B(0,r)} J(z,f)|\nabla\psi_r(f(z)))|^2\, dm_z\right)^{1/2} \qquad (3.10)\\
&= C_1\left(\int_{f(B(0,r))} |\nabla\, h(w)|^2\, dm_w\right)^{1/2}\left(\int_{f(B(0,r))} |\nabla\psi_r(w)|^2\, dm_w\right)^{1/2} \\
&\le C_1\,\varepsilon^{1/2}\left(\int_{f(B(0,r))} |\nabla h(w)|^2\, dm_w\right)^{1/2} \to 0
\end{aligned}$$

as $r\to 0$.

Now (3.7), (3.8), (3.9) and (3.10) imply (3.5).

To prove the last assertion note first that (3.3) implies (3.2) because $K_{f^{-1}}(w) \ge 1$ a.e. and hence it suffices to show that

$$\int_K |\nabla(h\circ f)(z)|^2\, dm_z < \infty$$

on each comapact set $K\subset D$ and for this it suffices to show that

$$\int_{B(0,r_0)} |\nabla(h\circ f)(z)|^2\, dm_z < \infty.$$

We estimate as above to obtain

$$\begin{aligned}
&\int_{B(0,r_0)} |\nabla(h\circ f)(z)|^2\, dm_z \le \int_{B(0,r_0)} K_f(z)^2 J(z,f)|\nabla(h(f(z))|^2\, dm_z \\
&\le \left(\int_{B(0,r_0)} J(z,f)dm_z\right)^{1/2}\left(\int_{B(0,r_0)} K_f(z)^4 J(z,f)|\nabla(h(f(z))|^4 dm_z\right)^{1/2} \\
&\le \pi^{1/2}\left(\int_{B(0,r_0)} K_{f^{-1}}(w)^4\,|\nabla h(w)|^4\, dm_w\right)^{1/2} < \infty. \qquad \square
\end{aligned}$$

We demonstrate Theorem 3.2 with the following example.

Example 3.1. Consider the following solution in $A(0)$ of the equation (1.1) with continuous extension to D

$$u(re^{i\theta}) = r\cos(\theta + 1/r^2 - 1), \quad u(0) = 0,$$

where $\mathcal{A}(z)$ has the form (1.6) with $\mu(z) = (1+i|z|^2)^{-1}z/\bar{z}$. Now $u \notin W^{1,1}_{\mathrm{loc}}(D)$ because $u(z) = \mathrm{Re} f(z)$ where

$$f(z) = ze^{i(1/|z|^2-1)}$$

is a homeomorphic solution of the Beltrami equation in D with given μ. A simple calculation shows that the partial derivatives

$$|f_{\bar{z}}| = 1/|z|^2, \quad |f_z| = \left|1 - \frac{i}{|z|^2}\right|$$

are not locally integrable near the origin and hence f and $u \notin W^{1,1}_{\mathrm{loc}}(D)$. Since $K_{f^{-1}}(w) = K_f(f^{-1}(w))$,

$$K_f(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{\sqrt{1+|z|^4}+1}{\sqrt{1+|z|^4}-1}$$

and $|f(z)| = |z|$, we see that $K_{f^{-1}}(w) \sim C/|w|^4$ as $w \to 0$. For the harmonic function $h(w) = \mathrm{Re}\, w^n$ we have $|\nabla h(w)|^2 = n^2|w|^{2(n-1)}$. If $n \geq 3$, then the condition (3.2) holds and, by Theorem 3.2, $u = h \circ f$ is solution of (1.1) in the whole disk D. This can be also verified by straightforward computation. Similarly the condition (3.3) can be illustrated by choosing $n \geq 5$ to obtain an ordinary solution $h \circ f \in C(\overline{D}) \cap W^{1,2}(D)$.

Note that in the example $K_f(z) \notin L^1_{\mathrm{loc}}(D)$.

4. Criteria for cavitation

The following two examples show that the cavitation problem requires more precise information on the behavior of $\mu(z)$ than merely on $K_f(z)$.

Example 4.1. Given the dilatation $\mu_1(z) = -\frac{1}{1+2|z|}\frac{z}{\bar{z}}$, the homeomorphism of the Beltrami equation $f_1(z) = \frac{1}{2}(z + z/|z|)$ maps $A(0)$ onto the annulus $A(1/2, 1)$. Hence f has cavitation at 0.

Example 4.2. Given the dilatation $\mu_2(z) = \frac{1}{1+2|z|}\frac{z}{\bar{z}}$, the automorphism $f_2(z) = ze^{1-1/|z|}$, $f_2(0) = 0$, of the disk D satisfies the Beltrami equation with such dilatation. Hence f has no cavitation at $z = 0$ in spite of the fact that $|\mu_1(z)| = |\mu_2(z)|$ and also $K_{f_1}(z) = K_{f_2}(z)$ in D.

