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Minimal degree rational open up mappings and related questions

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Dedicated to Vilmos Totik on his 70th birthday

Abstract. We establish the existence and uniqueness of rational conformal maps of minimal degree $n+1$ for opening up n arcs. In earlier results, the degree was exponential in n. We also discuss two related problems. (a) We establish existence of rational functions of minimal degree with prescribed critical values, and show that the number of (suitably normalized) rational functions is given in terms of the Hurwitz numbers. (b) We consider the problem of finding rational functions of minimal degree with prescribed critical points, where we establish existence of solutions by considering certain polynomial equations, and where the number of normalized solutions is bounded from above by a Catalan number. We illustrate our results with two examples.

Alimman asteen aukaisevia rationaalikuvauksia ja niihin liittyviä kysymyksiä

Tiivistelmä. Osoitamme, että on olemassa yksikäsitteinen rationaalinen konformikuvaus, jolla on alin mahdollinen aste $n+1$ ja joka aukaisee annetut n kaarta. Aiemmissa tuloksissa tarvittava aste oli eksponentiaalinen lukumäärän n suhteen. Lisäksi tarkastelemme kahta tähän liittyvää ongelmaa. (a) Osoitamme, että on olemassa rationaalifunktioita, joiden aste on alin mahdollinen ja joilla on annetut kriittiset arvot, ja näytämme, että (sopivasti normitettujen) rationaalifunktioiden lukumäärä vastaa Hurwitzin lukuja. (b) Tarkastelemme ongelmaa löytää rationaalifunktioita, joiden aste on alin mahdollinen ja joilla on annetut kriittiset pisteet. Osoitamme ratkaisujen olemassaolon tutkimalla tiettyjä polynomiyhtälöitä ja näytämme, että Catalanin luvut antavat ylärajan normitettujen ratkaisujen lukumäärälle. Havainnollistamme tuloksiamme kahdella esimerkillä.

1. Introduction

We call a rational function F of *type* (n,m) if it can be written as $F = P/Q$, where P and Q are coprime polynomials with $\deg(P) = n$ and $\deg(Q) = m$. The *degree* of F is $\deg(F) = \max\{n,m\}$. We denote the extended complex plane by $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$.

The original purpose of this research was to prove the following theorem.

Theorem 1.1. *Let $\gamma_1,\dots,\gamma_n$ be disjoint Jordan arcs in the complex plane. Then there exists a rational function F of type $(n+1,n)$ and a compact set $K \subset \mathbf{C}$ bounded by n disjoint Jordan curves such that F is a conformal map from $\mathbf{C}_\infty \setminus K$ onto $\mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ and $F(\infty) = \infty$. Moreover, F and K are unique up to pre-composition of F with a linear transformation. In particular, the normalization $F(z) = z + O(1/z)$ at infinity determines F and K uniquely.*

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Recall that Jordan arcs and Jordan curves are homeomorphic images of the interval $[0, 1]$ and the unit circle, respectively. Throughout this article, we always mean that a conformal map is also injective. Note that $F(\infty) = \infty$ always holds if F is of type $(n + 1, n)$. The inverse function $F^{-1}\colon \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j \to \mathbf{C}_\infty \setminus K$ "opens up" the arcs γ_j. We call F^{-1}, and for simpler language also F, an *open up mapping* for the arcs $\gamma_1, \dots, \gamma_n$.

Such results can be applied, among others, to prove asymptotically sharp Bernstein- and Markov-type inequalities on several Jordan arcs. In [19] and [20] it was shown how it works in the case of one arc.

For a rational function F in Theorem 1.1, the endpoints of the arcs are critical values (see Proposition 3.2 below). This observation leads to the problem of finding rational functions of minimal degree with prescribed critical values. As for terminology, the *critical points* are the points in the set $\{z \in \mathbf{C}\colon F'(z) = 0\}$, and the *critical values* are the elements of $\{F(z)\colon z \in \mathbf{C} \text{ with } F'(z) = 0\}$.

Theorem 1.2. *Let $\eta_1, \eta_2, \dots, \eta_{2n} \in \mathbf{C}$ be distinct. Then there exists a rational function F of type $(n+1, n)$ such that the set of critical values of F is $\{\eta_1, \eta_2, \dots, \eta_{2n}\}$. Moreover, each function can be normalized by $F(z) = z + O(1/z)$ at infinity. If $n = 1$, there is exactly one normalized function. If $n \geq 2$, the number of normalized functions is $(n + 1)H_n$ with the Hurwitz numbers*

$$H_n = \begin{cases} \frac{(2n)!}{n!}(n+1)^{n-3}, & \text{if } n \geq 2, \\ 1, & \text{if } n = 1. \end{cases} \tag{1}$$

In contrast to Theorem 1.1, the normalization $F(z) = z + O(1/z)$ at infinity does not uniquely determine a rational function with prescribed critical values when $n \geq 2$. We give an example for this in Section 7.

For each critical value η_j of F, there exists a critical point ζ_j with $F(\zeta_j) = \eta_j$ and $F'(\zeta_j) = 0$. This remark leads to the related problem of finding a rational function of minimal degree with prescribed critical points. It turns out that this is a simpler problem (with half as many equations and unknowns) than the previous one and it is answered by the following theorem.

Theorem 1.3. *Let $\zeta_1, \dots, \zeta_{2n} \in \mathbf{C}$ be distinct. Then there exists a rational function F of type $(n + 1, n)$ such that the set of critical points of F is $\{\zeta_1, \dots, \zeta_{2n}\}$. Moreover, each function F can be normalized by $F(z) = z + O(1/z)$ at infinity, and the number of normalized functions is bounded by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

That the degree is, indeed, minimal in Theorems 1.1–1.3 is shown next.

Proposition 1.4. *The rational functions in Theorems 1.1, 1.2 and 1.3 are minimal in the sense that if $F = P/Q$ with coprime polynomials P, Q, then neither $\deg(P) < n + 1$ nor $\deg(Q) < n$ can occur. Moreover, they have only simple poles.*

Proof. Since a rational function in Theorem 1.1 or 1.2 has $2n$ distinct critical points, it is enough to show the proposition for rational functions in Theorem 1.3 (i.e., with $2n$ distinct critical points). Let F be a rational function as in Theorem 1.3. Assume that $F = P/Q$ with coprime polynomials P, Q, and that

1. $\deg(F) \leq n$, or
2. $\deg(P) \leq n + 1$ and $\deg(Q) \leq n$, and we have strict inequality for P or Q.

Then $F' = (P'Q - PQ')/Q^2$ and $\deg(P'Q - PQ') < 2n$. In Theorem 1.3 if F has $2n$ distinct critical points then we obtain $F' \equiv 0$, i.e., F is constant, which is impossible.

It remains to show that F has simple poles. If $F = P/Q$ with $\deg(P) = n+1$ and $\deg(Q) = n$ has a pole of order $m \geq 2$, then, after cancelling common factors, the numerator of $F' = (P'Q - PQ')/Q^2$, has degree at most $2n+1-m < 2n$ but $2n$ zeros, which is impossible. □

Remark 1.5. Each function in Theorems 1.1, 1.2 or 1.3 (without normalization) yields infinitely many solutions of the respective problem by suitable affine transformations. Indeed, let $a, b \in \mathbf{C}$ with $a \neq 0$.

