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Quasiconformal solutions to elliptic partial differential equations

David Kalaj

Abstract. In this paper, we assume that G and Ω are two Jordan domains in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundaries, where $n \geq 2$, and prove that every quasiconformal mapping $f \in \mathcal{W}^{2,1+\epsilon}_{\mathrm{loc}}$ of G onto Ω, satisfying the elliptic partial differential inequality $|L_A[f]| \lesssim (\|Df\|^2 + |g|)$, with $g \in \mathcal{L}^p(G)$, where $p > n$, is Lipschitz continuous. The result is sharp since for $p = n$, the mapping f is not necessarily Lipschitz continuous. This extends several results for harmonic quasiconformal mappings.

Elliptisten osittaisdifferentiaaliyhtälöiden kvasikonformiset ratkaisut

Tiivistelmä. Tässä työssä oletamme, että G ja Ω ovat kaksi $\mathcal{C}^2$-reunaista Jordanin aluetta avaruudessa $\mathbb{R}^n$, missä $n \geq 2$, ja todistamme, että jokainen kvasikonformikuvaus $f \in \mathcal{W}^{2,1+\epsilon}_{\mathrm{loc}}$, joka kuva alueen G surjektiivisesti alueeksi Ω ja toteuttaa elliptisen osittaisdifferentiaaliepäyhtälön $|L_A[f]| \lesssim (\|Df\|^2 + |g|)$, missä $g \in \mathcal{L}^p(G)$ jollakin $p > n$, on Lipschitzin-jatkuva. Tulos on tarkka, sillä kuvauksen f ei tarvitse olla Lipschitzin-jatkuva, jos $p = n$. Tämä yleistää useita harmonisia kvasikonformikuvauksia koskevia tuloksia.

1. Introduction and statement of the main result

In the paper $\mathbb{B} = \mathbb{B}^n$ denotes the unit ball in $\mathbb{R}^n$, $n \geq 2$. For a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and a real matrix $A = (a_{ij})_{i,j=1}^n$, we consider the vector norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ and the matrix norms: Hilbert–Schmidt norm and induced norm

$$\|A\| := (\operatorname{trace} A^t A)^{1/2} = \left(\sum_{i,j=1}^n a_{ij}^2\right)^{1/2}$$

and

$$|A| = \sup\{|Ax| \colon |x| = 1\}.$$

We also consider the matrix function

$$l(A) = \inf\{|Ax| \colon |x| = 1\}.$$

For further details on these and the notation in the remainder of the text, we refer to [22].

Definition 1.1. A homeomorphism $f \colon G \to \mathbb{R}^n$, $n \geq 2$ of a domain G in $\mathbb{R}^n$ is called quasiconformal (q.c.) if f is in ACL^n, and there exists a constant K, $1 \leq K < \infty$ such that

$$|Df(x)|^n \leq K|J_f(x)|, \quad |Df(x)| = \max_{|h|=1} |Df(x)h|, \tag{1.1}$$

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a.e. in G, where $f'(x) = Df(x)$ is the formal derivative. The smallest $K \geq 1$ for which this inequality is true is called the outer dilatation of f and is denoted by $K_O(f)$. If f is quasiconformal, then the smallest $K \geq 1$ for which the inequality

$$|J_f(x)| \leq Kl(Df(x))^n,$$

holds a.e. in G is called the inner dilatation of f and denoted by $K_I(f)$. The maximal dilatation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, f is said to be K-quasiconformal. It is well-known that

$$K_I(f) \leq K_O^{n-1}(f), \quad K_O(f) \leq K_I^{n-1}(f),$$

and hence $K_I(f)$ and $K_O(f)$ are simultaneously finite.

Definition 1.2. A continuous and nonconstant mapping $f\colon G \to \mathbb{R}^n$, $n \geq 2$, in the local Sobolev space $\mathcal{W}^{1,n}_{\mathrm{loc}}(G, \mathbb{R}^n)$ is K-quasiregular, $K \geq 1$, if

$$|Df(x)|^n \leq KJ_f(x)$$

for almost every $x \in G$.

We refer also to the monographs [33, p. 128] for this definition and the basic theory of quasiregular mappings.

Notice that the condition $u \in ACL^n$ guarantees the existence of the first derivative of u almost everywhere (see [32]). Moreover, $J_u(x) = \det(Du(x)) \neq 0$ for a.e. $x \in \Omega$. For a continuous mapping u, the condition (i) is equivalent to the fact that u belongs to the Sobolev space $\mathcal{W}^{1,n}_{\mathrm{loc}}(\Omega)$.

The Sobolev space $\mathcal{W}^{k,p}(\Omega)$, $k \in \mathbb{N}$ and $p \geq 1$, is defined to be the set of all functions f on Ω such that for every multi-index η with $|\eta| \leqslant k$, the mixed partial derivative

$$f^{(\eta)} = \frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$$

exists in the weak sense and is in $\mathcal{L}^p(\Omega)$, i.e.

$$\left\| f^{(\eta)} \right\|_{\mathcal{L}^p} < \infty.$$

That is, the Sobolev space $\mathcal{W}^{k,p}(\Omega)$ is defined as

$$\mathcal{W}^{k,p}(\Omega) = \{u \in \mathcal{L}^p(\Omega)\colon D^\eta u \in \mathcal{L}^p(\Omega)\ \forall |\eta| \leqslant k\}.$$

The natural number k is called the order of the Sobolev space $\mathcal{W}^{k,p}(\Omega)$. There are several choices for a norm for $\mathcal{W}^{k,p}(\Omega)$. The following is one of the equivalent norms:

$$\|u\|_{k,p,\Omega} = \|u\|_{\mathcal{W}^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|^p_{\mathcal{L}^p(\Omega)}\right)^{\frac{1}{p}}, & 1 \leqslant p < \infty; \\ \max_{|\alpha|\leqslant k} \|D^\alpha u\|_{\mathcal{L}^\infty(\Omega)}, & p = \infty. \end{cases}$$

If $k = 0$, we use the notation $\|u\|_{p,\Omega} = \|u\|_{k,p,\Omega}$.

For a function (a mapping) u defined in a domain Ω, we define $|u|_\infty = \sup\{|u(x)|\colon x \in \Omega\}$. We say that $u \in \mathcal{C}^{k,\alpha}(\Omega)$, $0 < \alpha \leq 1$, $k \in \mathbb{N}$, if

$$\|u\|_{l,\alpha} := \sum_{|\eta|\leq l} |D^\eta u|_\infty + \sum_{|\eta|=l} \sup_{x,y\in\Omega} |D^\eta u(x) - D^\eta u(y)| \cdot |x-y|^{-\alpha} < \infty.$$

In this paper, we study quasiconformal solutions of differential inequalities of the type $|Lu| \leq M|Du|^2 + |g|$, with $\mathcal{L}^p$ integrable g. Here L is a homogeneous second-order uniformly elliptic linear operator with coefficients that are continuous up to the boundary; see Section 2 for details. In particular, we are interested in the global Lipschitz and Hölder regularity of solution u.