In the following we employ together with $|\mu(z)|$ and $\arg \mu(z)$ the so-called directional distortion functions, see e.g. [8, Section 3.9]. For a given μ in D the angular dilatation $D_\mu(z)$ is defined by

$$D_\mu(z) = \frac{\left|1 - \mu(z)\frac{\bar{z}}{z}\right|^2}{1 - |\mu(z)|^2}. \tag{4.1}$$

The angular dilatation is a measurable function in D and satisfies the inequality

$$1/K_f(z) \leq D_\mu(z) \leq K_f(z) \tag{4.2}$$

a.e. in D. The name of $D_\mu(z)$ comes from the following important relation: if f is a quasiconformal mapping in D with complex dilatation μ and if we write $z = re^{i\theta}$, then for almost all $z \in D$

$$\left|\frac{\partial f}{\partial \theta}(z)\right|^2 = r^2 D_\mu(z) J(z, f). \tag{4.3}$$

The radial dilatation of f at 0 is defined as $D_{-\mu}(z)$ and its name comes from the relation

$$\left|\frac{\partial f}{\partial r}(z)\right|^2 = D_{-\mu}(z)J(z,f) \quad \text{a.e. in } D. \tag{4.4}$$

The idea to employ the directional dilatations for the study of quasiconformal mappings in the plane is due to Andreian Cazacu [1], Reich and Walczak [20] and similar ideas are also used in [14].

Recall that a doubly connected domain $\mathcal{R}$ in $\mathbb{C}$ is called a ring domain provided that the components of its complement are non-degenerate, i.e. the components consists of more than one point. Let C_0 be the bounded and C_1 the unbounded component of its complement. A counterpart of Riemann's mapping theorem says that a ring domain $\mathcal{R}$ can always be mapped by a conformal mapping onto an annulus $A(t)$ where $0 < t < 1$. The number $1/t$ determines the equivalence class, and the quantity $\operatorname{mod}\mathcal{R} = \log 1/t$ is called *the modulus of the ring domain* $\mathcal{R}$. This definition can be easily extended to the case where the components are degenerate; then $\operatorname{mod}\mathcal{R} = \infty$.

The second tool, that we are going to employ for the study of the cavitation problem is the following statement on the distortion of the modulus of ring domains under quasiconformal mappings.

Proposition 4.1. *Let f be as in Theorem 1.1. Then for each annulus $A(r)$, $r \in (0,1)$ and each nonnegative measurable functions $q(t)$, $t \in (r,1)$ and $p(\theta)$, $\theta \in (0,2\pi)$, such that*

$$\int_r^1 q(t)\,dt = 1, \quad \frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,d\theta = 1, \tag{4.5}$$

the following inequalities hold

$$\left(\frac{1}{2\pi}\iint\limits_{A(r)} q^2(t)D_\mu(te^{i\theta})\,dm_z\right)^{-1} \le \operatorname{mod} f(A(r)) \le \frac{1}{2\pi}\iint\limits_{A(r)} p^2(\theta)D_{-\mu}(te^{i\theta})\frac{dm_z}{t^2}. \tag{4.6}$$

Proof. We may assume that $A' = f(A(r)) = A(\rho)$, $0 < \rho < 1$. Denote by γ_t the circle $|z| = t$, $r < t < 1$. Then for almost all $t \in (r,1)$, f is absolutely continuous on γ_t and totally differentiable at every point in γ_t except for a set of linear measure zero. By Fubini's theorem, $D_\mu(z)$ and $J(z,f)$ are integrable on γ_t for almost all $t \in (r,1)$. For such t we have

$$2\pi = \left|\int_{\gamma_t} d\arg f\right| \le \int_{\gamma_t} |d\arg f| \le \int_{\gamma_t}\frac{|df(z)|}{|f(z)|} \le \int_0^{2\pi}\frac{|f_\theta(te^{i\theta})|}{|f(te^{i\theta})|}\,d\theta.$$

By Schwarz's inequality and (4.3) we get

$$(2\pi)^2 \le t^2\int_0^{2\pi} D_\mu(te^{i\theta})\,d\theta\int_0^{2\pi}\frac{J_f}{|f|^2}(te^{i\theta})\,d\theta,$$

and hence

$$\frac{2\pi}{t\phi(t)} \le t\int_0^{2\pi}\frac{J_f}{|f|^2}(te^{i\theta})\,d\theta$$

for almost every $t \in (r,1)$, where

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} D_\mu(te^{i\theta})\,d\theta.$$

Integrating both sides with respect to t from r to 1, we obtain

$$2\pi\int_r^1 \frac{dt}{t\phi_\mu(t)} \le \int_r^1\int_0^{2\pi} \frac{J_f}{|f|^2} t\,d\theta\,dt = \int_A \frac{J_f}{|f|^2}\,dm_z$$
$$= \iint_{A'} \frac{dm_w}{|w|^2} = 2\pi\log 1/\rho = 2\pi \operatorname{mod} f(A(r))$$

and we arrive at the inequality

$$\operatorname{mod} f(A(r)) \ge \int_r^1 \frac{1}{t\phi(t)} dt.$$

For the other direction from

$$1 = \left(\int_r^1 q(t)\,dt\right)^2 \le \int_r^1 \frac{1}{t\phi(t)}\,dt \cdot \frac{1}{2\pi}\iint_{A(r)} q^2 D_\mu(z)\,dm_z,$$

we see that

$$\int_r^1 \frac{1}{\phi(t)}\,dt \ge \left[\frac{1}{2\pi}\iint_{A(r)} q^2 D_\mu(z)\,dm_z\right]^{-1}.$$

Thus

$$\operatorname{mod} f(A(r)) \ge \left[\frac{1}{2\pi}\iint_{A(r)} q^2(|z|) D_\mu(z)\,dm_z\right]^{-1}$$

and the proof of the left hand side of (4.6) is completed.

