On the Minkowski content of self-similar random homogeneous iterated function systems

Sascha Troscheit

Abstract. The Minkowski content of a compact set is a fine measure of its geometric scaling. For Lebesgue null sets it measures the decay of the Lebesgue measure of epsilon neighbourhoods of the set. It is well known that self-similar sets, satisfying reasonable separation conditions and non-log commensurable contraction ratios, have a well-defined Minkowski content. When dropping the contraction conditions, the more general notion of average Minkowski content still exists. For random recursive self-similar sets the Minkowski content also exists almost surely, whereas for random homogeneous self-similar sets it was recently shown by Zähle that the Minkowski content exists in expectation.

In this short note we show that the upper Minkowski content, as well as the upper average Minkowski content of random homogeneous self-similar sets is infinite, almost surely, answering a conjecture posed by Zähle. Additionally, we show that in the random homogeneous equicontractive self-similar setting the lower Minkowski content is zero and the lower average Minkowski content is also infinite. These results are in stark contrast to the random recursive model or the mean behaviour of random homogeneous attractors.

Satunnaisten itsesimilaarien homogeenisten iteroitujen funktiojärjestelmien Minkowskin sisältö

Tiivistelmä. Kompaktin joukon Minkowskin sisältö on joukon geometrisen skaalan hienovarainen mitta. Lebesguen nollajoukoille se mittaa joukon epsilon-ympäristöjen Lebesguen mitan vaimenemista. On tunnettua, että itsesimilaareilla joukoilla, joilla on kohtuulliset erillisyysominaisuudet ja logaritmisesti yhteismitattomat kutistussuhteet, on hyvin määritelty Minkowskin sisältö. Jos kutistussuhde-ehdosta luovutaan, on yleisempi Minkowskin keskisisältö edelleen olemassa. Satunnaisille rekursiivisille itsesimilaareille joukoille Minkowskin sisältö on olemassa melkein varmasti, kun taas satunnaisille homogeenisille itsesimilaareille joukoille Zähle on hiljattain osoittanut, että Minkowskin sisältö on olemassa odotusarvoisesti.

Tässä lyhyessä tutkimuksessa ratkaisemme Zählen esittämän konjektuurin osoittamalla, että sekä Minkowskin yläsisältö että satunnaisten homogeenisten itsesimilaarien joukkojen Minkowskin yläkeskisisältö ovat äärettömiä melkein varmasti. Lisäksi osoitamme, että satunnaisessa homogeenisessa yhtäkutistavassa itsesimilaarissa tilanteessa on Minkowskin alasisältö nolla ja Minkowskin alakeskisisältö ääretön. Nämä tulokset ovat täysin vastakkaisia satunnaiselle rekursiiviselle mallille tai satunnaisten homogeenisten kiintojoukkojen käytökselle.

1. Introduction

The s-dimensional Minkowski content of a compact set $K \subset \mathbb{R}^d$ is defined as the limit

$$(1.1) \qquad\qquad \mathcal{M}^s(K) = \lim_{\varepsilon \to 0} \varepsilon^{s-d} \mathcal{L}^d(\langle K \rangle_\varepsilon),$$

where $\langle K \rangle_\varepsilon = \{x \in \mathbb{R}^d \colon \inf_{y \in K} d(x,y) \leqslant \varepsilon\}$ is the (closed) ε-neighbourhood of K. If the limits exist for all $s \geqslant 0$, there exists a critical exponent $s \geqslant 0$ such that $\mathcal{M}^t(K) = 0$ for all $t < s$ and $\mathcal{M}^t(K) = \infty$ for all $t > s$. This critical exponent is known as the Minkowski dimension of K, which coincides with the box-counting dimension of K.

Note that the limit in (1.1) may not exist. Instead, we may take upper and lower limits giving the notions of upper and lower Minkowski content (denoted by $\overline{\mathcal{M}}^s$ and $\underline{\mathcal{M}}^s$, respectively). The upper and lower Minkowski contents have well-defined critical exponents that are referred to as the upper and lower Minkowski dimension (or upper/lower box-counting dimension). If these critical exponents coincide, we speak of the Minkowski dimension of K. However, even if the Minkowski dimension of K exists and therefore is the critical exponent of the Minkowski content, the limit in (1.1) may still not exist. Its existence is therefore a measure of "regularity" and constitutes an interesting property of a sets. We say that a set K is Minkowski measurable (with dimension s) if $\mathcal{M}^s(K) \in (0, \infty)$. For more background on the Minkowski content and its relation to Zeta functions, see [LRZ16].

Self-similar sets. Self-similar sets are compact sets that are invariant under a finite collection of contracting similarities $f_1, \ldots, f_n$. That is, for maps satisfying $|f_i(x) - f_i(y)| = r_i|x - y|$ for all $x, y \in \mathbb{R}^d$, the associated self-similar set F is the unique non-empty compact set that satisfies

$$F = \bigcup_{i=1}^{n} f_i(F).$$

These self-similar sets are the quintessential fractal sets that are widely studied, especially under assumptions that limit the overlaps $f_i(F) \cap f_j(F)$. We refer the reader to [Fal97] and [Fal14] for an overview of dimension theoretic properties.

Let $f_i \colon [0,1] \to [0,1]$ be a finite collection of contracting similarities on the unit interval indexed by $i = 1, \ldots, N$ with contraction rates $r_i \in (0,1)$. Assume that the maps satisfy the separation condition: $f_i([0,1]) \cap f_j([0,1]) = \varnothing$ whenever $i \neq j$. Under these assumptions, Falconer [Fal95] showed the following dichotomy: If two of the similarities have log-incommensurable contraction rates, i.e. $\log r_i / \log r_j \notin \mathbb{Q}$ for some i, j (the non-arithmetic case), then F is Minkowski measurable. Otherwise (the arithmetic case), there exists a non-trivial periodic function g, such that $|\varepsilon^{s-d} \mathcal{L}^s(\langle F \rangle_\varepsilon) - g(-\log \varepsilon)| \to 0$. In particular, the Minkowski content does not exist but the lower and upper Minkowski content are positive and finite. These results extend naturally to self-similar sets in higher dimensions, see [Fal97, Corollary 7.6] and hold under less restrictive separation conditions such as the open set condition[1], see Gatzouras [Gat00]. Further generalisations can be made such as replacing the d-dimensional Lebesgue measure with n-dimensional "curvatures" (or "intrinsic volumes") that capture the lower-order behaviour of fractal sets, see Winter [Win08] for details.

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[1] A collection of contractions (f_i) with attractor F satisfies the *open set condition* if there exists a non-empty open set O such that $f_i(O) \subseteq O$ and $f_i(O) \cap f_j(O) = \varnothing$ whenever $i \neq j$.

While the Minkowski content does not exist in the arithmetic case, the behaviour of $\varepsilon^{s-d} \mathcal{L}^d(\langle F \rangle_\varepsilon)$ is still very regular. To consolidate these behaviours, a slightly weaker definition of content is required, the average Minkowski content.