A mapping f of a set S in Euclidean n-space $\mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 2$, is said to belong to the Hölder class (Lipschitz class) $\mathrm{Lip}_\alpha(S)$, $0 < \alpha < 1$ ($\alpha = 1$), if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha \tag{1.2}$$

for all x and y in S. If G is a bounded domain in $\mathbb{R}^n$ and if f is quasiconformal in G with $f(G) \subset \mathbb{R}^n$, then f is in $\mathrm{Lip}_\alpha(S)$ for each compact $S \subset G$, where $\alpha = K_I(f)^{1/(1-n)}$ and $K_I(f)$ is the inner dilatation of f. Simple examples show that f need not be in $\mathrm{Lip}_\alpha(G)$ even when f is continuous in $\overline{G}$. On the other hand, Martio and Näkki in [25] showed that if f induces a boundary mapping that belongs to $\mathrm{Lip}_\alpha(\partial G)$, then f is in $\mathrm{Lip}_\beta(G)$, where

$$\beta = \min(\alpha, K_I(f)^{1/(1-n)}). \tag{1.3}$$

Moreover, the exponent β is sharp. In particular, when $G = \Omega = \mathbb{B}$ and $f\colon \mathbb{B} \to \mathbb{B}$ is K-quasiconformal and there is a constant $C = C(K, f(0), n)$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \quad x, y \in \mathbb{B},$$

where $\alpha = K^{1/(1-n)}$. The last result is due to Vuorinen and Fehlmann [8].

In a recent paper by Kalaj and Saksman [21] it was shown that if f is a quasiconformal mapping of the unit disk onto a Jordan domain with $\mathcal{C}^2$ boundary such that its weak Laplacian $\Delta f \in \mathcal{L}^p(\mathbb{B}^2)$, for $p > 2$, then f is Lipschitz continuous. The condition $p > 2$ is also necessary. Furthermore, it was also shown in the same paper that if $p = 1$, then f is absolutely continuous on the boundary of $\mathbb{B}^2$. In a certain sense, the results from [21] optimize the results of the Kalaj, Mateljević, Pavlović, Partyka, Sakan, Astala, Manojlović [16, 17, 19, 18, 27, 28, 29, 30, 14, 15, 5], since it does not assume that the mapping is harmonic, neither its weak Laplacian is bounded. Furthermore, the two-dimensional result by Kalaj and Saksman in [21] has been extended by Kalaj and Zlatičanin in [22] to a higher-dimensional case.

In [15], the study of the Lipschitz property of self-maps $u\colon \mathbb{B}^2 \to \mathbb{B}^2$ satisfying the elliptic partial differential inequality $|Lu| \lesssim (\|Du\|^2 + 1)$ was initiated. Here and in what follows, $a(u) \lesssim b(u)$ means that there is a constant C that does not depend on a function u such that $a(u) \leq Cb(u)$. We will also use the symbol $a(u) \simeq b(u)$ which, means that there is a constant $C \geq 1$ such that $a(u)/C \leq b(u) \leq Ca(u)$.

Generalizing these ideas to higher dimensions, we now proceed to consider quasiconformal mappings $u\colon G \to \Omega$ between two smooth domains G and Ω in $\mathbb{R}^n$, satisfying the elliptic partial differential inequality of the form $|Lu| \lesssim (\|Du\|^2 + |g|)$ for some $g \in \mathcal{L}^p(G)$, where $p > n/2$. We are interested in the conditions under which the quasiconformal mapping u is in $\mathrm{Lip}_\alpha(G)$, for $\beta < \alpha < 1$, where β is defined in (1.3).

It follows from our results that when G and Ω are diffeomorphic images of the closed n-ball, then the quasiconformality of u combined with the $\mathcal{L}^p$-integrability of $|Lu|$ for $p > n/2$ guarantee that u is in $\mathrm{Lip}_\alpha(G)$, where $\alpha = 2 - \frac{p}{n}$. In particular if $p = n$, then $u \in \mathrm{Lip}_\alpha(\mathbb{B})$ for $\alpha < 1$ and if $p > n$, then u is Lipschitz.

A Jordan domain G in the space is a diffeomorphic image of the open unit ball in n-dimensional space. We say that the Jordan domain is $\mathcal{C}^2$ smooth if the diffeomorphism is a twice differentiable function which, along with its inverse and has a $\mathcal{C}^2$ extension to the boundary. This is the main result of this paper.

Theorem 1.3. *Let $n \geq 2$ and $p > n/2$ and assume that $g \in \mathcal{L}^p(G)$, where G is a bounded Jordan domain with C^2 boundary in $\mathbb{R}^n$ and let Ω be a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary. Assume further that*

1. *$f\colon G \xrightarrow{\text{onto}} \Omega$ is K-quasiconformal,*
2. *for some $\epsilon > 0$, $f \in \mathcal{W}^{2,1+\epsilon}_{\text{loc}}(G)$ and*
3. *f has second order partial derivatives a.e. which satisfy the elliptic differential inequality inequality*

$$|Lf| \leq M\|Df\|^2 + N|g|, \tag{1.4}$$

where M and N are two constants.

Then

- *If $p < n$, then f is globally Hölder continuous with the Hölder exponent $\alpha = \max\{2 - \frac{n}{p}, K^{1/(1-n)}\}$.*
- *If $p = n$, then f is globally Hölder continuous for every $\alpha \in (0,1)$.*
- *If $p > n$, then f is globally Lipschitz continuous. In this case $f \in \mathcal{C}^{1,\alpha}_{\text{loc}}$, where $\alpha = 1 - n/p$.*

The proof of Theorem 1.3 uses a bootstrapping argument and Sobolev embedding theorem. The conclusion is optimal, and for $p = n$, the mapping need not be locally Lipschitz continuous (see Example 1.5 below). The condition of quasiconformality in our main result is also crucial. Indeed, there exist harmonic diffeomorphisms of the unit disk onto itself that have no Lipschitz extension up to the boundary.

Since harmonic functions, i.e. smooth solution to the PDE $\Delta f = 0$, are $\mathcal{C}^\infty$, the following corollary, which is new in the cases $n \geq 3$, is an immediate consequence of the main result.