Let γ_θ be the radial segment $z(t) = te^{i\theta}$, $r \le t \le 1$. Now f is absolutely continuous on γ_θ and totally differentiable at every point in γ_θ except for a set of linear measure 0. By Fubini's theorem $D_{-\mu}(z)$ and the Jacobian $J(z,f)$ are integrable on γ_θ for almost all $\theta \in [0,2\pi]$. Then the relation $|f_t(z)|^2 = D_{-\mu}(z)J(z,f)$ and Schwarz's inequality yields for such θ

$$\operatorname{mod}^2 f(A(r)) = \log^2\frac{1}{\rho} \le \left|\int_{\gamma_\theta} d\log f\right|^2 \le \left(\int_{\gamma_\theta} |d\log f|\right)^2 \le \left(\int_r^1 \left|\frac{f_t(te^{i\theta})}{f(te^{i\theta})}\right| dt\right)^2$$
$$\le \int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t} \cdot \int_r^1 J(te^{i\theta},f)|f(te^{i\theta})|^{-2} t\,dt$$

for almost all $\theta \in [0,2\pi]$. Hence

$$\int_r^1 J(z,f)|f|^{-2} t\,dt \ge \frac{\operatorname{mod}^2 f(A(r))}{\psi(\theta)},$$

where

$$\psi(\theta) = \int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}.$$

Integrating this inequality with respect to θ over $[0,2\pi]$ and using the integral transformation formula, we obtain

$$\operatorname{mod}^2 f(A(r)) \cdot \int_0^{2\pi} \frac{d\theta}{\psi(\theta)} \le \iint_{A(r)} J(z,f)|f(z)|^{-2}\,dx\,dy = \iint_{A'} \frac{dm_w}{|w|^2}$$
$$= 2\pi\log 1/\rho = 2\pi \cdot \operatorname{mod} f(A(r))$$

and hence

$$\frac{1}{\operatorname{mod} f(A(r))} \ge \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\psi(\theta)}.$$

Since

$$1=\left(\frac{1}{2\pi}\int_0^{2\pi}p(\theta)\,d\theta\right)^2\le\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\psi(\theta)}\cdot\frac{1}{2\pi}\int_0^{2\pi}p^2(\theta)\psi(\theta)\,d\theta,$$

we arrive at the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\psi(\theta)}\ge\left[\frac{1}{2\pi}\iint_{A(r)}p^2D_{-\mu}(z)\frac{dm_z}{|z|^2}\right]^{-1}.$$

Then

$$\operatorname{mod}f(A(r))\le\frac{1}{2\pi}\iint_{A(r)}p^2(\theta)D_{-\mu}(te^{i\theta})\frac{dm_z}{t^2}$$

and the proof is completed. □

The following statement is obvious.

Proposition 4.2. *Let f be a homeomorphism defined in Theorem 1.1. The mapping f has cavitation at $z=0$ if and only if*

$$\lim_{r\to 0}\operatorname{mod}(f(A(r)))=\log\frac{1}{r'},\quad 0<r'<1. \tag{4.7}$$

Combining Proposition 4.2 with the right hand side of the inequality (4.6), we arrive at the following sufficient condition for the mapping f in Theorem 1.1 to have cavitation at the point $z=0$.

Theorem 4.1. *If $p(\theta)$ is an admissible function in Proposition 4.1 and*

$$\frac{1}{2\pi}\iint_{|z|<1}p^2\frac{\left|1+\mu(z)\frac{\bar z}{z}\right|^2}{1-|\mu(z)|^2}\frac{dm_z}{|z|^2}<\infty, \tag{4.8}$$

then f has cavitation at the point $z=0$.

Note that $p(\theta)\equiv 1$, in particular, is an admissible function. It is easy to verify that the function

$$p_{\text{extr}}(\theta)=\frac{1}{I\cdot\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}$$

where

$$I=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}\,d\theta,$$

is also admissible for the inequality (4.6). On the other hand, it can be verified that the function $p_{\text{extr}}(\theta)$ solves the problem

$$\inf_p\iint_{|z|<1}p^2(\theta)D_{-\mu}(z)\frac{dx\,dy}{|z|^2}$$

where the infimum is taken over all admissible functions. The inequality

$$\frac{1}{2\pi}\int_{|z|<1}p^2(\theta)D_{-\mu}(z)\frac{dm_z}{|z|^2}\ge\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}\,d\theta\right]^{-1}=I^{-1} \tag{4.9}$$

follows from

$$1=\left(\frac{1}{2\pi}\int_0^{2\pi}p(\theta)\,d\theta\right)^2\le\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\psi(\theta)}\cdot\frac{1}{2\pi}\int_0^{2\pi}p^2(\theta)\psi(\theta)\,d\theta$$

where

$$\psi(\theta)=\int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}.$$

On the other hand it is easy to check that the equality sign in (4.9) holds just for the function $p_{\mathrm{extr}}(\theta)$.

We finally arrive at the inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\psi(\theta)} \geq \left[\frac{1}{2\pi}\iint_{A(r)} p^2(\theta)D_{-\mu}(z)\frac{dm_z}{|z|^2}\right]^{-1}$$

that holds for every $0<r<1$ and admissiple p.

Now we have a weaker estimate from (4.6)

$$\operatorname{mod} f(A(r)) \leq \frac{1}{\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}\,d\theta} \tag{4.10}$$

which is the best possible sufficient condition for cavitation achieved by this method.

Theorem 4.2. *If*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}} \neq 0, \tag{4.11}$$

then f has cavitation at the point $z=0$.