1. In Theorems 1.1 and 1.2, if F is a solution, then so is $F(az+b)$.
2. In Theorem 1.3, if F is a solution, then $aF(z)+b$ is also a solution.

In particular, if a solution exists, it is not unique. We call two solutions F, G equivalent, if there exist $a, b \in \mathbf{C}$ with $a \neq 0$ such that $G(z) = F(az+b)$ (in Theorems 1.1 and 1.2) or $G(z) = aF(z)+b$ (in Theorem 1.3). We can specify any normalization at infinity to obtain a unique representative of each equivalence class: Given $\alpha, \beta \in \mathbf{C}$ with $\alpha \neq 0$ and a solution F of Theorem 1.1, there exist unique a, b as above, such that $F(az+b) = \alpha z + \beta + O(1/z)$ for $z \to \infty$. A similar statement holds for the other two theorems.

The paper is organized as follows. We give an overview of known related results in Section 2. In Section 3, we show the existence and uniqueness in Theorem 1.1 using the theory of Riemann surfaces. This approach is purely geometric: it is short and intuitive, but not constructive. The results in Theorems 1.2 and 1.3 are of algebraic nature, and we reformulate both in terms of polynomial systems of equations. In Section 4, we prove Theorem 1.2. We give a geometric existence proof using the open up mapping, while the proof for the number of solutions builds on earlier results by Hurwitz and Mednykh on the number of Riemann surfaces with simple branch points. In Section 5, we prove Theorem 1.3 and use algebraic tools to show the existence of solutions. We conclude with a discussion in Section 6 and two examples in Section 7.

2. Overview of some known, earlier results

There are several results related to the three theorems above that are scattered through the literature. They appeared in various fields of mathematics and occurred in almost every decade in the last century. Let us recall some of them, not necessarily in chronological order.

Starting with Theorem 1.1, Widom's seminal paper [43] must be mentioned where he iterated Joukowskii mappings to construct a rational open up mapping, see [43, pp. 206–207]. In Widom's construction, the rational function has degree 2^n, i.e., it grows exponentially with the number of arcs. Later, Widom's iterated construction appeared in connection with Riemann surfaces, see the papers by Seppälä [39] and Hidalgo and Seppälä [15]. Seppälä attributes this approach to Myrberg [30], who, in turn, credits this idea to Poincaré, see [30, p. 4].

The question about the existence of rational functions with prescribed critical values can be considered in general: is it possible to cover Riemann surfaces with prescribed ramification sets? For results in this direction and going back to a problem of Hurwitz, we refer to Mednykh's paper [28] and the references therein.

These ramification sets or branching points naturally lead to Theorem 1.2. Instead of rational functions, polynomials with prescribed critical values were also investigated in Thom's paper [41], in which the existence of such polynomials was established. See also the papers by Mycielski and Paszkowski [29], Kammerer [21], Kuhn [24] and Kristiansen [23] for the real case and further references. Let us remark that Beardon, Carne, and Ng in [6] investigated the properties of the mapping from critical points to critical values, realizing and describing a natural connection between the two problems in the class of polynomials. This leads to Theorem 1.3.

Theorem 1.3 was proved by Goldberg in [12] using projective spaces, Grassmann manifolds, and homology classes. She also counted the number of solutions. Since we are also interested in obtaining the solutions, we show the existence by more constructive means.

For real rational functions with prescribed real critical points we refer to the paper of Eremenko and Gabrielov [10] and the recent article [34] where such rational functions are used in Schramm–Loewner evolution.

An interesting application of the open up mapping is the computation of the logarithmic capacity of a compact set E consisting of n disjoint arcs. The rational function maps the exterior of E to a domain with smooth boundary, from which the logarithmic capacity of E can be computed numerically with a conformal map of Walsh, as described by Nasser, Liesen and Sète in [31] and [26].

The necessity of obtaining conformal representations by rational functions also appeared in the study of multiple orthogonal polynomials, see [3, 4, 5, 27]. In general, such representations are different from open up mappings, but they are the same in the case of two arcs. The case of two real intervals was considered by López Lagomasino, Pestana, Rodríguez, and Yakubovich in [27].

Instead of (general) rational functions, similar questions can be considered among (finite) Blaschke products. See, e.g., [22, 38] for further references. A similar open problem (determining a Blaschke product or its zeros from critical values) is also of interest and is raised in [38].

We mention that Theorem 1.1 is more precise than [19, Prop. 5]. Note that there is a minor flaw in the proof of existence in [19, Prop. 5], while the remaining proof is correct. In our paper, we prove the existence of F and K with a different method.

3. Existence and uniqueness of the open up mapping

In this section, we prove Theorem 1.1 using the theory of Riemann surfaces. For background on Riemann surfaces and the Riemann–Roch theorem, we refer to the books [37] or [11]. If Γ is a Jordan curve, denote the bounded and unbounded components of $\mathbf{C} \setminus \Gamma$ by $\operatorname{int}(\Gamma)$ and $\operatorname{ext}(\Gamma)$, respectively.

Proof of the existence in Theorem 1.1. Let γ be a Jordan arc in $\mathbf{C}$. Every point of γ is accessible from $\mathbf{C} \setminus \gamma$; see [33, p. 164]. We will show that every point of γ that is not an endpoint of γ gives rise to two distinct accessible boundary points of $\mathbf{C}_\infty \setminus \gamma$, while an endpoint of γ gives rise to one accessible boundary point of $\mathbf{C}_\infty \setminus \gamma$. Recall that a boundary point z of a domain D is *accessible* from D if there exists a Jordan arc ℓ with one endpoint at z and otherwise contained in D; see, e.g., [33, p. 162] or [13, Ch. II, §3, p. 35]. Such a Jordan arc ℓ is also called an *end-cut* [33, p. 118]. Following Goluzin [13, pp. 36–37], two accessible boundary points z_1 and z_2 in ∂D are regarded as distinct, if either $z_1 \neq z_2$ or if $z_1 = z_2$ but given two end-cuts ℓ_1, ℓ_2, there exists a neighborhood U of z_1 such that ℓ_1, ℓ_2 cannot be joined in $U \cap D$. There exists a Jordan curve $\widetilde{\gamma}$ in $\mathbf{C}$ such that γ is an arc of $\widetilde{\gamma}$, i.e., $\gamma \subseteq \widetilde{\gamma}$; see [33, Ch. VI, Thm. 14.5] or [36, Cor. 17.23]. Let $\mathbb{D} = \{z \in \mathbf{C} \colon |z| < 1\}$ be the unit disk. By the Schoenflies theorem (see, e.g., [35, Cor. 2.9]), there exists a homeomorphism $f \colon \mathbf{C} \to \mathbf{C}$ such that $f(\widetilde{\gamma}) = \partial\mathbb{D}$ is the unit circle, $f(\operatorname{int}(\widetilde{\gamma})) = \operatorname{int}(\partial\mathbb{D}) = \mathbb{D}$ and

$f(\mathrm{ext}(\widetilde{\gamma})) = \mathrm{ext}(\partial\mathbb{D})$. It is not hard to see that every point $z \in \partial\mathbb{D}$ gives rise to one accessible boundary point in $\mathbb{D}$ and to one in $\mathrm{ext}(\partial\mathbb{D})$. By the homeomorphism f, also every $z \in \widetilde{\gamma}$ gives rise to one accessible boundary point in $\mathrm{int}(\widetilde{\gamma})$, denoted by z^+, and to one accessible boundary point in $\mathrm{ext}(\widetilde{\gamma})$, denoted by z^-. In the domain $\mathbf{C}_\infty\backslash\gamma$, a point $z \in \gamma$ that is not an endpoint of γ gives rise to two accessible boundary points: z^+ accessible from $\mathrm{int}(\widetilde{\gamma})$ and z^- accessible from $\mathrm{ext}(\widetilde{\gamma})$. An endpoint z of γ yields one accessible boundary point ($z^+ = z^-$ in $\mathbf{C}_\infty \setminus \gamma$), which follows from [33, Thm. 14.2, p. 162]. Set $\gamma^+ := \{z^+ : z \in \gamma\}$ and $\gamma^- := \{z^- : z \in \gamma\}$. As point sets, $\gamma^+ = \gamma^- = \gamma$, but as sets of accessible boundary points, γ^+ and γ^- are two Jordan arcs that are disjoint except at their endpoints.