The average Minkowski content. The average Minkowski content of a compact set K is given by

$$(1.2) \qquad \mathfrak{M}^s(K) = \lim_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \varepsilon^{s-d} \mathcal{L}^d(\langle K \rangle_\varepsilon) \frac{1}{\varepsilon} d\varepsilon .$$

The average Minkowski content is an averaging over the decay of the Lebesgue measure, that preserves the critical exponent of a set if it exists. That is, if the Minkowski dimension of a set K is $s > 0$, then $\mathfrak{M}^t(K) = 0$ for $t < s$ and $\mathfrak{M}^t(K) = \infty$ for $t > s$. Should the small scale behaviour be periodic, as is the case for some self-similar sets, the average Minkowski content still exists. As with the Minkowski content, we write $\underline{\mathfrak{M}}^s$ and $\overline{\mathfrak{M}}^s$ for the upper and lower average Minkowski content, obtained by taking the upper and lower limit in (1.2), respectively.

Random attractors and results. It is a general observation that introducing randomness can homogenise local structures. This is especially visible in stochastically self-similar sets, first considered in [Fal86, Gra87, MW86]. These sets satisfy a similar invariance to deterministic self-similar sets given by

$$F =_d \bigcup f_i(F)$$

where the equality holds in distribution, and the sets as well as the maps are independent. These sets are generally well-behaved, have positive and finite Hausdorff measure for an appropriate gauge function [GMW88] and are almost surely Minkowski measurable [Gat00]. We will refer to this model of randomness as the random recursive model.

Another important model of stochastic self-similarity is that of random homogeneous sets, also known as 1-variable sets (named after the more general notion of V-variable sets [BHS12]). In this model, at every iteration step only one family of functions is chosen (independently of other levels) and applied to every subset. This introduces geometric dependencies and was shown to behave rather differently to the random recursive model. In particular, the almost sure behaviour is determined by a "geometric expectation" $(\exp \mathbb{E}(\log(.)))$ as opposed to the "arithmetic expectation" $(\mathbb{E})$ for random recursive sets. The Hausdorff and Minkowski dimension of random homogeneous attractors are in general strictly smaller than their random recursive analogue [Ham92, RU11, Tro17] and there is no gauge function that gives positive and finite Hausdorff measure for random homogeneous sets [Tro21]. In [Zah20], Zähle investigated the Minkowski and average Minkowski content in the random homogeneous setting. If one assumes a non-arithmetic condition, which is e.g. satisfied if there is positive probability that an IFS is chosen with log-incommensurable contraction ratios, the Minkowski content of the random attractor exists in expectation. Zähle further showed that the average Minkowski content exists in expectation independent of the non-arithmetic condition. We point out that these results are only achieved in expectation, and not almost surely. In fact, the critical exponents no longer agree: the almost sure Minkowski dimension of the random homogeneous attractor is strictly smaller than the expected Minkowski dimension of the attractor. For related work, see also Zähle [Zah11] and Rataj, Winter, and Zähle [RWZ21]. In [Zah20], Zähle further conjectured that the almost sure Minkowski content does not

exist for random homogeneous self-similar sets and in this article we show that this is indeed the case.

In particular we show, under mild separation assumptions, but no assumptions on the non-arithmetic nature of the contractions, that the upper Minkowski and average Minkowski content of random homogeneous self-similar sets is infinite. In the special case where the random functions are equicontractive at each construction level, we additionally show that the lower Minkowski content is zero and that the lower average Minkowski content is infinite.

2. Definitions and results

2.1. Notation. Let $0 < r_{\min} < r_{\max} < 1$ and let S_d be the set of contracting similarities $f \colon \mathbb{R}^d \to \mathbb{R}^d$ that map the closed unit ball $K = B(0,1)$ into itself ($f(K) \subseteq K$) and have contraction ratio bounded above by $r_{\max}$ and below by $r_{\min}$. Equip S_d with the topology of pointwise convergence and write $\mathcal{B}_d$ for its Borel σ-algebra. Let $\Lambda = \bigcup_{k=1}^{\infty} S_d^k$ and let $\mathcal{B}_d^* = \{B \subset \Lambda \colon B \cap S_d^k \in (\mathcal{B}_d)^k, \forall k \in \mathbb{N}\}$ be the natural σ-algebra on Λ. Let $\mathbb{P}_1$ be a probability measure on $(\Lambda, \mathcal{B}_d^*)$.

The product space $(\Omega, \mathcal{B}, \mathbb{P}) = (\Lambda, \mathcal{B}_d^*, \mathbb{P}_1)^{\mathbb{N}}$ where each realisation

$$\Omega \ni \omega = (\omega_1, \omega_2, \dots) = ((f_{\omega_1}^1, f_{\omega_1}^2, .., f_{\omega_1}^{N_{\omega_1}}),\ (f_{\omega_2}^1, \dots, f_{\omega_2}^{N_{\omega_2}}), \dots)$$

is a sequence of N_{ω_i} many similarities denoted by $f_{\omega_i}^j$ for $j \in \{1, \dots, N_{\omega_i}\}$. We call $(\Omega, \mathcal{B}, \mathbb{P})$ a random iterated function system (RIFS). For convenience we will write $\Sigma_\lambda = \{1, \dots, N_\lambda\}$ and $\Sigma(\omega) = \Sigma_{\omega_1} \times \Sigma_{\omega_2} \times \dots$ to refer to the indices of the maps in $\lambda \in \Lambda$ and their infinite codings. For $v \in \Sigma_\lambda$, we write r_λ^v for the contraction ratio of f_λ^v, i.e. $|f_\lambda^v(x) - f_\lambda^v(y)| = r_\lambda^v |x - y|$ for all $x, y \in \mathbb{R}^d$.

To define the homogeneous random attractor F_ω we define the projection $\pi \colon \Omega \times \Sigma(\omega) \to \mathbb{R}^d$, given by

$$\pi((\omega, v)) = \lim_{n \to \infty} f_\omega^{v|n}(0) = \lim_{n \to \infty} f_{\omega_1}^{v_1} \circ \cdots \circ f_{\omega_n}^{v_n}(0).$$

Since all f_λ^w are strict contractions on a compact set, the limit is well-defined. The attractor F_ω is then given by the projection of all words

$$F_\omega = \bigcup_{v \in \Sigma(\omega)} \pi((\omega, v)).$$

Choosing ω with law $\mathbb{P}$, gives rise to the random attractor F_ω.

The set may equivalently be defined as the $\limsup$ set of covers of increasing levels. We set

$$F_\omega^n = \bigcup_{v \in \Sigma(\omega)} f_\omega^{v|n}(K) = \bigcup_{v \in \Sigma(\omega)} f_{\omega_1}^{v_1} \circ \cdots \circ f_{\omega_n}^{v_n}(K)$$

Then $F_\omega \subseteq F_\omega^n$ for all $n \in \mathbb{N}$ since $f_\lambda^j(K) \subseteq K$. Further, $d_H(F_\omega, F_\omega^n) \to 0$ as $n \to \infty$, where d_H is the Hausdorff distance. This gives the alternative definition

$$F_\omega = \lim_{n \to \infty} F_\omega^n = \bigcap_{n \in \mathbb{N}} F_\omega^n = \lim_{n \to \infty} \bigcap_{i=n}^{\infty} \bigcup_{v \in \Sigma(\omega)} f_\omega^{v|n}(K).$$

To study its Minkowski content, we make the following assumption on how images for distinct words are separated.