Corollary 1.4. *Let f be a quasiconformal harmonic mapping between two Jordan domains in $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Then f is Lipschitz continuous.*

Example 1.5. Let $f(w) = w\log^\alpha(1/|w|)$, $0 < \alpha < \frac{n-1}{n}$ and assume that $r < e^{-(n-1)/n}$. Then f is a quasicoformal mapping from the ball $\mathbb{B}(0,r) := r\mathbb{B}$ onto the ball $\mathbb{B}(0,\delta)$, $\delta = r\log^\alpha(1/r)$, such that $\Delta f \in \mathcal{L}^n(\mathbb{B}(0,r))$, and f is not Lipschitz continuous. For the sake of simplifying calculations, we set $n = 3$ and assume that $w = (x,y,z)$.

Then for $\rho = |w|$, by direct calculations we get

$$J(w,f) = \log^{-1+3\alpha}\left[\frac{1}{\rho}\right]\left(-\alpha + \log\left[\frac{1}{\rho}\right]\right)$$

and

$$\|Df\|^2 = 4^{-\alpha}\log^{-2+2\alpha}\left[\frac{1}{\rho^2}\right]\left(4\alpha^2 + 4\alpha\log\left[\rho^2\right] + 3\log^2\left[\rho^2\right]\right).$$

The second relation implies that f is not Lipschitz continuous near $w = 0$.

Then it is clear that

$$\frac{\|Df\|^3}{J(w,f)}$$

is bounded when $w \to 0$. This implies that f is quasiconformal in $\mathbb{B}(0,r)$ for $r \in (0,1)$. Furthermore

$$\rho^2|\Delta f(x,y,z)|^3 = \rho^2\left(\left|\frac{\alpha^2\log^{-4+2\alpha}[1/\rho]\left(-1+\alpha+\frac{3}{2}\log[\rho^2]\right)^2}{\rho^2}\right|\right)^{3/2}.$$

Since

$$\int_0^r \frac{\log^{-6+3\alpha}[\rho^{-2}]\log^3[1/\rho]}{\rho}\,d\rho = \frac{8^{-2+\alpha}(-\log r)^{-2+3\alpha}}{2-3\alpha},$$

it follows that

$$\int_0^r \rho^2|\Delta f(x,y,z)|^3\,d\rho$$

is convergent if and only if $\alpha < 2/3$. Moreover,

$$\int_0^r \rho^2|\Delta f(x,y,z)|^p\,d\rho$$

diverges for every $p > 3$. In other words $\Delta f \in \mathcal{L}^n(\mathbb{B}(0,r))$, and also we can easily show that $D^2 f \in \mathcal{L}^n(\mathbb{B}(0,r))$ ($f \in \mathcal{W}^{2,n}(\mathbb{B}(0,r))$) but $\Delta f \notin \mathcal{L}^p(\mathbb{B}(0,r))$ for $p > n$.

2. Preliminary results for elliptic operator

Let $A(x) = \{a_{ij}(x)\}_{i,j=1}^n$ be a symmetric matrix function defined in a bounded domain $\Omega \subset \mathbb{R}^n$: $a_{ij}(x) = a_{ji}(x)$ for every i,j and $x \in \Omega$. Assume that

$$\Lambda^{-1} \le \langle A(x)\xi,\xi\rangle \le \Lambda \quad \text{for} \quad |\xi| = 1, \tag{2.1}$$

where Λ is a constant ≥ 1 or written in coordinates

$$\Lambda^{-1} \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda \quad \text{for} \quad \sum_{i=1}^n \xi_i^2 = 1. \tag{2.2}$$

In addition, we suppose that

$$A \in \mathcal{C}^0(\overline{\Omega}). \tag{2.3}$$

For a function or mapping u that is twice weakly differentiable a.e. in Ω and which satisfies

$$L[u] = L_A[u] := \sum_{i,j=1}^n a_{ij}(x) D_{ij} u(x), \tag{2.4}$$

subjected to conditions (2.2) and (2.3), we consider the following differential inequality

$$|L[u]| \lesssim (\|Du\|^2 + |g|), \tag{2.5}$$

where $g \in \mathcal{L}^p(\Omega)$ and Du is the differential matrix of u. Here $p > 1$. The inequality (2.5) is considered in the strong sense, i.e. we consider the inequality for almost every $x \in \Omega$.

Lemma 2.1. *Assume that $H\colon \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^2$ differentiable function and let w be a strong solution to equation* (2.5)*. Let $h = H \circ w$. Then*

$$|L_A[h]| \le C_1(H,A)\|Dw\|^2 + C_2(H,A)|g|.$$

Moreover, if for some $p > n/2$, $w \in \mathcal{W}^{2,p}_{\mathrm{loc}}$, then $H \circ w \in \mathcal{W}^{2,p}_{\mathrm{loc}}$.

Proof. Because H and w are double differentiable for almost every $x = (x_1, \dots, x_n)$, by direct calculations we get

$$\begin{aligned}\frac{\partial^2 (H\circ w)(x)}{\partial x_k \partial x_\ell} &= \sum_{i=1}^n \frac{\partial[\frac{\partial H}{\partial w_i}\frac{\partial w_i}{\partial x_k}]}{\partial x_\ell} = \sum_{i=1}^n \left[\frac{\partial[\frac{\partial H}{\partial w_i}]}{\partial x_\ell}\frac{\partial w_i}{\partial x_k} + \frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}\right]\\ &= \sum_{i=1}^n \left[\left[\sum_{j=1}^n \frac{\partial^2 H}{\partial w_j \partial w_i}\frac{\partial w_j}{\partial x_k}\right]\frac{\partial w_i}{\partial x_\ell}\right] + \sum_{i=1}^n \frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}\\ &= \sum_{i,j=1}^n \frac{\partial^2 H}{\partial w_i \partial w_j}\left[\frac{\partial w_i}{\partial x_k}\frac{\partial w_j}{\partial x_\ell}\right] + \sum_{i=1}^n \frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}.\end{aligned}$$

So

$$\frac{\partial^2 (H\circ w)(x)}{\partial x_k \partial x_\ell} = \sum_{i=1}^n \left[\left[\sum_{j=1}^n \frac{\partial^2 H}{\partial w_j \partial w_i}\frac{\partial w_j}{\partial x_k}\right]\frac{\partial w_i}{\partial x_\ell}\right] + \sum_{i=1}^n \frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}. \tag{2.6}$$