The sufficient condition (4.11) for a locally quasiconformal mapping $f\colon D\backslash\{0\}\to D$ to have cavitation at the singular point $z=0$ is quite effective. For example, it successfully controls the mappings considered earlier in Example 4.1 and Example 4.2, for which the dilatation coefficient $K_f(z)=1+1/|z|$ is the same although they have different Beltrami coefficients μ_1 and μ_2. Indeed, if $\mu_1(z)=-\frac{1}{1+2|z|}\frac{z}{\bar z}$, we have that

$$D_{-\mu_1}(te^{it})=\frac{|1-\frac{1}{1+2t}|^2}{1-\frac{1}{(1+2t)^2}}=\frac{t}{1+t},$$

and hence

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{\int_0^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}=1/\log 2.$$

By Theorem 4.2, the mapping f_1 has cavitation at $z=0$.

However, the sufficient condition (4.11) for the mapping f to have cavitation at $z=0$ fails to be at the same time a necessary condition. Before to illustrate this, we will study a geometric meaning of the integral in (4.11). It will allow us to construct the required example of the mapping.

To this end, let us first recall the notion of the modulus $\mathcal{M}(\Sigma)$ of the path family Σ, which is defined as follows, see e.g. [8, Section 2.3]:

$$\mathcal{M}(\Sigma)=\inf_{\rho}\int_{\mathbb{C}}\rho^2(z)dm_z, \tag{4.12}$$

where the infinum is taken over all non-negative Borel measurable functions $\rho\colon\mathbb{C}\to\mathbb{R}$ such that

$$\int_\gamma \rho\,ds \geq 1, \tag{4.13}$$

for every locally rectifiable $\gamma\in\Sigma$.

The quantity $\mathcal{M}(\Sigma)$ is conformal invariant, for example, if $\mathcal{R}$ is a ring domain in the complex plane, then

$$\operatorname{mod}\mathcal{R}=\frac{2\pi}{\mathcal{M}(\Gamma)}=2\pi\mathcal{M}(\Sigma), \tag{4.14}$$

where Γ is the set of all curves joining the boundary components of $\mathcal{R}$ in $\mathcal{R}$ and Σ is the set of all curves separating the boundary components of $\mathcal{R}$ in $\mathcal{R}$, respectively.

Remark 4.1. The inequalities (4.6) can be written in the following equivalent form

$$2\pi\left[\frac{1}{2\pi}\iint\limits_{A(r)} p^2 D_{-\mu}(z)\frac{dm_z}{|z|^2}\right]^{-1} \le \mathcal{M}(f(\Gamma)) \le \iint\limits_{A(r)} q^2 D_\mu(z)\,dm_z, \tag{4.15}$$

where Γ stands for the family of curves joining the boundary components of $A(r)$ in $A(r)$.

The following statement, due to Ahlfors and Beurling [3] and Rodin [21], see also Brakalova [9], allows us to compute explicitly the moduli of images of some simple curve families under quasiconformal mappings and therefore to give a geometric interpretation of the preceding result.

Lemma 4.1. *Given $0 < r < 1$, let σ_r be the family of radial segments, joining the boundary circles of the annulus $A(r)$, and let Σ_r be the image of σ_r under a quasiconformal homeomorphism f, satisfying the Beltrami equation $f_{\bar z} = \mu(z) f_z$. Then the module of the curve family Σ_r is represented by the explicit formula*

$$\mathcal{M}(\Sigma_r) = \int_0^{2\pi} \frac{d\theta}{\int_r^1 \frac{|1+\mu(te^{i\theta})e^{-2i\theta}|^2}{1-|\mu(te^{i\theta})|^2}\frac{dt}{t}}. \tag{4.16}$$

Proof. The function $\rho_0(w) = \rho\circ f^{-1}(w)$, where

$$\rho(re^{i\theta}) = \frac{D_{-\mu}(re^{i\theta})}{r|f_r|\int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}} = \frac{|f_r|}{rJ(z,f)\int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}, \tag{4.17}$$

is admissible for the family of curves Σ_r. Indeed, let $\gamma_\theta(t) = f(te^{i\theta})$, $r \le t \le 1$, where $\theta \in [0, 2\pi)$ and is fixed. Then

$$\int_{\gamma_\theta} \rho_0(w)\,|dw| = \int_r^1 \rho(te^{i\theta})|f_t(te^{i\theta}|\,dt = 1 \tag{4.18}$$

and since

$$|f_r|^2 = D_{-\mu}(z)J(z,f), \tag{4.19}$$

we see that

$$\mathcal{M}(\Sigma_r) \le \int_{\mathbb{C}} \rho_0^2(w) dm_w = \int\limits_{r<|z|<1} \rho^2(z)J(z,f)\,dm_z = \int_0^{2\pi}\frac{d\theta}{\int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}. \tag{4.20}$$

On the other hand, it is easy to verify that if ρ_0^* is another admissible function, then

$$,\int_{\mathbb{C}} \rho_0^{*2}\,dm_w \ge \int_{\mathbb{C}} \rho_0^2\,dm_w \tag{4.21}$$

and thus

$$\mathcal{M}(\Sigma_r) = \int_0^{2\pi}\frac{d\theta}{\int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t}}. \tag{4.22}$$

In order to show that (4.21) holds, we start with the evident inequality

$$\int_{\gamma_\theta} (\rho_0^*(w) - \rho_0(w))\,|dw| = \int_r^1 (\rho^* - \rho)|f_t|\,dt \ge 0. \tag{4.23}$$