Definition 2.1. We say that a RIFS $(\Omega, \mathcal{B}, \mathbb{P})$ satisfies the *uniform cylinder separation condition* if there exist $\gamma > 0$ and $x_0 \in \mathbb{R}^d$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$ and for all $v, w \in \Sigma(\omega)$, $n, m \in \mathbb{N}$,

$$|f_\omega^{v|n}(x_0) - f_\omega^{w|m}(x_0)| \geqslant \gamma \min\{r_{\omega_1}^{v_1} \ldots r_{\omega_n}^{v_n}, r_{\omega_1}^{w_1} \ldots r_{\omega_m}^{w_m}\}$$

whenever $v|_k \neq w|_k$ for $k = \min\{n, m\}$.

Several common separation conditions such as the uniform strong separation condition as well as the uniform open set condition[2] satisfy this condition. It is an adaptation of the weak separation condition (as used in [AKT20]) for random sets. In the deterministic setting, the weak separation condition is an important generalisation of the open set condition, and we refer the reader to [KR16] for a discussion. In Lemma 3.5 we show that the uniform open set condition implies the uniform cylinder separation condition.

As a further assumption, we need to ensure that the random iterated function is indeed "random".

Definition 2.2. We say that an RIFS $(\Omega, \mathcal{B}, \mathbb{P})$ is *almost deterministic* if there exists $s \in \mathbb{R}$ such that

$$\mathbb{P}_1 \left\{ \lambda \in \Lambda \colon \sum_{i=1}^{N_\lambda} (r_\lambda^i)^s = 1 \right\} = 1$$

Conversely, an RIFS is *not almost deterministic* if there exists no such $s \in \mathbb{R}$.

Therefore, assuming that our RIFS is not almost deterministic means we get a bona-fide random set where coverings are (almost surely) not geometrically similar. To avoid trivial singleton attractors, and to simplify calculations we will also make the assumption that

$$(2.1) \qquad\qquad 2 \leqslant \operatorname*{ess\,sup}_{\lambda \sim \mathbb{P}_1\ \lambda \in \Lambda} N_\lambda < \infty.$$

Recall that the Minkowski dimension of all random homogeneous sets exists almost surely and coincides with the almost sure Hausdorff dimension irrespective of overlap conditions, see [Tro17]. We refer to this almost sure value as the *essential Minkowski dimension* of the RIFS $(\Omega, \mathcal{B}, \mathbb{P})$ and usually denote it by $s = \operatorname{ess}_{\omega \sim \mathbb{P}} \dim_M F_\omega$.

Our main result for general self-similar random iterated function systems is

Theorem 2.3. *Let F_ω be the attractor of the self-similar random iterated function system $(\Omega, \mathcal{B}, \mathbb{P})$. Write s for its essential Minkowski dimension. Assume that the RIFS satisfies the uniform cylinder separation condition, is not almost deterministic, and satisfies (2.1). Then, almost surely,*

$$\overline{\mathcal{M}}^s(F_\omega) = \infty \quad \text{and} \quad \overline{\mathfrak{M}}^s(F_\omega) = \infty$$

In particular, the attractor F_ω is almost surely not Minkowski measurable and does not have finite average Minkowski content.

If we further restrict the random iterated function system to be equicontractive for every $\lambda \in \Lambda$, we can say more on the Minkowski content and average Minkowski content.

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[2]A RIFS $(\Omega, \mathcal{B}, \mathbb{P})$ satisfies the uniform open set condition if there exists an non-empty open set O such that $\mathbb{P}_1$-almost every IFS $\lambda \in \Lambda$ satisfies the open set condition with O.

Theorem 2.4. *Let F_ω be the attractor of the self-similar random iterated function system $(\Omega, \mathcal{B}, \mathbb{P})$. Write s for its essential Minkowski dimension. Assume that the RIFS satisfies the uniform cylinder separation condition, is not almost deterministic, and satisfies (2.1). Assume further that for $\mathbb{P}_1$-almost all $\lambda \in \Lambda$ there exists r_λ such that $r_\lambda^i = r_\lambda$ for all $1 \leqslant i \leqslant N_\lambda$. Then, almost surely,*

$$\underline{\mathcal{M}}^s(F_\omega) = 0 \quad and \quad \underline{\mathfrak{M}}^s(F_\omega) = \infty.$$

Combining Theorems 2.3 and 2.4 we obtain

Corollary 2.5. *Let F_ω be the attractor of an RIFS satisfying the assumptions of Theorem 2.4. The upper and lower Minkowski content of F_ω are almost surely as divergent as possible*

$$0 = \underline{\mathcal{M}}^s(F_\omega) < \overline{\mathcal{M}}^s(F_\omega) = \infty$$

whereas the average Minkowski content is infinite, $\mathfrak{M}^s(F_\omega) = \infty$, almost surely.

Note that this is in stark contrast to the random recursive model where, almost surely, the stochastically self-similar set is Minkowski measurable, [Gat00]. Note also that this shows that the almost surely behaviour is drastically different to the behaviour in expectation, as analysed by Zähle [Zah20].

V-variable sets and random set with a neck structure. Random homogeneous iterated function systems are a special case of the more general set up of V-variable attractors, proposed by Barnsley et al. [BHS12]. This in turn can be generalised to code trees with a neck structure, see e.g. [JJWW17]. The defining feature of these is a relaxing of the condition that all subtrees at a level have to be identical, as is the case for random homogeneous (or 1-variable) attractors. V-variable fractals are conditioned to have at most V different subtrees at every construction level, whereas code trees with necks are those attractors where there are infinitely many levels (the necks) where all subtrees are identical.

It is this recurrent structure that was used in [Tro21] to show that the Hausdorff measure cannot be positive and finite, regardless of gauge functions. It appears as though there are no barriers to extending the observations in this paper to V-variable attractors and code trees with necks, and we conjecture that they, too, have infinite upper Minkowski and upper average Minkowski content.

3. Proofs

As remarked above, the essential Hausdorff and Minkowski dimensions of homogeneous random self-similar sets coincide and are given by the unique s for which

$$\mathbb{E}_{\mathbb{P}_1}\left(\log \sum_{i=1}^{N_\lambda} (r_\lambda^i)^s\right) := \int_\Lambda \log \sum_{i=1}^{N_\lambda} (r_\lambda^i)^s \, d\,\mathbb{P}_1(\lambda) = 0,$$

see e.g. [Ham92, Tro17]. To ease notation, we will refer to the Hutchinson sum above by

$$\mathfrak{S}_\lambda^s := \sum_{i=1}^{N_\lambda} (r_\lambda^i)^s.$$

If the random iterated function system is not almost deterministic, $\log \mathfrak{S}_\lambda^s$ is a random variable with mean 0 and there exists positive probability that $\log \mathfrak{S}_\lambda^s \neq 0$, i.e. it has

positive variance. To show that its variance is also finite, consider

$$\mathbb{E}_{\mathbb{P}_1}\left(\left(\log\sum_{v=1}^{N_\lambda}(r_{\omega_1}^v)^s\right)^2\right) \leqslant \max\left\{(\log r_{\min}^s)^2, (\log(Nr_{\max}^s))^2\right\} < \infty.$$

Recall the Lyapunov Central Limit Theorem (CLT) and the law of the iterated logarithm.

Theorem 3.1. (Lyapunov Central Limit Theorem) *Let X_n be a sequence of square integrable random variables with mean m_n and variance $v_n > 0$. Assume that there exists $\delta > 0$ such that*

$$\lim_{n\to\infty}\left(\sqrt{\sum_{i=1}^n v_i}\right)^{-(2+\delta)} \cdot \sum_{i=1}^n \mathbb{E}(|X_i - m_i|^{2+\delta}) = 0.$$

Then,

$$\frac{1}{\sqrt{\sum_{i=1}^n v_n}}\sum_{i=1}^n(X_i - m_i)$$

converges in distribution to the normal distribution with mean 0 and variance 1.