Further,

$$\begin{aligned}&\sum_{k,\ell} a_{k\ell}(x)\frac{\partial^2 (H\circ w)(x_1,\dots,x_n)}{\partial x_k \partial x_\ell}\\ &= \sum_{k,\ell} a_{k\ell}(x)\sum_{i,j=1}^n \frac{\partial^2 H}{\partial w_i \partial w_j}\left[\frac{\partial w_i}{\partial x_k}\frac{\partial w_j}{\partial x_\ell}\right] + \sum_{k,\ell} a_{k\ell}(x)\sum_{i=1}^n \frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}.\end{aligned}$$

Therefore

$$\begin{aligned}L_A[h](x) &= \sum_{k,\ell} a_{k\ell}(x)\sum_{i,j=1}^n \frac{\partial^2 H}{\partial w_i \partial w_j}\left[\frac{\partial w_i}{\partial x_k}\frac{\partial w_j}{\partial x_\ell}\right] + \sum_{i=1}^n\sum_{k,\ell} a_{k\ell}(x)\frac{\partial H}{\partial w_i}\frac{\partial^2 w_i}{\partial x_k \partial x_\ell}\\ &= \sum_{k,\ell} a_{k\ell}(x)\sum_{i,j=1}^n \frac{\partial^2 H}{\partial w_i \partial w_j}\left[\frac{\partial w_i}{\partial x_k}\frac{\partial w_j}{\partial x_\ell}\right] + \sum_{i=1}^n \frac{\partial H}{\partial w_i}L_A[w_i].\end{aligned}$$

Thus

$$\begin{aligned}|L_A[h]| &\le C(H,A)\|Dw\|^2 + |\langle \nabla H, L_A[w]\rangle|\\ &\le C(H,A)\|Dw\|^2 + |\nabla H|_\infty(\|Dw\|^2 + |g|).\end{aligned} \tag{2.7}$$

Hence

$$|L_A[h]| \le (C(H,A) + |\nabla H|_\infty)\|Dw\|^2 + |\nabla H|_\infty |g|. \tag{2.8}$$

In order to obtain the last claim, assume first that $n/2 < p < n$. Since $w \in \mathcal{W}^{2,p}_{\mathrm{loc}}$, it follows that $w \in \mathcal{W}^{1,q}_{\mathrm{loc}}$ for $q = np/(n-p)$, by Sobolev embedding theorem. Thus $q > 2p > n$. Hence w is also continuous up to choice of representative by Morrey's inequality, see e.g. Proposition 2.3 below. A vector-valued first order chain rule (of Ambrosio–Dal Maso [3]) thus gives $H \circ g \in \mathcal{W}^{1,q}_{\mathrm{loc}} \cap \mathcal{C}$ with $D(H\circ w) = DH(w)Dw$ almost everywhere. A similar first order chain rule using the fact that $DH \in \mathcal{C}^1$ gives $DH(w) \in \mathcal{W}^{1,q}_{\mathrm{loc}} \cap \mathcal{C}$. Now, since $DH(w) \in \mathcal{W}^{1,q}_{\mathrm{loc}} \cap \mathcal{L}^\infty_{\mathrm{loc}}$ and $Dw \in \mathcal{W}^{1,p}_{\mathrm{loc}} \cap \mathcal{L}^q_{\mathrm{loc}}$ a product rule of Sobolev functions yields that $DH(w)Dw \in \mathcal{W}^{1,s}_{\mathrm{loc}}$, where $s = \min\{(\frac{1}{p} + \frac{1}{\infty})^{-1}, (\frac{1}{q} + \frac{1}{q})^{-1}\} = p$. Thus $H \circ w \in \mathcal{W}^{2,p}_{\mathrm{loc}}$. If $p \ge n$, then we chose $q > n$ arbitrary and repeat the previous proof. □

Lemma 2.2. *Let $p \geq 1$. If g is a $\mathcal{C}^2$ bi-Lipschitz diffeomorphism between two domains G and Ω in $\mathbb{R}^n$ and if h is a strong solution of the strong elliptic partial differential inequality $|L_A[h](x)| \lesssim \|Dh(x)\|^2 + k(x)$, $x \in \Omega$, where $k \in \mathcal{L}^p(\Omega)$, then $f = h \circ g$ is a strong solution of the strong elliptic partial differential inequality*

$$|L_B[f](y)| \lesssim \|Df(y)\|^2 + k_1(y),$$

for $y \in G$ where $k_1 \in \mathcal{L}^p(G)$, and $B(y) = (g'(y))^{-1} \cdot A(g(y)) \cdot ((g'(y))^{-1})^T$, $y \in G$. Moreover, for every $p > n$, $h \in \mathcal{W}^{2,p}_{\mathrm{loc}}$ if and only if for every $p > n$, $f = h \circ g \in \mathcal{W}^{2,p}_{\mathrm{loc}}$.

Proof. Let $x \in \Omega$ and let $\tilde{g} = g^{-1}$. Then $h(x) = f(\tilde{g}(x))$, $y = (y_1, \dots, y_n) = \tilde{g}(x) = (g^1(x), \dots, g^n(x))$, $x = (x_1, \dots, x_n)$. Now we have

$$h_{x_i} = \sum_{j=1}^n f_{y_j} g^j_{x_i}. \tag{2.9}$$

This equality is in the weak and in the strong sense. In the weak sense because of [34, Theorem 2.2.2], and in the strong sense, because by assumption h is differentiable almost everywhere and g is smooth. This also follows from [11, eq. 7.19]. By applying again [34, Theorem 2.2.2] to $f_{y_j}(\tilde{g}(x)) \cdot g^j_{x_i}$, in view of $D(uv) = uDv + vDu$ (see e.g. [11, eq. 7.18]) we obtain the following equality:

$$h_{x_i x_\ell} = \sum_{j=1,k=1}^n f_{y_j y_k} g^j_{x_i} g^k_{x_\ell} + \sum_{j=1}^n f_{y_j} g^j_{x_i x_\ell}, \tag{2.10}$$

for almost every x (which is also in weak and strong sense). Moreover (2.10) implies that $f \in \mathcal{W}^{2,p}_{\mathrm{loc}} \Rightarrow h \in \mathcal{W}^{2,p}_{\mathrm{loc}}$. The opposite implication is also proven similarly.