If we set $\rho_0^*(w) = \rho^*(f^{-1}(w)$, and recall from (4.19) that $J(z,f) = |f_r|/r\rho\phi(\theta)$, where

$$\phi(\theta) = \int_r^1 D_{-\mu}(te^{i\theta})\frac{dt}{t},$$

then we get

$$\int\limits_{f(A_r)} (\rho_0^*\rho_0 - \rho_0^2)\,dm_w = \int\limits_{A_r} (\rho^*\rho - \rho^2)J(z,f)\,dm_z$$

$$= \int_0^{2\pi}\int_r^1 (\rho^*\rho - \rho^2)\frac{|f_t|}{\rho\phi(\theta)}\,dt\,d\theta = \int_0^{2\pi}\left(\frac{1}{\phi(\theta)}\int_r^1 (\rho^* - \rho)|f_t|\,dt\right)d\theta \geq 0.$$

It remains to apply the Cauchy–Schwarz inequality to arrive at (4.21) and therefore, to complete the proof of Lemma 4.1. □

Now, let us consider the spiral-like mapping

$$f(z) = \frac{1+|z|}{2}e^{i(\theta + \log|z|)}, \quad z = re^{i\theta}, \tag{4.24}$$

preserving all points of the unit circle. It maps locally quasiconformaly the punctured unit disk $D\setminus\{0\}$ onto the annulus $1/2 < |w| < 1$ and transforms the radial lines into spirals that infinitely winding approach the circle $|w| = 1/2$ and so f has cavitation at $z = 0$. Because the length of each of the spirals approaches the infinity, the modulus of this family goes to zero. By Lemma 4.1, the integral (4.11) should be equal to zero. We can verify it by straightforward computation. Indeed, the complex dilatation of the mapping is given by the formula

$$\mu(z) = e^{2i\theta}\frac{-1 + i(1+r)}{1 + 2r + i(1+r)}, \tag{4.25}$$

and therefore

$$D_{-\mu}(z) = \frac{|1 + e^{-2i\theta}\mu(z)|^2}{1 - |\mu(z)|^2} = \frac{r^2 + (1+r)^2}{r(1+r)}. \tag{4.26}$$

Now we see that

$$\lim_{r\to 0}\int_0^{2\pi} \frac{d\theta}{\int_r^1 \frac{t^2 + (1+t)^2}{t(1+t)}\frac{dt}{t}} = 0 \tag{4.27}$$

and so the sufficient condition (4.11) for cavitation is violated since f has cavitation at $z = 0$. The spiral-like mappings play an important role in the geometric function theory and its applications, see e.g. [12, 13].

Next we study necessary conditions for a mapping f to have cavitation at $z = 0$. Now we will employ the left hand side of the inequality (4.6) together with Proposition 4.2.

Theorem 4.3. *Let f be a homeomorphism defined in Theorem 1.1. If f has cavitation at $z = 0$, then*

$$\left[\iint\limits_{|z|<1} q^2\frac{\left|1 - \mu(z)\frac{\bar z}{z}\right|^2}{1 - |\mu(z)|^2}\,dm_z\right]^{-1} < \infty \tag{4.28}$$

for each admissible function $q(t)$ as in Proposition 4.1.

Proof. If f has cavitation at $z = 0$, then $\lim_{r\to 0}\operatorname{mod} f(A(r,1))$ is finite. The latter and the inequality (4.6) immediately imply (4.28). □

Theorem 4.4. *If f has cavitation at $z=0$, then*

$$\int\limits_0^1 \frac{1}{\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left|1-\mu(te^{i\theta})e^{-2i\theta}\right|^2}{1-|\mu(te^{i\theta})|^2}\,d\theta} \cdot \frac{dt}{t} < \infty. \tag{4.29}$$

Proof. The function $q(t) = 1/I \cdot t\phi(t)$ where

$$I = \int_0^1 \frac{dt}{t\phi(t)}, \quad \phi(t) = \frac{1}{2\pi}\int_0^{2\pi} D_\mu(te^{i\theta})\,d\theta,$$

satisfies the conditions of Proposition 4.1. It remains to transform the integral (4.28) with such q. Indeed, since

$$\begin{aligned} \frac{1}{2\pi}\iint_{|z|<1} q^2 D_\mu(te^{i\theta})t\,dt\,d\theta &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{1}{I^2t^2\phi^2(t)} D_\mu(te^{i\theta})t\,dt\,d\theta \\ &= \int_0^1 \frac{1}{I^2\phi^2(t)}\phi(t)\frac{dt}{t} = \frac{1}{I}, \end{aligned}$$

we see that

$$\iint_{|z|<1} q^2(|z|)\frac{\left|1-\mu(z)\frac{\bar z}{z}\right|^2}{1-|\mu(z)|^2}\,dm_z = \frac{2\pi}{I}.$$

Thus

$$\begin{aligned} \left[\iint_{|z|<1} q^2(|z|)\frac{\left|1-\mu(z)\frac{\bar z}{z}\right|^2}{1-|\mu(z)|^2}\,dm_z\right]^{-1} &= \frac{I}{2\pi} = \frac{1}{2\pi}\int_0^1 \frac{dt}{t\phi(t)} \\ &= \frac{1}{2\pi}\int\limits_0^1 \frac{1}{\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\left|1-\mu(te^{i\theta})e^{-2i\theta}\right|^2}{1-|\mu(te^{i\theta})|^2}\,d\theta} \cdot \frac{dt}{t}. \end{aligned}$$ □

The necessary condition in Theorem 4.4 for cavitation is not sufficient. To construct an example for such a mapping we first clarify the geometric meaning of the integral in formula (4.29).