For a proof and detailed discussion see, for example, [Bau96, §28]. Note that the theorem above makes the assumption that the variance is positive for all n. This can without loss of generality be relaxed to $v_n = 0$ for some n, so long as $\sum v_n \to \infty$. This is because if $\nu_n = 0$ then $X_n - m_n = 0$ almost surely.

We will also need the law of the iterated logarithm (LIL).

Theorem 3.2. (Wittmann Law of the Iterated Logarithm [Wit85]) *Let X_n be a sequence of square integrable random variables with mean m_n and variance $v_n > 0$. Assume that the sequence satisfies*

$$\sum_{i=1}^\infty \frac{\mathbb{E}(|X_i|^p)}{(2s_n \log\log s_n)^{p/2}} < \infty$$

for some $2 < p \leqslant 3$, where $s_n = \sum_{i=1}^n v_i$. Assume further that $s_n \to \infty$ and $\limsup_{n\to\infty} s_{n+1}/s_n < \infty$. Then,

$$\limsup_{n\to\infty}\frac{\sum_{i=1}^n X_i}{\sqrt{2s_n \log\log s_n}} = 1$$

and

$$\liminf_{n\to\infty}\frac{\sum_{i=1}^n X_i}{\sqrt{2s_n \log\log s_n}} = -1.$$

Again, we may let $v_n = 0$ for some n, given that this does not affect the limiting behaviour of the sum. We further note that for small n, the value of $\log\log n$ is not defined. Since we are only interested in limits, we may assume n is large enough such that this is well-defined.

We note that any sequence of random variables for which X_n and s_n are bounded immediately satisfies the conditions of both theorems.

3.1. Proof of Theorem 2.3. To prove the Theorem 2.3, we first construct a random measure μ_ω on the symbolic space $\Sigma(\omega)$.

Recall that the Hutchinson sum satisfies $\mathbb{E}(\log\mathfrak{S}_{\omega_1}^s) = 0$, where s is the essential Minkowski dimension. The fact that $s > 0$ follows directly from the assumptions

that $r_\lambda^i \geqslant r_{\min}$ and $\mathbb{P}_1(N_\lambda \geqslant 2) > 0$. The argument is standard and left to the reader. Note that the assumption that the RIFS is not almost deterministic is equivalent, by definition, to $\mathfrak{S}_\lambda^s$ not being a constant $\mathbb{P}_1$-almost surely. Equivalently, the variation of $\mathfrak{S}_\lambda^s$ is positive.

To every letter $i \in \Sigma_\lambda$ we associate probability $p_\lambda^i = (r_\lambda^i)^s / \mathfrak{S}_\lambda^s$. Then

$$\sum_{i \in \Sigma_\lambda} p_\lambda^i = \sum_{i \in \Sigma_\lambda} \frac{(r_\lambda^i)^s}{\mathfrak{S}_\lambda^s} = 1.$$

Hence, the measure induced by setting $\mu_\omega([v|_n]) = \prod_{i=1}^n p_{\omega_i}^{v_i}$ for cylinder $[v|_n] = \{w \in \Sigma(\omega) : (\forall 1 \leqslant i \leqslant n) w_i = v_i\}$ is a bona fide probability measure on $\Sigma(\omega)$.

Write

$$\overline{r}_\lambda = \exp \sum_{i=1}^{N_\lambda} p_\lambda^i \log r_\lambda^i$$

and

$$v_\lambda = \sum_{i=1}^{N_\lambda} p_\lambda^i (\log r_\lambda^i - \log \overline{r}_\lambda)^2.$$

The quantity $\overline{r}_\lambda$ is the geometric mean of the contraction rates with respect to choosing letters $i \in \Sigma_\lambda$ with probabilities $\{p_\lambda^i\}$. Observe that $\log \overline{r}_\lambda$ is the (arithmetic) mean of the logarithms of the contraction rates. The second quantity v_λ then denotes the variance of the logarithm of the contraction rate with respect to the same measure on Σ_λ. Note that v_λ may be zero. However, because we are considering systems which are not almost deterministic, there exists positive probability that $v_\lambda > 0$. This further implies that for generic $\omega \in \Omega$ the sum $\sum_{i=1}^n v_{\omega_i}$ eventually grows faster than ηn for some constant $\eta > 0$. The boundedness of the contraction ratios further imply

$$\lim_{k \to \infty} \frac{1}{\left(\sum_{k=1}^n v_{\omega_i}\right)^3} \sum_{k=1}^n \sum_{i=1}^{N_\lambda} p_\lambda^i |\log r_\lambda^i - \log \overline{r}_\lambda|^3 = 0$$

for generic $\omega \in \Omega$ and so the sum $\sum_{k=1}^n \log r_{\omega_k}^{v_n}$ satisfies the Lyapunov Central Limit Theorem with respect to μ_ω.

Now write M_n for the set

$$M_n = \left\{ v \in \Sigma(\omega) : \prod_{i=1}^n r_{\omega_i}^{v_i} \in \left[\prod_{i=1}^n \overline{r}_{\omega_i} \cdot e^{-\sqrt{\sum_{i=1}^n v_{\omega_i}}}, \prod_{i=1}^n \overline{r}_{\omega_i} \cdot e^{\sqrt{\sum_{i=1}^n v_{\omega_i}}} \right] \right\}$$

noting that

$$M_n = \left\{ v \in \Sigma(\omega) : \frac{\sum_{i=1}^n (\log r_{\omega_i}^{v_i} - \log \overline{r}_{\omega_i})}{\sqrt{\sum_{i=1}^n v_{\omega_i}}} \in [-1, 1] \right\}.$$

By the Lyapunov Central Limit Theorem,

$$\mu_\omega(M_n) \to (2\pi)^{-1/2} \int_{-1}^1 \exp(-x^2/2)dx =: q \quad \text{as } n \to \infty.$$

Hence, for large enough n, we have

$$q/2 \leqslant \mu_\omega(M_n) = \sum_{v \in M_n} \mu_\omega([v|_n]) = \sum_{v \in M_n} \frac{(r_{\omega_1}^{v_1} \dots r_{\omega_n}^{v_n})^s}{\mathfrak{S}_{\omega_1}^s \dots \mathfrak{S}_{\omega_n}^s} \leqslant \#M_n \frac{(\overline{r}_{\omega_1} \dots \overline{r}_{\omega_n})^s}{\mathfrak{S}_{\omega_1}^s \dots \mathfrak{S}_{\omega_n}^s} e^{\sqrt{\sum_{i=1}^n v_{\omega_i}}}$$

and so

$$\#M_n \geqslant \frac{q}{2} \frac{\mathfrak{S}_{\omega_1}^s \ldots \mathfrak{S}_{\omega_n}^s}{(\overline{r}_{\omega_1} \ldots \overline{r}_{\omega_n})^s} \cdot \exp\left[-\left(\sum_{i=1}^n v_{\omega_i} \right)^{1/2} \right].$$

Recall that we are considering generic $\omega \in \Omega$ and that $\sum_{i=1}^n v_{\omega_i} \geqslant \eta n$ for some $\eta > 0$ and large enough $n \in \mathbb{N}$. It is straightforward to show that all assumptions in the law of the iterated logarithm are satisfied for generic $\omega \in \Omega$. Thus there exists a constant $C > 0$ and a subsequence n_k such that $\mathfrak{S}_{\omega_{n_k}}^s \geqslant \exp(C\sqrt{n_k \log\log n_k})$. As a consequence,

$$\#M_{n_k} \geqslant C' \exp\left(C\sqrt{n_k \log\log n_k} - \sqrt{\eta\, n_k} \right) (\overline{r}_{\omega_1} \ldots \overline{r}_{\omega_{n_k}})^{-s}.$$

We can use the uniform cylinder separation condition to obtain a lower bound on the cardinality of a sufficiently separated set.