Furthermore,

$$\begin{aligned} \sum_{i,\ell=1}^n a_{i\ell}(x) h_{x_i x_\ell} &= \sum_{i,\ell=1}^n a_{i\ell}(x) \sum_{j=1,k=1}^n f_{y_j y_k} g^j_{x_i} g^k_{x_\ell} + \sum_{i,\ell=1}^n a_{i\ell}(x) \sum_{j=1}^n f_{y_j} g^j_{x_i x_\ell} \\ &= \sum_{j=1,k=1}^n \left[\sum_{i,\ell=1}^n a_{i\ell}(x) g^j_{x_i} g^k_{x_\ell} \right] f_{y_j y_k} + \sum_{i,\ell=1}^n a_{i\ell}(x) \sum_{j=1}^n f_{y_j} g^j_{x_i x_\ell}. \end{aligned} \tag{2.11}$$

Now if

$$B = B(y) = (b_{jk})^n_{j,k=1},$$

where

$$b_{jk} = \sum_{i,\ell=1}^n a_{i\ell}(x) g^j_{x_i} g^k_{x_\ell},$$

then

$$B(y) = D\tilde{g}(x) A(x) \left(D\tilde{g}(x)\right)^T.$$

Thus

$$\lambda_g^2 \lambda_A |\xi|^2 \leq \langle B\xi, \xi \rangle \leq \Lambda_g^2 \Lambda_A |\xi|^2, \xi \in \mathbb{R}^n,$$

where

$$\lambda_g = \inf_x l(\tilde{g}'(x))$$

and

$$\Lambda_g = \sup_x |\tilde{g}'(x)|.$$

By combining (2.9) and (2.11) we get

$$\begin{aligned}|L_B[f]| &\le |L_A[h]| + C\|Df\| \le C_1\|Dh\|^2 + C_2k + C_3\|Df\|\\ &\le C_4\|Df\|^2 + C_5k + C_6\|Df\| \lesssim (\|Df\|^2 + k_1).\end{aligned}$$

In the last inequality we used the simple inequality $a^2 + a \le 2(a^2+1)$, and $k_1(y) = k(x) + 1$. □

Proposition 2.3. (Morrey's inequality) *Assume that $n < p \le \infty$ and assume that U is a domain in $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary. Then there exists a constant C depending only on n, p, and U such that*

$$\|u\|_{C^{0,\alpha}(U)} \le C\|u\|_{\mathcal{W}^{1,p}(U)} \tag{2.12}$$

for every $u \in \mathcal{C}^1(U) \cap \mathcal{L}^p(U)$, where

$$\alpha = 1 - \frac{n}{p}.$$

Remark 2.4. Since $i\colon \mathcal{W}^{1,p}(U) \to \mathcal{C}^\alpha(U)$ defined by $i(u)(x) = u(x)$ for almost every $x \in U$, is a continuous embedding, provided that U has $\mathcal{C}^1$ boundary it follows that (2.12) does hold for every $u \in \mathcal{W}^{1,p}(U)$.

Proposition 2.5. [11, Theorem 9.15, Lemma 9.17] *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and let L be a strictly elliptic operator in Ω with coefficients $a_{ij} \in C(\overline{\Omega})$. Then, if $f \in \mathcal{L}^p(\Omega)$ with $1 < p < \infty$, the Dirichlet problem $Lu = f$ in Ω with the boundary condition $u \in W_0^{1,p}(\Omega)$ has a unique solution $u \in W^{2,p}(\Omega)$. Moreover,*

$$\|u\|_{2,p,\Omega} \le C\|f\|_{p,\Omega}. \tag{2.13}$$

Note that functions on $\mathcal{W}^{1,p}(\Omega)$ that vanish continuously in $\partial\Omega$ are in $\mathcal{W}_0^{1,p}(\Omega)$ (See [11, p. 154]). Thus

Corollary 2.6. *Under assumptions of Proposition 2.5, if the condition $u \in \mathcal{W}_0^{1,p}(\Omega)$ is replaced by the condition that u vanishes continuously in $\partial\Omega$, we have*

$$\|u\|_{2,p,\Omega} \le C\|f\|_{p,\Omega}. \tag{2.14}$$

Proposition 2.7. *Let Ω be a domain in $\mathbb{R}^n$ with a (possible empty) $\mathcal{C}^{1,1}$ boundary portion $T \subset \partial\Omega$ and assume that $\Omega_1 \Subset \Omega \cup T$ is a relatively compact domain with smooth boundary in $\Omega \cup T$. Let $u \in \mathcal{W}^{2,p}(\Omega)$, $1 < p < \infty$, be a strong solution of $Lu = f \in \mathcal{L}^p(\Omega)$ in Ω with $u = 0$ in T, in the sense of $\mathcal{W}^{1,p}$, where L is a strongly elliptic operator with $a^{ij} \in \mathcal{C}^0(\overline{\Omega})$.*

(a) *Then*

$$\|u\|_{2,p,\Omega_1} \le C_1\left(\|u\|_{p,\Omega} + \|f\|_{p,\Omega}\right), \tag{2.15}$$

where C_1 depends on $n, p, \Lambda, \Omega, \Omega_1, \omega_A$.

(b) *Moreover, if $f \in \mathcal{L}^q$, with some $q > p$, then $u \in \mathcal{W}^{2,q}_{\mathrm{loc}}(\Omega \cup T)$, $u = 0$ in T in the sense of $\mathcal{W}^{1,q}$, and u satisfies the estimate (2.15) with p replaced by q:*

$$\|u\|_{2,q,\Omega_1} \le C_1\left(\|u\|_{q,\Omega} + \|f\|_{q,\Omega}\right). \tag{2.16}$$

(c) *Furthermore, if $p > n$, then there exists $\tilde{u} \in \mathcal{C}^{1,\alpha}$, such that $u = \tilde{u}$ almost everywhere, and*

$$\|\tilde{u}\|_{\mathcal{C}^{1,\alpha}(\Omega_1)} \le C_2\left(\|\tilde{u}\|_{p,\Omega} + \|f\|_{p,\Omega} + 1\right), \tag{2.17}$$

where $\alpha = 1 - \frac{n}{p}$, and C_2 depend on $n, p, \Lambda, \Omega, \Omega_1, \omega_A$. In particular, u is Lipschitz continuous and has an essentially bounded gradient in Ω_1.

Proof of Proposition 2.7. First of all the part (a) follows from [11, Theorem 9.13]. Moreover, the part (b) follows from [11, Lemma 9.16].