Lemma 4.2. *Given $0 < r < 1$, let c_r be the family of circles, centered at the origin, that separate the boundary circles of the annulus $A(r)$, and let C_r be the image of c_r under a quasiconformal homeomorphism f, satisfying the Beltrami equation $f_{\bar z} = \mu(z)f_z$. Then the modulus $\mathcal{M}(C_r)$ of the curve family C_r has the formula*

$$\mathcal{M}(C_r) = \int_r^1 \frac{dt}{t\int_0^{2\pi} \frac{|1-\mu(te^{i\theta})e^{-2i\theta}|^2}{1-|\mu(te^{i\theta})|^2}\,d\theta}. \tag{4.30}$$

Proof. The function $\rho_0(w) = \rho\circ f^{-1}(w)$, where

$$\rho(re^{i\theta}) = \frac{D_\mu(re^{i\theta})}{|f_\theta|\int_0^{2\pi} D_\mu(te^{i\theta})\,d\theta} = \frac{|f_\theta|}{r^2J(z,f)\int_0^{2\pi} D_\mu(te^{i\theta})\,d\theta}, \tag{4.31}$$

is admissible for the family of curves C_r. Indeed, let $\gamma_t(\theta) = f(te^{i\theta})$, $r \le t \le 1$, is fixed and $\theta\in[0,2\pi)$. Then the computation shows that

$$\int_{\gamma_t} \rho_0(w)\,|dw| = \int_0^{2\pi} \rho(te^{i\theta})|f_\theta(te^{i\theta}|\,d\theta = 1. \tag{4.32}$$

Since $|f_\theta|^2 = r^2 D_\mu(z) J(z,f)$, we see that

$$\mathcal{M}(C_r) \le \int_{\mathbb{C}} \rho_0^2(w)\, dm_w = \int\limits_{r<|z|<1} \rho^2(z) J(z,f)\, dm_z = \int_r^1 \frac{dt}{t\int_0^{2\pi} D_\mu(te^{i\theta})\, d\theta}. \tag{4.33}$$

On the other hand, it is easy to verify that if ρ_0^* is another admissible function, then

$$\int_{\mathbb{C}} \rho_0^{*2}\, dm_w \ge \int_{\mathbb{C}} \rho_0^2\, dm_w \tag{4.34}$$

and thus

$$\mathcal{M}(C_r) = \int_r^1 \frac{dt}{t\int_0^{2\pi} D_\mu(te^{i\theta})\, d\theta}. \tag{4.35}$$

In order to show that (4.34) holds, we start with the evident inequolity

$$\int_{\gamma_t} (\rho_0^*(w) - \rho_0(w))\, |dw| = \int_0^{2\pi} (\rho^* - \rho)|f_\theta|\, d\theta \ge 0. \tag{4.36}$$

If we set $\rho_0^*(w) = \rho^*(f^{-1}(w)$, and recall that $J(z,f) = |f_\theta|/r^2\rho\psi(t)$, where

$$\psi(t) = \int_0^{2\pi} D_\mu(te^{i\theta})\, d\theta,$$

then we get

$$\begin{aligned}\int\limits_{f(A_r)} (\rho_0^*\rho_0 - \rho_0^2)\, dm_w &= \int\limits_{A_r} (\rho^*\rho - \rho^2) J(z,f)\, dm_z \\ &= \int_0^{2\pi}\int_r^1 (\rho^*\rho - \rho^2)\frac{|f_t|}{\rho t\psi(t)}\, dt\, d\theta \ge 0.\end{aligned}$$

It remains to apply the Cauchy–Schwarz inequality to arrive at (4.34) and therefore, to complete the proof of Lemma 4.2. □

Our goal now is to construct a homeomorphism f of D onto D that is locally quasiconformal in $\overline{D}\setminus\{0\}$, $f(0)=0$, and such that it transforms the family of circles $c_r(\theta) = re^{i\theta}$, $\theta \in (0,2\pi]$, $0<r<1$, to the family of curves $C_r = f(re^{i\theta})$ whose lengths $\gamma(r)$ are not less than $\gamma_0 > 0$ for all $r>0$. Thus in this case we have $\mathcal{M}(C_r) < \infty$ and therefore the necessary condition (4.29) for cavitation is not sufficient.

Proposition 4.3. *There is a homeomorphism $f\colon D \to D$ that is locally quasiconformal in the punctured unit disk $\overline{D}\setminus\{0\}$, $f(0)=0$, whose complex dilatation μ satisfies condition (4.29).*

Proof. We construct the required mapping in the form

$$f(z) := e^{i\theta} R(r,\theta), \quad z = re^{i\theta},\ \ r\in(0,1),\ \ \theta\in[-\pi,\pi] \tag{4.37}$$

for a positive function $R(r,\theta)$ with $R(r,-\pi) = r = R(r,\pi)$ that has small oscillation in the variable θ about the value r for every fixed $r\in(0,1)$. Moreover, $R(r,\theta)\to 0$ as $r\to 0$ uniformly with respect to $\theta\in[-\pi,\pi]$ but the lengths of the images of all circles $|z| = r$ under the mapping f are greater than 2π for all $r\in(0,1)$, furthermore, the lengths converge to ∞ as $r\to 0$. Then by Lemma 3.2 the mapping f satisfies (4.29).

For this purpose, we start with the following basic saw-like piecewise linear $2\pi-$periodic oscillation $\varphi\colon \mathbb{R}\to[-1,1]$ that is similar to the function $\sin\theta$ but more convenient because of $|\dot\varphi(\theta)| = 2/\pi$ for all $\theta\in\mathbb{R}$ except the countable collection of the points of its maximum $\theta_k^+ := \frac{\pi}{2} + 2\pi k$ and minimum $\theta_k^- := -\frac{\pi}{2} + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$.