Lemma 3.3. *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a RIFS as in Theorem 2.3. Then there exists a constant $C_d > 0$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$ the following holds. Let A be a finite collection finite length words that are not prefixes of each other, i.e.*

$$A \subseteq \bigcup_{n=1}^\infty \Sigma_{\omega_1} \times \cdots \times \Sigma_{\omega_n} \quad \text{with distinct } v, w \in A \Rightarrow v|_n \neq w|_n, \ n = \min\{|v|, |w|\}.$$

Then there exists a $(\gamma \min_{v \in A} r_\omega^v)$-separated set $F_A \subset F_\omega$ with $\#F_A \geqslant C_d \#A$.

Proof. Let A be given and assume that $\omega \in \Omega$ is generic. We define A^* by considering descendants of A such that the associated contraction is comparable to the minimal contraction in A. Concretely, writing 1^k for the word $(1, 1, \ldots, 1)$ of length k and $\underline{r}_A = \min_{v \in A} r_\omega^v$,

$$A^* = \big\{ v1^k \in \Sigma_{\omega_1} \times \cdots \times \Sigma_{\omega_{|v|+k}} : v \in A \quad \text{and}$$
$$r_{\omega_1}^{v_1} \ldots r_{\omega_{|v|}}^{v_{|v|}} r_{\omega_{|v|+1}}^1 \ldots r_{\omega_{|v|+k+1}}^1 < \underline{r}_A \leqslant r_{\omega_1}^{v_1} \ldots r_{\omega_{|v|}}^{v_{|v|}} r_{\omega_{|v|+1}}^1 \ldots r_{\omega_{|v|+k}}^1 \big\}.$$

We note that since the contractions are uniformly bounded from above, there must exist such $k \geqslant 0$ for every $v \in A$. Further, since elements in A are not prefixes of each other, A^* must also have this property and $\#A^* = \#A$. Additionally, the contractions are uniformly bounded from below, which implies that the contraction ratios satisfy

$$\underline{r}_A \leqslant r_\omega^w \leqslant \frac{\underline{r}_A}{r_{\min}}.$$

for all $w \in A^*$. The uniform cylinder separation condition implies that there exists $x_0 \in \mathbb{R}^d$ such that

$$|f_\omega^v(x_0) - f_\omega^w(x_0)| \geqslant \gamma \underline{r}_A$$

where $v, w \in A^*$. Therefore the balls $B(f_\omega^v(x_0), \gamma \underline{r}_A/2)$ are pairwise disjoint.

Recall that by assumption $F_\omega \subset K = B(0, 1)$. Therefore, $F_\omega \subset B(x_0, 1 + |x_0|)$ and for all $v \in A^*$,

$$F_\omega \cap B\left(f_\omega^v(x_0), \frac{\underline{r}_A}{r_{\min}}(1 + |x_0|) \right) \neq \varnothing.$$

We can use a volume argument to obtain an upper bound on how many of those balls may intersect F_ω, hence giving an upper bound to the cardinality of A^*. Let

$A^*(v) = \{w \in A^* \colon |f_\omega^v(x_0) - f_\omega^v(x_0)| \leqslant 3(\underline{r}_A/r_{\min})(1 + x_0)\}$ Then,

$$
\begin{aligned}
V_d\left(4\frac{\underline{r}_A}{r_{\min}}(1 + |x_0|)\right)^d &= \mathcal{L}^d\left(B\left(f_\omega^v(x_0),\ 4\frac{\underline{r}_A}{r_{\min}}(1 + |x_0|)\right)\right) \\
&\geqslant \mathcal{L}^d\left(\bigcup_{w \in A^*(v)} B\left(f_\omega^w(x_0),\ \gamma\underline{r}_A\right)\right) \\
&= \sum_{w \in A^*(v)} \mathcal{L}^d\left(B\left(f_\omega^w(x_0),\ \gamma\underline{r}_A\right)\right) \\
&= \#A^*(v)V_d(\gamma\underline{r}_A)^d
\end{aligned}
$$

and so

$$
\#A^*(v) \leqslant \left(4\frac{1 + |x_0|}{r_{\min}\gamma}\right)^d =: C_0.
$$

Therefore, we conclude that there exists a subset of F_ω with cardinality $\#A/C_0$ consisting of points separated by $(\underline{r}_A/r_{\min})(1 + |x_0|) \geqslant \gamma\underline{r}_A$, as claimed. $\qquad\square$

By the uniform cylinder separation condition and Lemma 3.3, there are at least $C_0\#M_{n_k}$ many elements in F_ω that are $\gamma' \prod_{i=1}^{n_k} \overline{r}_{\omega_i} \cdot e^{-\sqrt{\sum_{i=1}^{n_k} v_{\omega_i}}}$ separated, since elements in M_{n_k} are distinct. Hence, setting $\varepsilon_{n_k} = \gamma' \prod_{i=1}^{n_k} \overline{r}_{\omega_i} \cdot e^{-\sqrt{\sum_{i=1}^{n_k} v_{\omega_i}}}$ we can find a lower bound by finding the Lebesgue measure of a disjoint union,

$$
\begin{aligned}
\varepsilon_{n_k}^{s-d} \mathcal{L}^d(\langle F_\omega\rangle_{\varepsilon_{n_k}}) &\geqslant \varepsilon_{n_k}^{s-d} \mathcal{L}^d\left(\bigcup_{v \in M_{n_k}} B(f_\omega^{v|_{n_k}}(0), \varepsilon_{n_k})\right) \geqslant \varepsilon_{n_k}^{s-d} V_d \varepsilon_{n_k}^d C_0 \#M_{n_k} \\
&\geqslant \varepsilon_{n_k}^s V_d C_0 C' \exp\left(C\sqrt{n_k \log\log n_k} - \gamma\sqrt{n_k}\right)(\overline{r}_{\omega_1}\ldots\overline{r}_{\omega_{n_k}})^{-s} \\
&\geqslant \gamma'^s V_d C_0 C' \frac{1}{2^s} e^{-\gamma(1+s)\sqrt{n_k}} e^{C\sqrt{n_k \log\log n_k}} \\
&\geqslant C'' e^{(C/2)\sqrt{n_k \log\log n_k}}.
\end{aligned}
$$

Hence,

$$
(3.1) \qquad \varepsilon_{n_k}^{s-d} \mathcal{L}^d(\langle F_\omega\rangle_{\varepsilon_{n_k}}) \to \infty \quad \text{along the subsequence} \quad n_k \to \infty.
$$

This shows that $\overline{\mathcal{M}}^s(F_\omega) = \infty$.