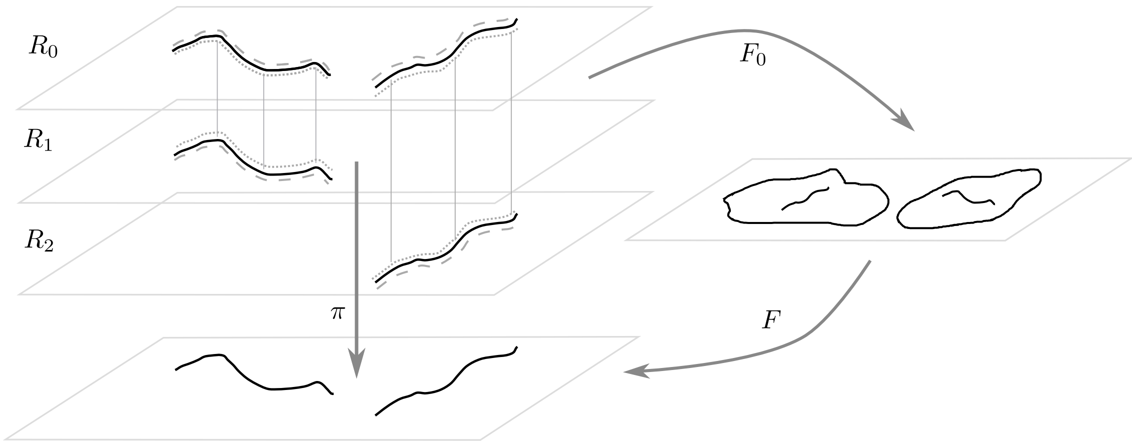

Figure 1. The Riemann surface R and the maps F_0, π, and F in the proof of the existence in Theorem 1.1 in the case of two arcs.

Take $n+1$ copies of the Riemann sphere, denoted by $R_0, R_1, \ldots, R_n$. Cut R_0 along all the arcs $\gamma_1, \ldots, \gamma_n$. For $j = 1, \ldots, n$, cut R_j along γ_j and join it crosswise to R_0 along the arc γ_j, i.e., for $z \in \gamma_j$ we identify $z^+ \in \gamma_j^+$ in R_0 with $z^- \in \gamma_j^-$ in R_j, and $z^- \in \gamma_j^-$ in R_0 with $z^+ \in \gamma_j^+$ in R_j. (See, e.g., [18, Ch. 4.3, p. 83] for the identification.) This results in a Riemann surface which we denote by R; see, e.g., [32, p. 8] or the Russian translation of Hurwitz and Courant [17, pp. 383–384 and p. 579]. Figure 1 illustrates the construction of R for $n = 2$. Let us sketch why R is a Riemann surface. The identification yields a topological surface, on which we define the following charts: $w = z$ in a neighborhood of a finite point that is not a branch point (not an endpoint of an arc $\gamma_1, \ldots, \gamma_n$), $w = \sqrt{z - \eta_k}$ in a neighborhood of a branch point η_k (an endpoint of one of the arcs $\gamma_1, \ldots, \gamma_n$), and $w = 1/z$ in a neighborhood of $z = \infty$.

For each $j = 1, \ldots, n$, the above identification of γ_j^+ in R_0 with γ_j^- in R_j yields a simple arc $\hat{\gamma}_j^+$ in R, and the identification of γ_j^- in R_0 with γ_j^+ in R_j yields a simple arc $\hat{\gamma}_j^-$ in R. Since $\hat{\gamma}_j^+$ and $\hat{\gamma}_j^-$ are disjoint except for their endpoints, $\hat{\gamma}_j := \hat{\gamma}_j^+ \cup \hat{\gamma}_j^-$ is a simple closed curve in R.

Note that R is simply connected (i.e., has genus 0) and compact. One of the corollaries of the Riemann–Roch theorem (see e.g. [11, pp. 130–131]) says that there is a biholomorphic mapping F_0 from R onto the Riemann sphere $\mathbf{C}_\infty$. (Alternatively, since R is compact and simply connected, the uniformization theorem implies the existence of F_0.) We choose F_0 such that $\infty \in R_0$ is mapped to ∞ (if needed, this can be achieved by postcomposition with a Möbius transformation).

Let $\pi : R \to \mathbf{C}_\infty$ be the canonical projection from R onto the Riemann sphere, i.e., with $\pi(w^{(k)}) = w$, where $w \in \mathbf{C}$, and $w^{(k)} \in R_k$ is above w. Then $F :=$

$\pi \circ F_0^{-1}\colon \mathbf{C}_\infty \to \mathbf{C}_\infty$ is a meromorphic function and hence F is a rational function (see e.g. [11, p. 11] or [42, Thm. 3.5.8]). Because of the projection, F is an $(n+1)$-to-1 mapping and thus F has degree $n+1$. The poles of F are the images under F_0 of $\infty \in R_j$, $j = 0, \ldots, n$. In particular, F has $n+1$ distinct poles and these are simple. Since $F_0^{-1}(\infty) = \infty \in R_0$, also $F(\infty) = \infty$ and F is of type $(n+1, n)$.

Consider the simply connected domains $R_j \setminus \gamma_j = \mathbf{C}_\infty \setminus \gamma_j$, $j = 1, \ldots, n$, in R and the n-connected domain $R_0 \setminus \bigcup_{j=1}^n \gamma_j = \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ in R. Since F_0 is biholomorphic,

$$G_j := F_0(R_j \setminus \gamma_j) \subseteq \mathbf{C}_\infty, \quad j = 1, \ldots, n,$$

are simply connected domains and

$$G_\infty := F_0(R_0 \setminus \textstyle\bigcup_{j=1}^n \gamma_j) \subseteq \mathbf{C}_\infty$$

is an n-connected domain, and $\infty \in G_\infty$ by the above choice of F_0. The domains $G_\infty, G_1, \ldots, G_n$ are disjoint since the sheets $R_0, R_1, \ldots, R_n$ are disjoint and F_0 is biholomorphic. In particular, $G_1, \ldots, G_n$ are bounded.