Let us prove the part (c). By abusing the notation $\tilde{u} = u$ and using Morrey's inequality, Remark 2.4, and (2.15) we have that

$$\begin{aligned}\|u\|_{\mathcal{C}^{1,\alpha}(\Omega_1)} &\le C(\|\nabla u\|_{\mathcal{C}^{\alpha}(\Omega_1)} + 1)\\ &\le C(\|\nabla^2 u\|_{\mathcal{L}^p(\Omega_1)} + 1) \le C(\|u\|_{2,p,\Omega_1} + 1) \le C\left(\|u\|_{p,\Omega} + \|f\|_{p,\Omega} + 1\right).\end{aligned}$$

Here, C can vary from one occurrence to the next but depends on the constants $n, p, \Lambda, \Omega, \omega_A$. □

Proposition 2.8. [11, Corollary 9.18]. *Let Ω be a $\mathcal{C}^{1,1}$ domain in $\mathbb{R}^n$, and let the operator L be strictly elliptic in Ω with coefficients $a_{ij} \in \mathcal{C}(\overline{\Omega})$. Then if $f \in \mathcal{L}^p(\Omega)$, $p > n/2$, $\varphi \in \mathcal{C}(\partial\Omega)$, the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$, has a unique solution $u \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega})$.*

Proposition 2.9. [10, Sobolev embedding theorem] *Let $p < n$ and assume that Ω is a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary. Furthermore, let*

$$\frac{1}{p} - \frac{2}{n} = \frac{1}{q} - \frac{1}{n},$$

i.e. $q = np/(n-p)$. If $u \in \mathcal{W}^{2,p}(\Omega)$, then

$$\|u\|_{1,q,\Omega} \le C\|u\|_{2,p,\Omega}. \tag{2.18}$$

3. Proof of the main result

We begin with

Lemma 3.1. *Under conditions of Theorem 1.3 we have*

$$f \in \mathcal{W}^{2,p}_{\mathrm{loc}}(G). \tag{3.1}$$

Proof of Lemma 3.1. Here we use the bootstrapping argument improving the regularity of f until we reach the right exponent in the right-hand of (3.10). Assume that $a \in G$ and assume that $\overline{\mathbb{B}}(a,r) \subset \Omega$. Assume w.l.o.g. that $r = 1$ and $a = 0$. Let $0 < r_0 < 1$. In this case we repeat the "cycling" argument two times. First of all we use (1.4) and $f \in \mathcal{W}^{1,n}_{\mathrm{loc}}$ to conclude that $L[f] \in \mathcal{L}^{q_1}(\mathbb{B})$, where $q_1 = \min\{n/2, p\}$. We assume w.l.o.g. that $q_1 > 1 + \epsilon$. Now Proposition 2.7, b) implies that $f \in \mathcal{W}^{2,q_1}_{\mathrm{loc}}(r_1\mathbb{B})$ for some $r_1 \in (r_0, 1)$. Now the Sobolev embedding theorem gives $f \in \mathcal{W}^{1,\min(n,np/(n-p))}_{\mathrm{loc}}(r_1\mathbb{B})$. Then (3.10) gives $L[f] \in \mathcal{L}^{\min(n,p)}_{\mathrm{loc}}(r_1\mathbb{B})$. Furthermore from Proposition 2.8, $f \in \mathcal{W}^{2,\min(n,p)}_{\mathrm{loc}}(r_2\mathbb{B})$ for some $r_2 \in (r_0, r_1)$. Then $f \in \mathcal{W}^{1,p}_{\mathrm{loc}}(r_2\mathbb{B})$. Now we use Proposition 2.8 again to conclude that $f \in \mathcal{W}^{2,p}_{\mathrm{loc}}(r_3\mathbb{B})$ for some $r_3 \in (r_0, r_2)$. Since every $K \Subset G$ can be covered by a finite number of such balls $\mathbb{B}(a, r_3)$, we obtain that $f \in \mathcal{W}^{2,p}_{\mathrm{loc}}(G)$. □

Proof of Theorem 1.3. By our assumption on the domain and image domain, we may fix diffeomorphisms $\psi\colon \overline{\Omega} \to \overline{\mathbb{B}}$ and $\varphi\colon \overline{\mathbb{B}} \to \overline{G}$ that are $\mathcal{C}^2$ up to the boundary. We then define $h\colon \mathbb{B} \to [0,1]$ by setting

$$h(x) := 1 - |\psi(f(\varphi(x)))|^2 \quad \text{for} \quad x \in \mathbb{B}.$$

As $\omega(x) = \psi(f(\varphi(x)))$ is a homeomorphism of the unit ball onto itself, for $\delta = 1/2$ there exists $r_0 < 1$, such that

$$|x| \ge r_0 \implies |\omega(x)| \ge 1/2. \tag{3.2}$$

Now define $w(x) = f(\varphi(x))$. Since f is K-quasiconformal, it follows from Definition 1.1, that

$$\frac{|l(Df(x))|}{|Df(x)|} \geq \frac{1}{K^{2/n}}.$$

Then for

$$l(\varphi) := \inf_{x\in\mathbb{B}} l(D\varphi(x))$$

we have

$$\frac{l(Dw(x))}{\|Dw(x)\|} \geq \frac{l(\varphi)}{\sqrt{n}K^{2/n}\||D\varphi(x)|\|_\infty}. \tag{3.3}$$

Furthermore, $h(x) = 1 - |\psi(w(x))|^2$ and we have

$$\nabla h(x) = -2(D\psi(w(x)) \cdot Dw(x))^T \omega(x).$$

Since ψ is a diffeomorphism up to the boundary, we get

$$|\nabla h(x)| \leq C_1(\psi, \varphi)|Dw(x)|, \quad x \in \mathbb{B} \tag{3.4}$$

where C_1 depends on ψ and φ. Then we get from (3.2) and (3.3)

$$\begin{aligned} |\nabla h(x)| &\geq \left(\min_{|x|\geq r_0} |\omega(x)| l(D(\psi(w(x))))\right) \cdot l(Dw(x)) \\ &> C(\psi, \varphi, K, r_0)|Dw(x)|, \quad |x| \geq r_0 \end{aligned} \tag{3.5}$$

where

$$C(\psi, \varphi, K, r_0) = \frac{l(\varphi)}{2\sqrt{n}K^{2/n}\||D\varphi(x)|\|_\infty} \inf_{r_0\leq|x|<1} l(D(\psi(w(x)))).$$

From (3.4) and (3.5) we get for $r_0 \in (0, 1)$ the estimate

$$|\nabla h(x)| \simeq |Dw(x)| \quad \text{for } r_0 \leq |x| < 1. \tag{3.6}$$

By Lemma 3.1 we have $f \in \mathcal{W}^{2,p}_{\mathrm{loc}}$. Furthermore Lemma 2.2 implies that $w \in \mathcal{W}^{2,p}_{\mathrm{loc}}$. By (3.4), (3.1) and (2.18) for $\Omega_1 = r_0\mathbb{B}$ we have for $n/2 < p < n$

$$\|\nabla h(x)\|_{\mathcal{L}^{np/(n-p)}(r_0\mathbb{B})} \lesssim \|Dw(x)\|_{\mathcal{L}^{np/(n-p)}(r_0\mathbb{B})} \leq C;$$

and for $p > n$ we have from (2.17) that

$$\|\nabla h(x)\|_{\mathcal{L}^\infty(r_0\mathbb{B})} \lesssim \|Dw(x)\|_{\mathcal{L}^\infty(r_0\mathbb{B})} \leq C.$$