We now show that the upper average Minkowski content also diverges. We define $\delta_{n_k} = \gamma' \prod_{i=1}^{n_k} \overline{r}_{\omega_i} \cdot e^{-\sqrt{\sum_{i=1}^{n_k} v_{\omega_i}}}$. Then,

$$
\begin{aligned}
\frac{1}{|\log \delta_{n_k}/2|} &\int_{\delta_{n_k}/2}^1 \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega\rangle_\varepsilon)\frac{1}{\varepsilon}d\varepsilon \\
&\geqslant \frac{1}{|\log \delta_{n_k}/2|} \int_{\delta_{n_k}/2}^{\delta_{n_k}} (\delta_{n_k})^{s-d} \mathcal{L}^d\left(\bigcup_{v \in M_{n_k}} B(f_\omega^{v|_{n_k}}(0), \delta_{n_k}/2)\right)\frac{1}{\varepsilon}d\varepsilon \\
&\geqslant \frac{1}{|\log \delta_{n_k}/2|}(\delta_{n_k})^{s-d} V_d(\delta_{n_k}/2)^d \#M_{n_k} \frac{\delta_{n_k} - \delta_{n_k}/2}{\delta_{n_k}} \\
&\geqslant C_1 \frac{1}{n_k}(\delta_{n_k})^s \#M_{n_k} \geqslant C_2 \frac{1}{n_k} e^{(C/2)\sqrt{n_k \log\log n_k}},
\end{aligned}
$$

where n_k is the same subsequence as in (3.1). We conclude that $\overline{\mathfrak{M}}^s(F_\omega) = \infty$, proving our claim. □

3.2. Proof of Theorem 2.4. By assumption there exist r_λ such that $r_\lambda^v = r_\lambda$ for all $v \in \Sigma_\lambda$. This greatly simplifies the expression for the number and size of covering sets of the attractor. Indeed, the k level set F_ω^k is a cover consisting of exactly $N_{\omega_1}N_{\omega_2}\ldots N_{\omega_k}$ balls of diameter exactly $r_{\omega_1}r_{\omega_2}\ldots r_{\omega_k}$. By Lemma 3.3 this also means that F_ω contains at least $C_0 N_{\omega_1}N_{\omega_2}\ldots N_{\omega_k}$ many points separated by $\gamma' r_{\omega_1}r_{\omega_2}\ldots r_{\omega_k}$. The Hutchinson sum reduces to $\mathfrak{S}_\lambda^s = N_\lambda(r_\lambda)^s$ in the equicontractive setting.

Since the sizes of n level cylinders are the same, we can get improved approximations for the Lebesgue measure of $\mathcal{L}^d(\langle F_\omega\rangle_\varepsilon)$. Fix a realisation $\omega \in \Omega$ and size $0 < \varepsilon < 1$. Set n such that $r_{\omega_1}r_{\omega_2}\ldots r_{\omega_n} < \varepsilon \leqslant r_{\omega_1}\ldots r_{\omega_{n-1}}$.

Recall that by definition

$$F_\omega = \bigcap_{k=1}^\infty F_\omega^k \subseteq F_\omega^m \quad \text{and so} \quad \langle F_\omega\rangle_\varepsilon \subseteq \langle F_\omega^m\rangle_\varepsilon$$

for all $m \in \mathbb{N}$. Since F_ω^n consists of $N_{\omega_1}\ldots N_{\omega_n}$ (possibly overlapping) images of the unit ball K,

$$\mathcal{L}^d(\langle F_\omega\rangle_\varepsilon) \leqslant \mathcal{L}^d(\langle F_\omega^n\rangle_\varepsilon) \leqslant \mathcal{L}^d\left(\bigcup_{i=1}^{N_{\omega_1}\ldots N_{\omega_n}} B(x_i, r_{\omega_1}\ldots r_{\omega_n} + \varepsilon)\right)$$

$$= N_{\omega_1}\ldots N_{\omega_n} \cdot V_d \cdot (r_{\omega_1}\ldots r_{\omega_n} + \varepsilon)^d$$

$$\leqslant N_{\omega_1}\ldots N_{\omega_n} \cdot V_d \cdot (r_{\omega_1}\ldots r_{\omega_n} + r_{\omega_1}\ldots r_{\omega_{n-1}})^d$$

$$\leqslant \left(1 + \frac{1}{r_{\min}}\right)^d V_d \cdot \prod_{i=1}^n r_{\omega_i}^d N_{\omega_i}.$$

Conversely F_ω contains $C_0 N_{\omega_1}\ldots N_{\omega_k}$ many points that are separated by $\gamma' r_{\omega_1}\ldots r_{\omega_k}$ for some uniform $\gamma' \leqslant 1$. Hence, for $\varepsilon \leqslant \gamma' r_{\omega_1}\ldots r_{\omega_k}$,

$$\mathcal{L}^d(\langle F_\omega\rangle_\varepsilon) \geqslant C_0 N_{\omega_1}\ldots N_{\omega_k} V_d \,\varepsilon^d.$$

In particular, for n such that $r_{\omega_1}\ldots r_{\omega_n} < \varepsilon \leqslant r_{\omega_1}\ldots r_{\omega_{n-1}}$, we let $k \leqslant n-2$ be the largest integer such that $k \leqslant n - \lceil\log(\gamma')/\log(r_{\max})\rceil - 1$. Then,

$$\frac{r_{\omega_1}\ldots r_{\omega_{n-1}}}{r_{\omega_1}\ldots r_{\omega_k}} = r_{\omega_{k+1}}\ldots r_{\omega_{n-1}} \leqslant r_{\max}^{n-k-1} \leqslant \gamma'$$

and so

$$\varepsilon \leqslant r_{\omega_1}\ldots r_{\omega_{n-1}} \leqslant \gamma' r_{\omega_1}\ldots r_{\omega_k}$$

as required. This gives

$$\mathcal{L}^d(\langle F_\omega\rangle_\varepsilon) \geqslant C_0 N_{\omega_1}\ldots N_{\omega_k} V_d \,\varepsilon^d$$

$$\geqslant C_0 N_{\omega_1}\ldots N_{\omega_k} V_d (r_{\omega_1}\ldots r_{\omega_{n-1}})^d$$

$$\geqslant C_0 V_d \,(\text{ess sup } N_\lambda)^{-(n-k-1)} \prod_{i=1}^n r_{\omega_i}^d N_{\omega_i}.$$

Thus, almost surely, there exists a universal constant $C_1 > 0$ such that

$$\frac{1}{C_1}\prod_{i=1}^n r_{\omega_i}^d N_{\omega_i} \leqslant \mathcal{L}^d(\langle F_\omega\rangle_\varepsilon) \leqslant C_1 \prod_{i=1}^n r_{\omega_i}^d N_{\omega_i} \quad \text{for } n \text{ s.t.} \quad \prod_{i=1}^n r_{\omega_i} < \varepsilon \leqslant \prod_{i=1}^{n-1} r_{\omega_i}.$$

Equivalently[3],

$$\mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) \approx \prod_{i=1}^n r_{\omega_i}^d N_{\omega_i} = \exp \sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s \quad \text{for } n \text{ s.t.} \quad \varepsilon \approx \prod_{i=1}^n r_{\omega_i}.$$

Again, let $s > 0$ be the unique exponent such that $\mathbb{E}_{\mathbb{P}_1}(\log \mathfrak{S}_\lambda^s) = 0$ and recall that the RIFS is not almost deterministic. In particular this means that $v = \mathrm{Var}(\log \mathfrak{S}_\lambda^s) > 0$. Hence $\sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s$ is a symmetric random walk. It immediately follows that

$$-\infty = \liminf_{n \to \infty} \sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s < \limsup_{n \to \infty} \sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s = \infty.$$

and so

$$\overline{\mathcal{M}}^s = \limsup_{\varepsilon \to 0} \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) = \infty \quad \text{and} \quad \underline{\mathcal{M}}^s = \liminf_{\varepsilon \to 0} \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) = 0,$$

proving the first conclusion.