By construction, the boundary of $R_j \setminus \gamma_j$ in R is the simple closed curve $\hat{\gamma}_j$, and the boundary of $R_0 \setminus \bigcup_{j=1}^n \gamma_j$ in R consists of the simple closed curves $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$. Since $F_0\colon R \to \mathbf{C}_\infty$ is biholomorphic, $\Gamma_j := F_0(\hat{\gamma}_j)$, $j = 1, \ldots, n$, are disjoint Jordan curves in $\mathbf{C}_\infty$. Moreover, Γ_j is the boundary of G_j and one boundary component of G_∞. Together, we obtain that $\partial G_\infty = \bigcup_{j=1}^n \Gamma_j$ consists of n Jordan curves, and

$$K := \mathbf{C}_\infty \setminus G_\infty = \textstyle\bigcup_{j=1}^n (G_j \cup \Gamma_j) = \bigcup_{j=1}^n \overline{G}_j$$

is a compact set with n components, each of which is the closure of a simply connected Jordan domain. By construction, $F = \pi \circ F_0^{-1}\colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ is bijective and conformal. Similarly, $F\colon G_j \to \mathbf{C}_\infty \setminus \gamma_j$ is bijective and conformal. This completes the proof. $\square$

Remark 3.1. If the Jordan arcs γ_j are C^{k+} smooth, then the Jordan curves making up ∂K are C^{k+} smooth too, see [43, p. 206], where C^{k+} means k times continuously differentiable and the k-th derivative is Lipschitz α for some $\alpha > 0$. Moreover, analyticity is also preserved, that is, if γ_j are analytic Jordan arcs, then ∂K consists of analytic Jordan curves, see [19, p. 879]. Both assertions are also clear from the proof of the existence in Theorem 1.1, since $\Gamma_j = F_0(\hat{\gamma}_j)$ is the image of a simple closed curve under a biholomorphic map.

Next, we show that the critical values of an open up mapping as in Theorem 1.1 are precisely the endpoints of the arcs.

Proposition 3.2. *Let $\gamma_1, \ldots, \gamma_n$ be disjoint Jordan arcs in the complex plane. Denote the endpoints of γ_j by η_{2j-1} and η_{2j}, $j = 1, 2, \ldots, n$. If F is a rational open up mapping of type $(n+1, n)$ for $\gamma_1, \ldots, \gamma_n$, then the set of critical values of F is $\{\eta_1, \ldots, \eta_{2n}\}$. Moreover, $F^{-1}(\{\eta_j\})$ consists of n distinct points, one of them has multiplicity two and is a critical point of F, and the others have multiplicity one.*

Proof. Let F be a rational open up mapping of type $(n+1, n)$ for $\gamma_1, \ldots, \gamma_n$. In particular, $F\colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ is bijective and conformal. Denote the boundary curves of K by $\Gamma_1, \ldots, \Gamma_n$, labelled such that $F(\Gamma_j) = \gamma_j$ for $j = 1, 2, \ldots, n$.

Fix $k \in \{1, 2, \ldots, 2n\}$ and let η_k be an endpoint of γ_j (i.e. $k = 2j-1$ or $k = 2j$). Since F is of type $(n+1, n)$, there are $n+1$ pre-images of η_k under F in $\mathbf{C}$: There is exactly one in $\operatorname{int}(\Gamma_\ell)$ for each $\ell \neq j$, and there are two on Γ_j. (The pre-images cannot be in $\operatorname{int}(\Gamma_j)$, since $F\colon \operatorname{int}(\Gamma_j) \to \mathbf{C}_\infty \setminus \gamma_j$.) Let $\zeta_k \in \Gamma_j$ with $F(\zeta_k) = \eta_k$. Since η_k are the endpoints of arcs, we must have $F'(\zeta_k) = 0$ (otherwise F is locally bijective at ζ_k and $\eta_k = F(\zeta_k)$ is an interior point of γ_j), hence η_k is a critical value of F and ζ_k is a double pre-image of η_k under F. The set of critical values of F is $\{\eta_1, \dots, \eta_{2n}\}$, since F is of type $(n+1, n)$ and cannot have any further critical values. $\square$

Next, we show the uniqueness up to a linear transformation in Theorem 1.1, thus completing the proof.

Proof of the uniqueness in Theorem 1.1. Let $F, \widetilde{F}$ be rational functions that are open up mappings. We construct an analytic map $\varphi\colon \mathbf{C}_\infty \to \mathbf{C}_\infty$ with $\widetilde{F}(z) = F(\varphi(z))$ and show that $\varphi(z) = az + b$.

Let $G_\infty := \mathbf{C}_\infty \setminus K$ be the region that is mapped by F onto $\mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$. For $j = 1, \dots, n$, denote by Γ_j the boundary curve of K that is mapped by F onto γ_j, and let $G_j = \operatorname{int}(\Gamma_j)$. Then $F\colon G_j \to \mathbf{C}_\infty \setminus \gamma_j$ for $j = 1, \dots, n$ and $F\colon G_\infty \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ are conformal (and bijective). Introduce the same notation for $\widetilde{F}$, but with tildes. Then $\varphi := F^{-1} \circ \widetilde{F}\colon \widetilde{G}_j \to G_j$ is conformal (and bijective) for all $j = 1, \dots, n, \infty$, and $\widetilde{F}(z) = F(\varphi(z))$ for $z \in \widetilde{G}_1 \cup \ldots \cup \widetilde{G}_n \cup \widetilde{G}_\infty$.

We extend φ to an analytic function on $\mathbf{C}$ with a simple pole at ∞. Since $\widetilde{G}_1, \dots, \widetilde{G}_n$ are Jordan regions, φ extends to a homeomorphism $\widetilde{G}_j \cup \widetilde{\Gamma}_j \to G_j \cup \Gamma_j$ by the Osgood–Carathéodory theorem; see [14, Thm. 5.10e] or [42, Thm. 6.5.1]. This is also true for $\widetilde{G}_\infty$ by subdividing the relevant parts in Jordan regions, similar to [14, p. 385].

Let $j \in \{1, \dots, n\}$ and $\widetilde{z}_0 \in \widetilde{\Gamma}_j$. Then $\varphi\colon \widetilde{G}_j \cup \widetilde{\Gamma}_j \to G_j \cup \Gamma_j$ maps $\widetilde{z}_0$ to a point $z_0 \in \Gamma_j$. First, consider $\widetilde{z}_0 \in \widetilde{\Gamma}_j$ that is not a critical point of $\widetilde{F}$, so that $\widetilde{F}(\widetilde{z}_0)$ is not an endpoint of γ_j (see Proposition 3.2). Then $z_0 \in \Gamma_j$ is not a critical point of F. (Otherwise $F(z_0) = \widetilde{F}(\widetilde{z}_0)$ would be an endpoint of γ_j.) Then there exist open neighborhoods U of z_0, $\widetilde{U}$ of $\widetilde{z}_0$, and V and $\widetilde{V}$ of $F(z_0)$, such that $F\colon U \to V$ and $\widetilde{F}\colon \widetilde{U} \to \widetilde{V}$ are conformal (and bijective). Without loss of generality, we have $V = \widetilde{V}$. Then $F^{-1} \circ \widetilde{F}\colon \widetilde{U} \to U$ is analytic, maps $\widetilde{U} \cap \widetilde{G}_k \to U \cap G_k$ for $k \in \{j, \infty\}$ and $\widetilde{U} \cap \widetilde{\Gamma}_j \to U \cap \Gamma_j$, and coincides with φ in $\widetilde{U} \cap \widetilde{G}_j$ and $\widetilde{U} \cap \widetilde{G}_\infty$. Therefore, φ extends to an analytic function in $\widetilde{U}$.