It follows that for any $p \in (n/2, n)$ and $q \in (1, np/(n-p)]$ we have that

$$\nabla h \in \mathcal{L}^q(\mathbb{B}) \quad \text{if and only if } \; Dw \in \mathcal{L}^q(\mathbb{B}). \tag{3.7}$$

For $p = n$ and $q \in (1, +\infty)$ we have that

$$\nabla h \in \mathcal{L}^q(\mathbb{B}) \quad \text{if and only if } \; Dw \in \mathcal{L}^q(\mathbb{B}). \tag{3.8}$$

Also for $p > n$ and $q \in (1, +\infty]$ we have that

$$\nabla h \in \mathcal{L}^q(\mathbb{B}) \quad \text{if and only if } \; Dw \in \mathcal{L}^q(\mathbb{B}). \tag{3.9}$$

By Lemma 2.2 we obtain

$$|L_B[w]| \lesssim \|Dw\|^2 + |g \circ \varphi|, \tag{3.10}$$

where

$$B(y) = (g'(y))^{-1} \cdot A(g(y)) \cdot \left((g'(y))^{-1}\right)^T,$$

and then by Lemma 2.1, we have

$$|L_B[h]| \lesssim \|Dw\|^2 + |g \circ \varphi|. \tag{3.11}$$

We use a bootstrapping argument as in [5, 21, 22], based on the following implication:

$$\text{if } Dw \in \mathcal{L}^q(\mathbb{B}) \text{ with } n < q < 2n, \quad \text{then } Dw \in \mathcal{L}^{na/(2n-a)}(\mathbb{B}), \tag{3.12}$$

where $a = \min\{q, 2p\}$. To prove the implication (3.12), assume that $Dw \in \mathcal{L}^q(\mathbb{B})$ for an exponent $q \in (n, 2n)$. Then (3.11) and our assumption on g verify that $L_B[h] \in \mathcal{L}^{\min\{q/2,p\}}(\mathbb{B})$. Since h vanishes continuously on the boundary $\partial\mathbb{B}$, we may apply (2.14), Lemma 2.1 and (2.18) to obtain that $\nabla h \in \mathcal{L}^{na/(2n-a)}(\mathbb{B})$ which yields the claim according to (3.7), (3.8) and (3.9).

We then claim that in our situation one has $Dw \in \mathcal{L}^q(\mathbb{B})$ with some exponent

$$q \begin{cases} = 2p, & \text{if } p \le n; \\ > 2n, & \text{if } p > n. \end{cases} \tag{3.13}$$

The higher integrability of quasiconformal self-maps of $\mathbb{B}$ [26, Sect. 2.15]; see also [4], makes sure that $Dw = D(f \circ \varphi) \in \mathcal{L}^q(\mathbb{B})$ for some $q > n$. To prove (3.13) fix an exponent $q_0 > n$ obtained from the higher integrability of the quasiconformal map w such that $Dw \in \mathcal{L}^{q_0}(\mathbb{B})$. By reducing q_0 if necessary, we may well assume that $q_0 \in (n, 2n)$ and

$$q_0 \notin \{2^m/(2^{m-1} - 1),\ m = 3, 4, \ldots\}. \tag{3.14}$$

For example, we can choose $q_0 = n + \frac{1}{j\pi}$, for a big enough integer j.

If $q_0 \ge 2p$, then we are necessarily in the case $p \le n$ by our assumed $q_0 < 2n$, and (3.13) immediately follows. If $q_0 < 2p$ instead, we iterate (3.12) and deduce inductively that $Dw \in \mathcal{L}^{a_k}(\mathbb{B})$ for $k = 0, 1, 2, \ldots, k_0$, where $a_0 = q_0$ and a_k satisfy the recursion $a_{k+1} = \frac{na_k}{2n-a_k}$ until we reach a constant that is bigger or equal to $2n$. Namely, by solving the previous recursion we get

$$a_k = \frac{nq_0}{q_0 - 2^k(q_0 - n)},$$

which are real numbers, for every positive integer k, in view of (3.14). So $a_k > 2n$ with $q_0 - 2^k(q_0 - n) > 0$ if and only if

$$2^{1+k}(q_0 - n) > q_0. \tag{3.15}$$

Let k_0, be the first index such that (3.15) holds. Then $a_{k_0} > 2n$. Let

$$q = \min\{a_{k_0}, 2p\}. \tag{3.16}$$

Thus we may assume that $Dw \in \mathcal{L}^q(\mathbb{B})$ with q satisfying (3.13).

Now if $p < n$. Then $q = a_{k_0} = 2p$ and so $Dw \in \mathcal{L}^{2p}(\mathbb{B})$. Since $a = \min\{q, 2p\} = 2p$, we get from (3.12) that $Dw \in \mathcal{L}^{np/(n-p)}(\mathbb{B})$. Now by Morrey's inequality, w is Hölder continuous with the Hölder exponent $\alpha = 2 - \frac{n}{p}$. The mapping w is also Hölder continuous with the exponent $K^{1/(1-n)}$ by the result of Vuorinen and Fehlmann [8]. So it is Hölder continuous with the Hölder exponent $\alpha' = \max\{2 - \frac{n}{p}, K^{1/(1-n)}\}$. The same holds for the mapping f as claimed. For $p = n$ we use the previous case by choosing $p' < p$ close enough to p.