To show that the lower average Minkowski content also diverges to infinity we establish that the average Minkowski content behaves like the arithmetic average of the content at geometric scales. Let $\delta > 0$ be given and set k such that $\prod_{i=1}^k r_{\omega_i} \approx \delta$. Then,

$$\frac{1}{|\log \delta|} \int_\delta^1 \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) \frac{1}{\varepsilon} d\varepsilon = \frac{1}{|\log \delta|} \sum_{n=1}^k \int_{\prod_{i=1}^n r_{\omega_i}}^{\prod_{i=1}^{n-1} r_{\omega_i}} \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) \frac{1}{\varepsilon} d\varepsilon$$

$$\approx \frac{-1}{\log \prod_{i=1}^k r_{\omega_i}} \sum_{n=1}^k \exp\left(\sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s\right),$$

where the last line follows as $\varepsilon \approx \prod_{i=1}^n r_{\omega_i} \approx \prod_{i=1}^n r_{\omega_i} - \prod_{i=1}^{n-1} r_{\omega_i}$.

We can further bound the integral by noting that $r_{\min} \leqslant r_{\omega_i} \leqslant r_{\max}$ and we have

$$(3.2) \qquad \frac{1}{|\log \delta|} \int_\delta^1 \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) \frac{1}{\varepsilon} d\varepsilon \approx \frac{1}{k} \sum_{n=1}^k \exp\left(\sum_{i=1}^n \log \mathfrak{S}_{\omega_i}^s\right).$$

This shows again that the upper limit is infinite, since by the central limit theorem, there exist infinitely many k such that $\sum_{i=1}^k \log \mathfrak{S}_{\omega_i}^s > \sqrt{k}$.

To show that the lower limit is also infinite requires a little more effort and we need the following lemma on random walks.

Lemma 3.4. *Let $0 < t < \frac{1}{2}$. Write $W_n = \sum_{i=1}^n X_i$ for a random walk with i.i.d. increments. Assume that $\mathbb{E}(X_i) = 0$ and $0 < \mathrm{Var}(X_i) < \infty$. Let n_m be the unique integer sastifying $e^m \leqslant n_m < e^m + 1$. Then, almost surely, there exists (random) $m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$ there exists $k \in [n_m, n_{m+1} - 1]$ with $W_k > k^t$.*

---

[3]Let $a(x) = a(x, \omega)$ and $b(x) = b(x, \omega)$ be (random) functions. We write $a(x) \approx b(x)$ if there exists a constant $C > 0$ independent of x and ω, such that $0 < 1/C < a(x)/b(x) < C < \infty$ for all x and almost all ω.

Proof. We estimate

$$
\begin{aligned}
&\mathbb{P}(W_k \leqslant k^t \colon \forall k \in [n_m, n_{m+1} - 1]) \\
&= \mathbb{P}\left(W_k \leqslant k^t \colon \forall k \in [n_m, n_{m+1} - 1] \mid W_{n_m} > n_m^t \right) \cdot \mathbb{P}(W_{n_m} > n_m^t) \\
&\qquad + \mathbb{P}\left(W_k \leqslant k^t \colon \forall k \in [n_m, n_{m+1} - 1] \mid W_{n_m} \leqslant n_m^t \right) \cdot \mathbb{P}(W_{n_m} \leqslant n_m^t) \\
&\leqslant \mathbb{P}\left(W_k \leqslant k^t \colon \forall k \in [n_m, n_{m+1} - 1] \mid W_{n_m} \leqslant n_m^t \right) \\
&\leqslant \mathbb{P}\left(W_k \leqslant (n_{m+1} - 1)^t - n_m^t \colon \forall k \in [0, n_{m+1} - 1 - n_m] \right) \\
&= 1 - \mathbb{P}\left(\sup\{W_k \colon k \in [0, n_{m+1} - 1 - n_m]\} > (n_{m+1} - 1)^t - n_m^t \right)
\end{aligned}
$$

and using the reflection principle gives

$$
\begin{aligned}
&= 1 - 2\,\mathbb{P}\left(W_{n_{m+1}-1-n_m} > (n_{m+1} - 1)^t - n_m^t \right) \\
&= \mathbb{P}\left(\left| W_{n_{m+1}-1-n_m} \right| \leqslant (n_{m+1} - 1)^t - n_m^t \right) \\
&\leqslant \frac{2}{\sqrt{2\pi v_m}} \int_{-(n_{m+1}-1)^t + n_m^t}^{(n_{m+1}-1)^t - n_m^t} \exp\left(-\frac{x^2}{2v_m} \right) dx = 2\,\mathrm{Erf}\left(\tfrac{(n_{m+1}-1)^t - n_m^t}{\sqrt{2v_m}} \right),
\end{aligned}
$$

where $v_m = (n_{m+1} - 1 - n_m)\,\mathrm{Var}(X)$ and m is assumed sufficiently large for the Gaussian approximation to hold. Then, using Taylor series,

$$
\begin{aligned}
&= \frac{2\sqrt{2}\left((n_{m+1} - 1)^t - n_m^t \right)}{\sqrt{\pi v_m}} + O\left(\frac{(n_{m+1} - 1)^t - n_m^t}{\sqrt{2v_m}} \right)^3 \\
&\leqslant \frac{4}{\sqrt{\pi\,\mathrm{Var}(X)}} \cdot \frac{(n_{m+1} - 1)^t - n_m^t}{\sqrt{n_{m+1} - 1 - n_m}} \leqslant \frac{4}{\sqrt{\pi\,\mathrm{Var}(X)}} \frac{e^{tm}(e^t - 1)}{\sqrt{e^m(e - 2e^{-m})}} \\
&\leqslant C e^{-(\frac{1}{2} - t)m}
\end{aligned}
$$

for some uniform $C > 0$. Thus, the probability that W_k does not exceed k^t in $[n_{m+1}, n_m - 1]$ is summable in m. Hence, by the Borel–Cantelli lemma there are only finitely such m. This proves the lemma. $\qquad\square$