Next, let $\widetilde{z}_0 \in \widetilde{\Gamma}_j$ be a critical point of $\widetilde{F}$, then $z_0 \in \Gamma_j$ is a critical point of F. By the above extension, φ is analytic in a punctured neighborhood of $\widetilde{z}_0$ and continuous at $\widetilde{z}_0$, hence also analytic at $\widetilde{z}_0$.

Therefore, φ is analytic in $\mathbf{C}$ with a simple pole at ∞, hence $\varphi(z) = az + b$ with $a \neq 0$. Then $\widetilde{F}(z) = F(\varphi(z)) = F(az + b)$. This shows uniqueness up to a linear transformation. Finally, if F and $\widetilde{F}$ both have the form $z + O(1/z)$ at infinity, then $\varphi(z) = z$ and $F = \widetilde{F}$. $\square$

The open up mapping in Theorem 1.1 depends only on the endpoints of the arcs and the topology of $\mathbf{C} \setminus \bigcup_{j=1}^n \gamma_j$, but does not depend on the specific shape of the arcs. This is shown in the next theorem, which is formulated for one arc, but can be applied iteratively to allow deformation of all arcs.

Theorem 3.3. *Let $\gamma_1, \dots, \gamma_n$ be disjoint Jordan arcs in the complex plane, and let $F\colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ be an open up mapping as in Theorem 1.1. Let $\widetilde{\gamma}_1$ be a Jordan arc with same endpoints as γ_1 which is homotopic with fixed endpoints to γ_1 in $\mathbf{C} \setminus (\gamma_2 \cup \ldots \cup \gamma_n)$. Then F is also an open up mapping for $\widetilde{\gamma}_1, \gamma_2, \dots, \gamma_n$,* *i.e., there exists a compact set* $\widetilde{K}$ *bounded by* n *disjoint Jordan curves such that* $F\colon \mathbf{C}_\infty \setminus \widetilde{K} \to \mathbf{C}_\infty \setminus (\widetilde{\gamma}_1 \cup \gamma_2 \cup \ldots \cup \gamma_n)$ *is conformal and bijective.*

Proof. Since $F\colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ is an open up mapping, ∂K consists of n disjoint Jordan curves $\Gamma_1, \ldots, \Gamma_n$, labeled such that $F(\Gamma_j) = \gamma_j$ for $j = 1, 2, \ldots, n$. By Proposition 3.2, the endpoints η_1, η_2 of γ_1 are critical values of F. Let $\zeta_1, \zeta_2 \in \Gamma_1$ be the critical points of F with $F(\zeta_j) = \eta_j$ for $j = 1, 2$.

Let $f_0\colon [0,1] \to \widetilde{\gamma}_1$, $f_1\colon [0,1] \to \gamma_1$, be continuous and bijective functions with $f_0(0) = f_1(0) = \eta_1$, $f_0(1) = f_1(1) = \eta_2$. By assumption of the theorem, f_0, f_1 are homotopic with fixed endpoints in $D = \mathbf{C} \setminus (\gamma_2 \cup \cdots \cup \gamma_n)$, i.e., there exists a continuous function $H\colon [0,1] \times [0,1] \to D$ with $H(0,t) = f_0(t)$ and $H(1,t) = f_1(t)$ for all $t \in [0,1]$ and $H(s,0) = \eta_1$ and $H(s,1) = \eta_2$ for all $s \in [0,1]$. Then $H([0,1] \times [0,1]) \subseteq D$ is compact.

Let γ be a positively oriented Jordan curve in D such that $H([0,1] \times [0,1]) \subseteq \operatorname{int}(\gamma)$ and $\gamma_2, \ldots, \gamma_n \subseteq \operatorname{ext}(\gamma)$. Let $\Gamma^{(e)}$ be the pre-image in $\mathbf{C} \setminus K$ of γ under F. Then $\Gamma^{(e)}$ is a positively oriented Jordan curve with Γ_1 in its interior (i.e., $\Gamma_1 \subseteq \operatorname{int}(\Gamma^{(e)})$), since $F\colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus \bigcup_{j=1}^n \gamma_j$ is conformal and bijective. Since $F\colon \operatorname{int}(\Gamma_1) \to \mathbf{C}_\infty \setminus \gamma_1$ is also conformal and bijective, there exists a Jordan curve $\Gamma^{(i)} \subseteq \operatorname{int}(\Gamma_1)$ such that $F\colon \Gamma^{(i)} \to \gamma$ is bijective. Note that $\Gamma^{(i)}$ is negatively oriented.

Let A denote the (open) ring-domain bounded by $\Gamma^{(e)}$ and $\Gamma^{(i)}$. Then F is holomorphic on A, and $\Gamma_1 \subseteq A$. For $w \in \operatorname{int}(\gamma)$, the winding of $F - w$ along $\Gamma^{(i)} \cup \Gamma^{(e)}$ is $W(F - w; \Gamma^{(i)} \cup \Gamma^{(e)}) = 2$, hence $F\colon A \to \operatorname{int}(\gamma)$ is 2 to 1 by the argument principle.

We have $\widetilde{\gamma}_1 \subseteq \operatorname{int}(\gamma)$ by the definition of γ. Hence the pre-image of $\widetilde{\gamma}_1$ under F in A consists of two Jordan arcs connecting ζ_1, ζ_2. Since F is conformal in $A \setminus \{\zeta_1, \zeta_2\}$, the two arcs cannot intersect except at ζ_1, ζ_2, and hence form a Jordan curve $\widetilde{\Gamma}_1$. We orient $\widetilde{\Gamma}_1$ in the negative sense. We then have for $w \in \operatorname{int}(\gamma) \setminus \widetilde{\gamma}_1$ that $W(F - w; \Gamma^{(e)} \cup \widetilde{\Gamma}_1) = 1$, hence F maps the ring domain bounded by $\widetilde{\Gamma}_1$ and $\Gamma^{(e)}$ bijectively onto $\operatorname{int}(\gamma) \setminus \widetilde{\gamma}_1$. This implies that $F\colon \mathbf{C}_\infty \setminus \widetilde{K} \to \mathbf{C}_\infty \setminus (\widetilde{\gamma}_1 \cup \bigcup_{j=2}^n \gamma_j)$ is conformal and bijective and hence an open up mapping, where $\mathbf{C}_\infty \setminus \widetilde{K}$ is the unbounded domain with boundary $\widetilde{\Gamma}_1 \cup \bigcup_{j=2}^n \Gamma_j$. $\square$

We saw in Proposition 3.2 that an open up mapping has the endpoints of the arcs as critical values. Theorem 3.3 clarifies the difference between an open up mapping and rational functions with critical values at the endpoint of the arcs. While the information on the critical values is present in both problems, the difference is the additional "topological" information about the arcs in Theorem 1.1, which is not present in Theorem 1.2. We give an example for this difference in Section 7.

4. Rational functions with prescribed critical values

In this section, we first prove Theorem 1.2. Afterwards, we also consider an equivalent polynomial formulation, which can be suitable for the computation of rational functions with prescribed critical values.

The existence of solutions in Theorem 1.2 readily follows from Theorem 1.1 and Proposition 3.2. The exact number of solutions is derived with results on ramified coverings of the Riemann sphere.

Proof of Theorem 1.2. *Step 1: Existence.* Connect the points $\eta_1, \ldots, \eta_{2n}$ pairwise by Jordan arcs that do not intersect each other. By Theorem 1.1 there exists a rational open up mapping F of type $(n+1, n)$, which by Proposition 3.2 is a solution in Theorem 1.2.