By means of the basic oscillation we generate the following system of damped oscillations

$$\varphi_{m,n}(\theta) := 2^{-n}\varphi\left(2^m\theta\right), \quad m, n = 1, 2, \ldots \tag{4.38}$$

where indices n and m control the amplitudes and frequencies, respectively. Further, in order to satisfy the requirements for the function $R(r,\theta)$, we apply the special system of such oscillations

$$\varphi_l(\theta) := \varphi_{3l+1,2l+1}(\theta) = 2^{-(2l+1)}\varphi\left(2^{3l+1}\theta\right), \quad l = 1, 2, \ldots. \tag{4.39}$$

The length of the graph of $\varphi_*(\theta) := 2^{-1}\varphi(2\theta)$ over the segment $[-\pi,\pi]$ can be estimated from below by the sum of projections of its linear segments onto the vertical axis as 4 and coincides with such estimate for φ because of the amplitude of φ_* in comparison with φ decreases 2 times and the frequency increases 2 times, too. We see also that the amplitude of each next oscillation φ_{l+1} in comparison with the former φ_l decreases 4 times and the frequency increases 8 times. Thus, the similar estimate of the length of the graph of φ_{l+1} from below over $[-\pi,\pi]$ increases 2 times in comparison with φ_l, i.e., the latter is equal to $2^{l+2} \geq 8 > 2\pi$ for all $l = 1, 2, \ldots$. Note also that this estimate of the length L_l of the graph of φ_l from below over $[-\pi,\pi]$ can be calculated in the explicit integral form:

$$L_l \;\geq\; \int\limits_{-\pi}^{\pi} |\dot{\varphi}_l(\theta)|\, d\theta, \quad l = 1, 2, \ldots. \tag{4.40}$$

Now, let us set $\varphi_0(\theta) \equiv 0$ and obtain the following decreasing sequence

$$R(r_l,\theta) := R_l(\theta) := r_l + \varphi_l(\theta), \quad r_l := 2^{-l}, \;\; l = 0, 1, 2, \ldots \tag{4.41}$$

because of $\max\limits_{\theta\in[-\pi,\pi]} R(r_1,\theta) = 2^{-1} + 2^{-3} = 5/8 < 1$ and, for all $l = 1, 2, \ldots$,

$$\min_{\theta\in[-\pi,\pi]} R(r_l,\theta) = 2^{-l} - 2^{-(2l+1)} > 2^{-(l+1)} + 2^{-(2l+3)} = \max_{\theta\in[-\pi,\pi]} R(r_{l+1},\theta) \tag{4.42}$$

Then we define $R(r,\theta)$ for all $r \in (0,1)$ through the convex combinations

$$R(r,\theta) := \lambda_l(r)\cdot R_l(\theta) + (1-\lambda_l(r))\cdot R_{l+1}(\theta), \tag{4.43}$$

where

$$\lambda_l(r) = \frac{r - r_{l+1}}{r_l - r_{l+1}} = 2^{l+1}(r - r_{l+1}), \quad r \in [r_{l+1}, r_l], \;\; l = 0, 1, 2, \ldots. \tag{4.44}$$

By the definition, we see that for all given r and l

$$\begin{aligned} R(r,\theta) &:= r + \lambda_l(r)\cdot\varphi_l(\theta) + (1-\lambda_l(r))\cdot\varphi_{l+1}(\theta) \\ &= r + \lambda_l(r)\cdot[\varphi_l(\theta) - \varphi_{l+1}(\theta)] + \varphi_{l+1}(\theta), \end{aligned} \tag{4.45}$$

consequently, $R(r,\theta) = r + 2(1-r)\varphi_1(\theta) \to 1$ as $r \to 1$ as well as $R(r,\theta) \leq r + 5\cdot 2^{-(2l+3)} \to 0$ as $r \to 0$ and, moreover,

$$R_r = 1 + 2^{l+1}\cdot[\varphi_l(\theta) - \varphi_{l+1}(\theta)], \tag{4.46}$$

$$R_\theta = \lambda_l(r)\cdot\dot{\varphi}_l(\theta) + (1-\lambda_l(r))\cdot\dot{\varphi}_{l+1}(\theta). \tag{4.47}$$

By (4.46) we have, in particular, that

$$R_r \geq 1 - 2^{l+1}\cdot\left[2^{-(2l+1)} + 2^{-(2l+3)}\right] \geq 1 - \frac{1}{2} - \frac{1}{8} = \frac{3}{8} > 0, \tag{4.48}$$

i.e., the function $R(r,\theta)$ is increasing in the variable $r \in (0,1)$. Thus, by the above remarks the mapping f is a homeomorphism of D onto itself.

Note that the sign of $\dot{\varphi}_{l+1}(\theta)$ coincide with the sign of $\dot{\varphi}_l(\theta)$ on the half of the segments of monotonicity of the oscillation $\varphi_{l+1}(\theta)$ over $[-\pi,\pi]$. Next, $f_\theta = ie^{i\theta}R + e^{i\theta}R_\theta = e^{i\theta}(R_\theta + iR)$ and hence $|f_\theta| \geq R_\theta$. Thus, arguing as in (4.40), we obtain by (4.47) the estimate for the length $L(r)$ of the image of the circle $|z| = r \in [r_{l+1}, r_l]$, under the mapping f from below $L(r) \geq 2^{l+1}$ and hence $L(r) \to \infty$ as $r \to \infty$.