We now show that the right hand side of (3.2) diverges to infinity using Lemma 3.4. Fix a generic $\omega \in \Omega$ and let $t < 1/2$. Let n_m be the unique integer satisfying $e^m \leqslant n_m < e^m + 1$. Applying Lemma 3.4, we get

$$
(3.3) \qquad \sum_{k=n_m}^{n_{m+1}-1} \log \mathfrak{S}_{\omega_k}^s \geqslant \sup_{k \in [n_m, n_{m+1}-1]} k_m^t \geqslant n_m^t \geqslant e^{mt}
$$

for all large enough $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ and let m be such that $k \in [n_m, n_{m+1} - 1]$. Then, using (3.3),

$$\frac{1}{k} \sum_{j=1}^{k} \exp\left(\sum_{i=1}^{j} \log \mathfrak{S}_{\omega_i}^s\right) \geqslant \frac{1}{n_{m+1}-1} \sum_{j=1}^{n_m} \exp\left(\sum_{i=1}^{j} \log \mathfrak{S}_{\omega_i}^s\right)$$

$$\geqslant \frac{1}{n_{m+1}-1} \exp\left(\sum_{i=n_{m-1}}^{n_m-1} \log \mathfrak{S}_{\omega_i}^s\right)$$

$$\geqslant e^{-(m+1)} \exp\left(e^{mt}\right) \to \infty$$

as $m \to \infty$. But then, using (3.2),

$$\underline{\mathfrak{M}}^s(F_\omega) = \liminf_{\delta \to 0} \frac{1}{|\log \delta|} \int_\delta^1 \varepsilon^{s-d} \mathcal{L}^d(\langle F_\omega \rangle_\varepsilon) \frac{1}{\varepsilon} d\varepsilon = \infty,$$

showing that $\mathfrak{M}^s(F_\omega) = \infty$ almost surely. $\square$

3.3. Separation conditions. In this last section we prove that the uniform open set condition implies the uniform cylinder separation condition.

Lemma 3.5. *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a RIFS that satisfies the uniform open set condition. Then $(\Omega, \mathcal{B}, \mathbb{P})$ satisfies the uniform cylinder separation condition.*

Proof. Let O be the set guaranteed by the uniform strong open set condition and let $x_0 \in O$ and $\rho > 0$ be small enough such that the closed ball $B(x, \rho)$ is contained in O. Let $v, w \in \Sigma(\omega)$ be such that $v|_k \neq w|_k$ but $v|_{k-1} = w|_{k-1}$ for some $k \in \mathbb{N}$. By the uniform open set condition, $f_\omega^{v|n}(O) \cap f_\omega^{w|m}(O) = \varnothing$ for all $m, n \geqslant k$ since $v_k \neq w_k$. Now, $B(x, \rho) \subset O$ and we further have $f_\omega^{v|n}(B(x_0, \rho)) \cap f_\omega^{w|m}(B(x_0, \rho)) = \varnothing$. This implies that

$$|f_\omega^{v|n}(x_0) - f_\omega^{w|m}(x_0)| \geqslant \tfrac{1}{2} \operatorname{diam}(f_\omega^{v|n}(B(x_0, \rho))) + \tfrac{1}{2} \operatorname{diam}(f_\omega^{w|m}(B(x_0, \rho)))$$

$$= \tfrac{1}{2}\rho \left(r_{\omega_1}^{v_1} \ldots r_{\omega_n}^{v_n} + r_{\omega_1}^{w_1} \ldots r_{\omega_m}^{w_m}\right)$$

$$\geqslant \tfrac{1}{2}\rho \min\left\{r_{\omega_1}^{v_1} \ldots r_{\omega_n}^{v_n}, r_{\omega_1}^{w_1} \ldots r_{\omega_m}^{w_m}\right\}$$

and we see that the uniform cylinder separation condition holds for $\gamma = \tfrac{1}{2}\rho$. $\square$

References

[AKT20]   ANGELEVSKA, J., A. KÄENMÄKI, and S. TROSCHEIT: Self-conformal sets with positive Hausdorff measure. - Bull. London Math. Soc. 52:1, 2020, 200–223.

[BHS12]   BARNSLEY, M., J. HUTCHINSON, and Ö. STENFLO: *V*-variable fractals: dimension results. - Forum Math. 24, 2012, 445–470.

[Bau96]   BAUER, H.: Probability theory. - de Gruyter Stud. Math. 23, de Gruyter, Berlin, 1996.

[Fal86]   FALCONER, K.: Random fractals. - Math. Proc. Camb. Phil. Soc. 100, 1986, 559–582.

[Fal95]   FALCONER, K.: On the Minkowski measurability of fractals. - Proc. Amer. Math. Soc. 123, 1995, 1115–1124.

[Fal97]   FALCONER, K.: Techniques in fractal geometry. - John Wiley Sons, Chichester, UK, 1997.

[Fal14]   FALCONER, K.: Fractal geometry. - John Wiley & Sons, 3rd ed., 2014.

[Gat00]   GATZOURAS, D.: Lacunarity of self-similar and stochastically self-similar sets. - Trans. Amer. Math. Soc. 352, 2000, 1953–1983.

[Gra87]   GRAF, S.: Statistically self-similar fractals. - Probab. Theory Related Fields 74, 1987, 357–392.

[GMW88]   GRAF, S., R. D. MAULDIN, and S. C. WILLIAMS: The exact Hausdorff dimension of random recursive constructions. - Mem. Amer. Math. Soc. 71, 1988.

[Ham92]   HAMBLY, B.: Brownian motion on a homogeneous random fractal. - Probab. Theory Related Fields 94, 1992, 1–38.

[JJWW17]   JÄRVENPÄÄ, E., M. JÄRVENPÄÄ, M. WU, and W. WU: Random affine code tree fractals: Hausdorff and affinity dimensions and pressure. - Math. Proc. Camb. Phil. Soc. 162:2, 2017, 367–382.

[KR16]   KÄENMÄKI, A., and E. ROSSI: Weak separation condition, Assouad dimension, and Furstenberg homogeneity. - Ann. Acad. Sci. Fenn. Math. 41, 2016, 465–490.

[LRZ16]   LAPIDUS, M. L., G. RADUNOVIĆ, and ŽUBRINIĆ, D.: Fractal Zeta functions and fractal drums. Higher-dimensional theory of complex dimensions. - Springer International Publishing, Basel, Switzerland, 2016.

[MW86]   MAULDIN, R. D., S. C. WILLIAMS: Random recursive constructions: asymptotic geometric and topological properties. - Trans. Amer. Math. Soc. 295, 1986, 325–346.

[RWZ21]   RATAJ, J., S. WINTER, and M. ZÄHLE: Mean Lipschitz-Killing curvatures for homogeneous random fractals. - J. Fractal Geom. (to appear).

[RU11]   ROY, M., and M. URBANSKI: Random graph directed Markov systems. - Discrete Contin. Dyn. Syst. 30, 2011, 261–298.

[Tro17]   TROSCHEIT, S.: On the dimension of attractors of random self-similar graph directed iterated function systems. - J. Fractal Geom. 4, 2017, 257–303.

[Tro21]   TROSCHEIT, S. Exact Hausdorff and packing measures for random self-similar code-trees with necks. - Studia Math. 257:3, 2021, 253–285.

[Win08]   WINTER, S.: Curvature measures and fractals. - Diss. Math. 453, 2008, 1–66.

[Wit85]   WITTMANN, R. A general law of the iterated logarithm. - Z. Wahrsch. Verw. Gebiete 68, 1985, 521–543.

[Zah11]   ZÄHLE, M.: Lipschitz-Killing curvatures of self-similar random fractals. - Trans. Amer. Math. Soc. 363, 2011, 2663–2684.

[Zah20]   ZÄHLE, M.: The mean Minkowski content of homogeneous random fractals. - Mathematics 8:6, 2020, 883.

Sascha Troscheit

University of Oulu

Research Unit of Mathematical Sciences

P.O. Box 8000, FI-90014 University of Oulu, Finland

sascha.troscheit@oulu.fi