Step 2: Number of normalized solutions. Let F, F_1 be two rational functions of type $(n+1,n)$ with critical values $\eta_1, \dots, \eta_{2n}$ (we do not impose the normalization at infinity yet). Following Mednykh [28], we call F, F_1 equivalent, if there exists a homeomorphism $\varphi\colon \mathbf{C}_\infty \to \mathbf{C}_\infty$ with $F = F_1 \circ \varphi$. It follows that φ is a Möbius transformation.

By assumption, the critical values $\eta_1, \dots, \eta_{2n} \in \mathbf{C}$ are distinct and hence are ramification points of order 2 of the rational functions. In this case, the number of non-equivalent coverings (i.e., of equivalence classes) in the sense of Mednykh is given by the Hurwitz numbers H_n in (1), see [28] and Hurwitz' original article [16, p. 22]; see also [25, p. 290] and [8, Eqn. (4.10)].

In the second step, we estimate the number of normalized functions in Theorem 1.2 in each equivalence class. Note first that, by Remark 1.5, each function is equivalent to one that is normalized at infinity. Next, let F, F_1 be *two equivalent* rational functions of type $(n+1,n)$ with critical values $\eta_1, \dots, \eta_{2n}$ and normalized at infinity by $z + O(1/z)$. Let us write

$$F(z) = z + \sum_{j=1}^{n} \frac{r_j}{z - p_j},$$

where $p_1, \dots, p_n \in \mathbf{C}$ are distinct, and $r_j \neq 0$ for $j = 1, \dots, n$. As shown above, $F = F_1 \circ \varphi$ with a Möbius transformation φ. In particular, p is a pole of F if and only if $\varphi(p)$ is a pole of F_1. In particular, $\varphi^{-1}(\infty)$ is a pole of F and we distinguish two cases.

If $\varphi^{-1}(\infty) = \infty$, then φ is a linear transformation of the form $\varphi(z) = az + b$ with nonzero $a \in \mathbf{C}$. The normalization of F and F_1 at infinity and $F = F_1 \circ \varphi$ imply $\varphi(z) = z$, hence $F = F_1$.

Otherwise, $\varphi^{-1}(\infty)$ is a finite pole of F and there exists $j_0 \in \{1, \dots, n\}$ such that $\varphi^{-1}(\infty) = p_{j_0} \in \mathbf{C}$. Then φ has the form

$$\varphi(z) = \frac{az + b}{z - p_{j_0}}, \quad \varphi^{-1}(z) = \frac{p_{j_0} z + b}{z - a}, \tag{2}$$

with $a, b \in \mathbf{C}$ and $-ap_{j_0} - b \neq 0$. We compute

$$\begin{aligned} F_1(z) = F(\varphi^{-1}(z)) &= \frac{p_{j_0} z + b}{z - a} + \sum_{j=1}^{n} r_j \frac{z - a}{(p_{j_0} z + b) - p_j(z - a)} \\ &= p_{j_0} + \frac{ap_{j_0} + b}{z - a} + \sum_{j=1}^{n} r_j \frac{z - a}{(p_{j_0} - p_j) z + ap_j + b}. \end{aligned} \tag{3}$$

We distinguish the cases $j = j_0$ and $j \neq j_0$ in the sum. If $j \neq j_0$ then

$$\frac{z - a}{(p_{j_0} - p_j) z + ap_j + b} = \frac{1}{p_{j_0} - p_j} \cdot \frac{z - a}{z - \varphi(p_j)} = \frac{1}{p_{j_0} - p_j} + \frac{\frac{\varphi(p_j) - a}{p_{j_0} - p_j}}{z - \varphi(p_j)}.$$

Inserting this in (3) yields

$$F_1(z) = \frac{r_{j_0}}{ap_{j_0} + b}(z - a) + p_{j_0} + \frac{ap_{j_0} + b}{z - a} + \sum_{j \neq j_0} \frac{r_j}{p_{j_0} - p_j} + \sum_{j \neq j_0} \frac{r_j \frac{\varphi(p_j) - a}{p_{j_0} - p_j}}{z - \varphi(p_j)}.$$

By assumption, $F(z) = z + O(1/z)$ at ∞. Comparing the coefficient of z yields

$$\frac{r_{j_0}}{ap_{j_0} + b} = 1, \tag{4}$$

and comparing the constant coefficient yields

$$a = p_{j_0} + \sum_{j \neq j_0} \frac{r_j}{p_{j_0} - p_j}. \tag{5}$$

Moreover, by (4), $b = r_{j_0} - ap_{j_0}$. Thus, a and b are uniquely determined by F and p_{j_0}. This shows that φ is fully determined by F and a choice of one of the $n+1$ poles of F. Thus, each of the H_n many equivalence classes contains at most $n+1$ distinct normalized functions, which establishes the *bound* $(n+1)H_n$ on the number of normalized rational functions in Theorem 1.2. We proceed to derive the *exact* number.

If $n = 1$, there is one equivalence class since $H_1 = 1$. Thus the bound gives 2, but there is only one solution in Theorem 1.2, which can be seen as follows; see also Proposition 6.1 where the solution is derived explicitly. If $F(z) = z + \frac{r_1}{z - p_1}$ and if we choose $j_0 = 1$, then $a = p_1$ by (5) and thus $F_1(z) = z + \frac{r_1}{z-a} = F(z)$, so that there is only a single normalized solution in the only equivalence class.

If $n \geq 2$, the $n+1$ normalized functions in each equivalence class are distinct, which we show next. Consider as above $F = F_1 \circ \varphi$ with a Möbius transformation φ of the form (2) for some $j_0 \in \{1, \ldots, n\}$. We show by contradiction that F and F_1 are distinct, and therefore assume that $F = F_1$. This leads to

$$F(z) = (F \circ \varphi)(z). \tag{6}$$

In particular, $\varphi(\{p_1, \ldots, p_n, p_\infty = \infty\}) = \{p_1, \ldots, p_n, p_\infty = \infty\}$, that is φ permutes the poles of F. Since the set of poles of F is finite, there exists a positive integer N such that φ^N, the N-th iteration of φ, satisfies $\varphi^N(p_j) = p_j$ for $j = 1, \ldots, n, \infty$, hence the Möbius transformation φ^N has $n + 1 \geq 3$ fixed points and thus is the identity, $\varphi^N = \mathrm{id}$. Note that $\varphi \neq \mathrm{id}$, since φ has a finite pole at $p_{j_0} \in \mathbf{C}$.

Next, we show that φ has two distinct fixed points. Since $\varphi \neq \mathrm{id}$, it has one or two fixed points. Let us assume that φ has only one fixed point $A \in \mathbf{C}_\infty$, we shall reach a contradiction. Let $z = \chi(u)$ be a Möbius transformation with $\chi(\infty) = A$. The Möbius transformation $\chi^{-1} \circ \varphi \circ \chi$ fixes infinity and has only one fixed point, hence $(\chi^{-1} \circ \varphi \circ \chi)(u) = u + \beta$ for some $\beta \in \mathbf{C}$. Moreover, the set $\chi^{-1}(\{p_1, \ldots, p_n, \infty\})$ is invariant under $\chi^{-1} \circ \varphi \circ \chi$, hence $\beta = 0$ and $\varphi = \mathrm{id}$, a contradiction. Thus, φ has two distinct fixed points $A, B \in \mathbf{C}_\infty$, $A \neq B$.