To conclude that f is locally quasiconformal in the punctured unit disc $D \setminus \{0\}$ we first show that the restriction of f in each ring $r_{l+1} < |z| < r_l$, $l = 0,1,2,\ldots$, is a quasiconformal mapping and then by gluing these quasiconformal mappings in each ring $r_l < |z| < 1$, $l = 0,1,2,\ldots$ we obtain the local quasiconformality of f. Note that by the construction f has a.e. the first partial derivatives, consequently, the total differential by the Gehring–Lehto–Men'shov theorem, see e.g. [11] and [18], [17, Theorem III.3.1], and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$, see e.g. [17, (9.4)].

Indeed, by formulas (7.5.10) and (7.5.11) in [15, p. 141], we have that

$$\frac{f_{\bar z}}{f_z} = e^{2i\theta}\frac{irf_r - f_\theta}{irf_r + f_\theta} = e^{2i\theta}\frac{irR_r - (R_\theta + iR)}{irR_r + (R_\theta + iR)} = -e^{2i\theta}\frac{R_\theta - i(R - rR_r)}{R_\theta + i(R + rR_r)}, \tag{4.49}$$

consequently, f is sense-preserving, see [17, I.1.6], since $J_f(z) > 0$ a.e.,

$$|\mu(z)|^2 = \frac{R_\theta^2 + (R - rR_r)^2}{R_\theta^2 + (R + rR_r)^2} = 1 - \frac{4rR_rR}{R_\theta^2 + (R + rR_r)^2} < 1 \quad \text{a.e. in } D \tag{4.50}$$

because $R_r \geq 3 \cdot 2^{-3} > 0$ by (4.48) and, furthermore, by (4.46) and (4.47)

$$|\mu(z)|^2 \leq 1 - \frac{2^2 \cdot 2^{-(l+1)} \cdot 3 \cdot 2^{-3} \cdot 2^{-(l+2)}}{\left[\frac{2}{\pi} \cdot 2^{l+1}\right]^2 + (1+3)^2} \leq 1 - 2^{-2(l+m+3)} \tag{4.51}$$

a.e. in the ring $r_{l+1} < |z| < r_l$, where $m = \max\{1, l\}$ for $l = 0,1,2,\ldots$ Finally, by removability of analytic arcs, see e.g. [17, Theorem I.8.3], the mapping f is quasiconformal in all rings $r_l < |z| < 1$, $l = 0,1,2,\ldots$ □

We complete this section with a criterion on cavitation in the class of mappings whose dilatation coefficients are of the form

$$\mu(z) = k(|z|)\frac{z}{\bar z}, \tag{4.52}$$

where $k(t)$ stands for an arbitrary measurable complex valued function, such that $|k(t)| \leq q(r) < 1$ for $0 < r \leq t \leq 1$ and $|k(t)| \to 1$ as $t \to 0$. For every such coefficient μ the homeomorphic normalized solutions $f\colon D \setminus \{0\} \to D$ to the Beltrami equation have the following explicit representation, see, e.g. [8, p. 82]:

$$f(z) = \frac{z}{|z|}e^{\int_1^{|z|}\frac{1+k(t)}{1-k(t)}\frac{dt}{t}}. \tag{4.53}$$

It immediately leads to the following statement.

Theorem 4.5. *Let μ have the form (4.52). Then the mapping f in (4.53) has cavitation at $z = 0$ if and only if*

$$I(k) = \int_0^1 \operatorname{Re}\frac{1+k(t)}{1-k(t)}\frac{dt}{t} < \infty \tag{4.54}$$

and maps the punctured unit disk $A(0)$ onto the annulus $A(\rho)$ where

$$\log\frac{1}{\rho} = I(k). \tag{4.55}$$

Proof. The mapping f of the form (4.53) has cavitation at $z = 0$ if and only if $\lim_{z\to 0} |f(z)| \neq 0$. The latter holds if and only if $1/D_\mu$ satisfies the Dini condition, i.e.

$$\int_0^1 \frac{1}{D_\mu(te^{i\theta})}\frac{dt}{t} = \int_0^1 \frac{1-|\mu(te^{i\theta})|^2}{|1-\mu(te^{i\theta})e^{-2i\theta}|^2}\frac{dt}{t} = \int_0^1 \operatorname{Re}\frac{1+k(t)}{1-k(t)}\frac{dt}{t} < \infty. \tag{4.56}$$

Now we have that

$$\lim_{r\to 0} |f(z)| = \exp\left\{-\int_0^1 \frac{1-|\mu(te^{i\theta})|^2}{|1-\mu(te^{i\theta})e^{-2i\theta}|^2}\frac{dt}{t}\right\} = \rho < 1, \tag{4.57}$$

and therefore f maps the punctured unit disk $A(0)$ onto the annulus $A(\rho, 1)$, where

$$\log\frac{1}{\rho} = \int_0^1 \operatorname{Re}\frac{1+k(t)}{1-k(t)}\frac{dt}{t}. \tag{4.58}$$

□

Remark 4.2. We see that in the case (4.53) $\lim_{z\to 0} f(z) = 0$ if and only if

$$\lim_{r\to 0}\int_r^1 \operatorname{Re}\frac{1+k(t)}{1-k(t)}\frac{dt}{t} = \lim_{r\to 0}\int_r^1 \frac{1-|\mu(te^{i\theta})|^2}{|1-\mu(te^{i\theta})e^{-2i\theta}|^2}\frac{dt}{t} = +\infty. \tag{4.59}$$

Thus, under the condition (4.59), the mapping f is an automorphism of D and locally quasiconformal in $A(0)$.

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