Assume now that $p > n$. We know that $Dw \in \mathcal{L}^q(\mathbb{B})$ with some $q > 2n$. Furthermore (3.11) shows that $Lh \in \mathcal{L}^{\min\{p,(q/2)\}}(\mathbb{B})$. As $\min\{p, (q/2)\} > n$, we get from (2.17) that

$$\nabla h \in \mathcal{L}^\infty(\mathbb{B}). \tag{3.17}$$

Thus by (3.9) $Dw \in \mathcal{L}^\infty(\mathbb{B})$ and hence w is Lipschitz continuous. This implies that f is Lipschitz continuous. In particular $L_A[f] \in \mathcal{L}^p$. Now from Proposition 2.7, by taking $T = \emptyset$, and Ω_1 any compact set of Ω, we get that $f = \tilde{f} \in \mathcal{C}^{1,\alpha}(\Omega_1)$, where $\alpha = 1 - n/p$. $\square$

Remark 3.2. The conclusion of our main result (Theorem 1.3) will still hold (with very little modification) if we assume that the domains have merely $\mathcal{C}^{1,1}$ boundaries, and we expect that the same is true, provided that they have $\mathcal{C}^{1,\alpha}$ boundaries for $\alpha \in (0,1)$. The same proof works for quasiregular mappings having a continuous extension up to the boundary, or even more general, in Theorem 1.3 the condition "f is quasiconformal" can be replaced by the condition "f is a proper quasiregular mapping". The case $n = 2$ for quasiconformal harmonic mappings between Jordan domains in the plane with $\mathcal{C}^{1,\alpha}$ boundaries is settled in [13]. We notice that this is not true for domains with merely $\mathcal{C}^1$ boundaries as it is known for the planar case by some classical results. More precisely there is a univalent conformal mapping of the unit disk onto a Jordan domain with $\mathcal{C}^1$ boundary such that f is not globally Lipschitz continuous. We also refer to related results for biharmonic mappings [6].

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References

[1] Agmon, S., A. Douglis, and L. Nirenberg: Estimates near the boundary for elliptic partial differential equations satisfying general bounday conditions. I. - Comm. Pure Appl. Math. 12, 1959, 623–727.

[2] Ahlfors, L.: Lectures on quasiconformal mappings. - Van Nostrand Mathematical Studies, D. Van Nostrand, 1966.

[3] Ambrosio, L., and G. Dal Maso: A general chain rule for distributional derivatives. - Proc. Amer. Math. Soc. 108:3, 1990, 691–702.

[4] Astala, K., and P. Koskela: Quasiconformal mappings and global integrability of the derivative. - J. Anal. Math. 57, 1991, 203–220.

[5] Astala, K., and V. Manojlović: On Pavlović theorem in space. - Potential Anal. 43:3, 2015, 361–370.

[6] Chen, Sh., and X. Wang: Bi-Lipschitz characteristic of quasiconformal self-mappings of the unit disk satisfying bi-harmonic equation. - Indiana Univ. Math. J. 70:3, 2021, 1055–1086.

[7] Evans, L. C.: Partial differential equations. 2nd edition. - Grad. Stud. Math. 19, Amer. Math. Soc., Providence, RI, 2010.

[8] Fehlmann, R., and M. Vuorinen: Mori's theorem for n-dimensional quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 13:1, 1988, 111–124.

[9] Gehring, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. - Acta Math. 130, 1973, 265–277.

[10] Giaquinta, M., and L. Martinazzi: An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs. 2nd edition. - Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) 11. Pisa: Edizioni della Normale, 2012.

[11] Gilbarg, D., and N. Trudinger: Elliptic partial differential equations of second order. Second edition. - Springer, 1983.

[12] Hörmander, L.: The boundary problems of physical geodesy. - Arch. Ration. Mech. Anal. 62, 1976, 1–52.

[13] Kalaj, D.: Quasiconformal and harmonic mappings between Jordan domains. - Math. Z. 260:2, 2008, 237–252.

[14] Kalaj, D.: Harmonic mappings and distance function. - Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 10, 2011, 669–681.

[15] Kalaj, D.: On quasi-conformal self-mappings of the unit disc and elliptic PDEs in the plane. - Proc. R. Soc. Edinb. Sect. A Math. 143:4, 2013, 831–849.

[16] Kalaj, D.: A priori estimate of gradient of a solution to certain differential inequality and quasiregular mappings. - J. Anal. Math. 119, 2013, 63–88.

[17] Kalaj, D.: Quasiconformal harmonic mappings between Dini's smooth Jordan domains. - Pacific J. Math. 276:1, 2015, 213–228.

[18] Kalaj, D., and M. Mateljević: Inner estimate and quasiconformal harmonic maps between smooth domains. - J. Anal. Math. 100, 2006, 117–132.

[19] Kalaj, D., and M. Mateljević: (K, K')-quasiconformal harmonic mappings. - Potential Anal. 36:1, 2012, 117–135.

[20] Kalaj, D., and M. Pavlović: On quasiconformal self-mappings of the unit disk satisfying the Poisson equation. - Trans. Amer. Math. Soc. 363, 2011, 4043–4061.

[21] Kalaj, D., and E. Saksman: Quasiconformal mappings with controlled Laplacian. - J. Anal. Math. 137:1, 2019, 251–268.

[22] Kalaj, D., and A. Zlatičanin: Quasiconformal mappings with controlled Laplacian and Hölder continuity. - Ann. Acad. Sci. Fenn. Math. 44:2, 2019, 797–803.

[23] Martio, O.: On harmonic quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 425, 1968, 3–10.

[24] Martio, O., and R. Näkki: Continuation of quasiconformal mappings. - Sibirsk. Mat. Zh. 28:4, 1987, 162–170 (in Russian).

[25] Martio, O., and R. Näkki: Boundary Hölder continuity and quasiconformal mappings. - J. London Math. Soc. (2) 44:2, 1991, 339–350.

[26] Martio, O., and J. Väisälä: Global L^p-integrability of the derivative of a quasiconformal mapping. - Complex Var. Theory Appl. 9:4, 1988, 309–319.

[27] Partyka, D., and K. Sakan: Heinz type inequalities for Poisson integrals. - Comput. Methods Funct. Theory 14:2-3, 2014, 219–236.

[28] Partyka, D., and K. Sakan: On bi-Lipschitz type inequalities for quasiconformal harmonic mappings. - Ann. Acad. Sci. Fenn. Math. 32, 2007, 579–594.

[29] Partyka, D., and K. Sakan: Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains. - Ann. Acad. Sci. Fenn. Math. 39:2, 2014, 811–830.

[30] Pavlović, M.: Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc. - Ann. Acad. Sci. Fenn. Math. 27, 2002, 365–372.

[31] Pommerenke, C.: Boundary behaviour of conformal maps. - Grundlehren Math. Wiss. 299, Springer-Verlag, Berlin, 1992.

[32] Reshetnyak, Yu. G.: Generalized derivatives and differentiability almost everywhere. - Mat. Sb. (N.S.) 75:117, 1968, 323–334 (in Russian).

[33] Vuorinen, M.: Conformal geometry and quasiregular mappings. - Lecture Notes in Math. 1319, Springer-Verlag, Berlin, 1988.

[34] Ziemer, W. P.: Weakly differentiable functions. Sobolev spaces and functions of bounded variation. - Grad. Texts Math. 120, Springer-Verlag, New York, 1989.

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David Kalaj
University of Montenegro
Faculty of Natural Sciences and Mathematics
81000, Podgorica, Montenegro
davidk@ucg.ac.me