There exists a Möbius transformation $z = \psi(w)$ such that $\psi(0) = A$ and $\psi(\infty) = B$. Therefore $\psi^{-1}(\varphi(\psi(w))) = \lambda w$ for some $\lambda \in \mathbf{C}$, where $\lambda \neq 1$ since $\varphi \neq \mathrm{id}$. Transforming φ to the w-plane, we can write

$$w = \psi^{-1}\big(\varphi^N(\psi(w))\big) = (\psi^{-1} \circ \varphi \circ \psi)^N(w) = \lambda^N w$$

which implies that $\lambda^N = 1$. Since $\lambda \neq 1$, we have $N \geq 2$, and we can choose the smallest number $\nu \geq 2$ with $\lambda^\nu = 1$. We transform F to the w-plane and define $G(w) := F(\psi(w))$. Then (6) yields $G(w) = F(\psi(w)) = F(\varphi(\psi(w))) = F(\psi(\lambda w)) = G(\lambda w)$. Note that F and G have the same critical values $\eta_1, \ldots, \eta_{2n}$. Let $w_1, \ldots, w_{2n} \in \mathbf{C}$ be critical points of G with $G(w_j) = \eta_j$ for $j = 1, \ldots, 2n$. Since $\deg(G) = \deg(F) = n+1$, we have $\deg(G') \leq 2n+1$, i.e., G has at most $2n+1$ critical points. Now, $G'(w) = \lambda G'(\lambda w)$, implies that with w_j, also $\lambda w_j, \lambda^2 w_j, \ldots, \lambda^{\nu-1} w_j$ are critical points corresponding to the critical value η_j. If $w_j \neq 0$, these are pairwise distinct since $\nu \geq 2$ is minimal with the property $\lambda^\nu = 1$. Since critical points corresponding to different critical values are distinct, we obtain that G has strictly more than $2n+1$ critical points, a contradiction. Therefore, the assumption $F = F_1$ is wrong, and there are indeed exactly $n+1$ distinct normalized rational functions in each equivalence class. This completes the proof of Theorem 1.2. $\square$

In view of computing the rational functions with prescribed critical values, we reformulate Theorem 1.2 in terms of polynomials and show the equivalence of the two formulations in Proposition 4.2.

Theorem 4.1. *Let $\eta_1, \eta_2 \dots, \eta_{2n} \in \mathbf{C}$ be distinct. Then there exist polynomials*

$$P(z) = z^{n+1} + \sum_{j=0}^{n-1} p_j z^j \quad \text{and} \quad Q(z) = z^n + \sum_{j=0}^{n-1} q_j z^j, \tag{7}$$

i.e., with $p_{n+1} = q_n = 1$ and $p_n = 0$, and points $\zeta_1, \zeta_2, \dots, \zeta_{2n} \in \mathbf{C}$ such that

$$P(\zeta_j) - \eta_j Q(\zeta_j) = 0, \quad j = 1, 2, \dots, 2n, \tag{8}$$

$$P'(\zeta_j)Q(\zeta_j) - P(\zeta_j)Q'(\zeta_j) = 0, \quad j = 1, 2, \dots, 2n, \tag{9}$$

and

$$Q(\zeta_j) \neq 0, \quad j = 1, 2, \dots, 2n. \tag{10}$$

If $n = 1$, the solution (P, Q) is unique. If $n \geq 2$, the number of solutions (P, Q) is $(n+1)H_n$ with the Hurwitz number H_n in (1).

There are $4n$ equations in (8) and (9) for the $4n$ unknowns $p_{n-1}, \dots, p_1, p_0$, $q_{n-1}, \dots, q_1$, q_0 and $\zeta_1, \zeta_2, \dots, \zeta_{2n}$ in Theorem 4.1, which is twice as many as in Theorem 5.1.

Equation (8) prescribes the values of $F = P/Q$ at the points $\zeta_1, \dots, \zeta_{2n}$, provided that (10) holds, and $\zeta_1, \dots, \zeta_{2n}$ are the critical points of F by (9). If P, Q in (7) satisfy (8) and (9) but $Q(\zeta_j) = 0$ for some j, then also $P(\zeta_j) = 0$ and $F = P/Q$ does not satisfy Theorem 1.2 since the degree of F is too small; see Proposition 1.4.

We prove Theorem 4.1 by showing that the solutions (P, Q) correspond to the rational functions $F = P/Q$ in Theorem 1.2. The different normalization of F and (P, Q) is not essential; see Remark 1.5.

Proposition 4.2. *Let $\eta_1, \dots, \eta_{2n} \in \mathbf{C}$ be distinct.*

1. *If (P, Q) satisfy the equations in Theorem 4.1, then $F = P/Q$ is a rational function of type $(n+1, n)$ with critical values $\eta_1, \dots, \eta_{2n}$, i.e., F satisfies Theorem 1.2, P and Q are coprime and the points $\zeta_1, \dots, \zeta_{2n}$ are distinct.*
2. *If F is a function in Theorem 1.2, then for all $a, b \in \mathbf{C}$ with $a \neq 0$, also $F(az+b)$ is a function as in Theorem 1.2 and there exist unique a, b such that $F(az+b) = z + O(1/z)$ for $z \to \infty$. Moreover, there exist a, b such that $F(az+b) = P(z)/Q(z)$ and (P, Q) satisfy Theorem 4.1.*

Proof. Part 1. is obvious. For part 2., let F be of type $(n+1, n)$ such that

$$F(\zeta_j) = \eta_j, \quad F'(\zeta_j) = 0, \quad j = 1, 2, \dots, 2n.$$

Let $G(z) = F(az+b)$, with $a, b \in \mathbf{C}$ and $a \neq 0$, then

$$G((\zeta_j - b)/a) = \eta_j, \quad G'((\zeta_j - b)/a) = F'(\zeta_j)a = 0, \quad j = 1, 2, \dots, 2n,$$

and G also is as in Theorem 1.2.

If $F = P/Q$, then (P, Q) satisfies (8)–(9) and $\deg(P) = n+1$ and $\deg(Q) = n$ (see Proposition 1.4), but the normalization (7) is not necessarily satisfied. However, by a suitable affine transformation of the argument, $G(z) = F(az + b)$, we obtain $G = P_1/Q_1$, such that (P_1, Q_1) satisfies (7). Moreover, $Q_1(\zeta_j) \neq 0$ (otherwise, $P_1(\zeta_j) = 0$ by (8), and G would not be of type $(n + 1, n)$). Hence (P_1, Q_1) is a solution in Theorem 4.1. The remaining assertions are clear. □

5. Rational functions with prescribed critical points

We first reformulate the task of finding a rational function $F = P/Q$ with prescribed critical points in terms of the polynomials P and Q, and show the equivalence of the two formulations.

Theorem 5.1. *Let $\zeta_1, \zeta_2, \dots, \zeta_{2n} \in \mathbf{C}$ be distinct. Then there exist polynomials*

$$P(z) = \sum_{j=0}^{n+1} p_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{n} q_j z^j, \tag{11}$$

where $p_0, p_1, \dots, p_{n+1}, q_0, q_1, \dots, q_n \in \mathbf{C}$ and $p_{n+1} \neq 0$ and $q_n \neq 0$, such that

$$P'(\zeta_j)Q(\zeta_j) - P(\zeta_j)Q'(\zeta_j) = 0, \quad j = 1, \dots, 2n. \tag{12}$$

Moreover, each solution (P, Q) can be normalized by

$$p_{n+1} = q_n = 1 \quad \text{and} \quad p_n = 0. \tag{13}$$

The number of normalized solutions is bounded by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Remark 5.2. Each solution in Theorem 5.1 yields infinitely many solutions, which can be seen as follows. Let (P, Q) be a solution in Theorem 5.1, i.e., P and Q are polynomials of degrees $\deg(P) = n + 1$ and $\deg(Q) = n$ that satisfy (12). Then, for each $c, d \in \mathbf{C} \setminus \{0\}$ and $p \in \mathbf{C}$, also $(cP + pQ, dQ)$ is a solution in Theorem 5.1. We introduce an equivalence relation on the solutions: $(R, S) \sim (P, Q)$ if and only if $R = cP + pQ$, $S = dQ$ for some $c, d \in \mathbf{C} \setminus \{0\}$ and $p \in \mathbf{C}$. In particular, the equivalence class of (P, Q) contains a unique representative satisfying (13), which is obtained by $c = 1/p_{n+1}$, $d = 1/q_n$ and $p = -p_n/(q_n p_{n+1})$.

First, we show that a rational function $F = P/Q$ in Theorem 1.3 corresponds to a solution (P, Q) in Theorem 5.1.

Proposition 5.3. *Let $\zeta_1, \dots, \zeta_{2n} \in \mathbf{C}$ be distinct. Then (P, Q) is a solution in Theorem 5.1 if, and only if, $F = P/Q$ is a rational function as in Theorem 1.3, i.e., F is of type $(n + 1, n)$ with critical points $\zeta_1, \dots, \zeta_{2n}$.*

Proof. If (P, Q) is a solution in Theorem 5.1, then the polynomials P and Q are coprime, which can be seen as follows. Assume to the contrary that P, Q have a non-constant common factor C and write $P = CR$, $Q = CS$. Then $P'Q - PQ' = C^2(R'S - RS')$, which has $2n$ distinct zeros by (12). Since C has at most $\deg(C) \geq 1$ distinct zeros, $R'S - RS'$ has at least $2n - \deg(C)$ many distinct zeros. On the other hand, $\deg(R'S - RS') = 2n - 2\deg(C)$. This implies $R'S - RS' \equiv 0$, and thus $R = cS$ with $c \in \mathbf{C} \setminus \{0\}$, in contradiction to $\deg(P) = n + 1 > \deg(Q) = n$. Then $F = P/Q$ is of type $(n + 1, n)$ and is a solution in Theorem 1.3. The converse is clear. □

Thus, Theorem 1.3 and Theorem 5.1 are equivalent. In particular, the number of equivalence classes of solutions is the same in both theorems. We show the existence of solutions for Theorem 5.1 using algebraic tools, in particular Hilbert's Nullstellensatz. The bound on the number of solutions will be established for Theorem 1.3 using a result of Goldberg.

Proof of Theorems 1.3 and 5.1. *Step 1: Existence.* We show that there are polynomials P and Q as in (11) satisfying

$$P'(z)Q(z) - P(z)Q'(z) = \prod_{j=1}^{2n}(z-\zeta_j). \tag{14}$$

Then $\deg(P) = n+1$, $\deg(Q) = n$ and (P,Q) is a solution in Theorem 5.1.

We write

$$\prod_{j=1}^{2n}(z-\zeta_j) = \sum_{\ell=0}^{2n} c_\ell z^\ell \tag{15}$$

with constants $c_0,\dots,c_{2n} \in \mathbf{C}$ given by $\zeta_1,\dots,\zeta_{2n}$. Note that $c_{2n}=1$. We also write

$$P'(z)Q(z) - P(z)Q'(z) = \sum_{\ell=0}^{2n} \rho_\ell z^\ell, \tag{16}$$

where the coefficients are

$$\begin{aligned}\rho_\ell &:= \sum_{\substack{j+k=\ell\\ j\le n, k\le n}} (j+1)p_{j+1}q_k - \sum_{\substack{j+k=\ell\\ j\le n+1, k\le n-1}} (k+1)p_j q_{k+1}\\ &= \sum_{\substack{j+k=\ell+1\\ 0\le j\le n+1, 0\le k\le n}} (j-k)p_j q_k.\end{aligned} \tag{17}$$

In particular, $\rho_{2n} = p_{n+1}q_n$. The structure of the coefficients ρ_ℓ is important and we will use the following fact later: For each ℓ, ρ_ℓ is a homogeneous polynomial of order 2 in the variables $p_0,p_1,\dots,p_{n+1}$, $q_0,q_1,\dots,q_n$ and contains only products p_jq_k with $j+k-1=\ell$. Conversely, each product p_jq_k (with $j\ne k$) appears only in ρ_{j+k-1} and with coefficient $j-k$.

Then (14) is equivalent to the system of polynomial equations

$$\rho_\ell = c_\ell, \quad \ell = 0,1,\dots,2n. \tag{18}$$

To show that (18) has a solution, we consider the ideal generated by $\rho_0 - c_0,\dots,$ $\rho_{2n}-c_{2n}$:

$$\begin{aligned}I &:= \langle \rho_0 - c_0,\dots,\rho_{2n}-c_{2n}\rangle\\ &= \left\{\sum_{\ell=0}^{2n} A_\ell(\rho_\ell - c_\ell) \colon A_\ell \in \mathbf{C}[p_{n+1},\dots,p_1,p_0,q_n,\dots,q_1,q_0], \ell = 0,\dots,2n\right\}.\end{aligned}$$

By the weak form of Hilbert's Nullstellensatz, (18) has a solution if and only if

$$I \ne \mathbf{C}[p_{n+1},\dots,p_1,p_0,q_n,\dots,q_1,q_0];$$

see, e.g., [1, Thm. 2.2.3] or [7, Thm. 1, p. 177]. Hence, we are going to show that there exists a polynomial $Y \notin I$, i.e., such that

$$Y = \sum_{\ell=0}^{2n} A_\ell(\rho_\ell - c_\ell) \tag{19}$$

does not hold for any $A_0,\dots,A_{2n} \in \mathbf{C}[p_{n+1},\dots,p_1,p_0,q_n,\dots,q_1,q_0]$.

Consider the equations in (18) with nonzero right hand side

$$L_1 := \{\ell \in \{0,1,\dots,2n\} \colon c_\ell \ne 0\}.$$

Note that $|L_1| > 1$. Indeed, $c_{2n} = 1$ so $2n \in L_1$, and the assumption $|L_1| = 1$ would imply $\prod_{j=1}^{2n}(z - \zeta_j) = z^{2n}$, which contradicts that $\zeta_1, \dots, \zeta_{2n}$ are distinct. We determine the minimal element in these equations with respect to the ordering

$$p_{n+1} > p_n > p_{n-1} > \dots > p_1 > p_0 > q_0 > q_1 > \dots > q_{n-1} > q_n$$

of the variables with some order, e.g., the degree lexicographic order (also called graded lexicographic order, see, e.g. [7, p. 58])

$$p_{j_1} q_{k_1} = \min\{p_r q_s \colon \operatorname{coeff}(p_r q_s, \rho_\ell) \neq 0 \text{ and } \ell \in L_1\},$$

where $\operatorname{coeff}(p_r q_s, \rho_\ell)$ denotes the coefficient of $p_r q_s$ in ρ_ℓ. Let $\ell_1 = j_1 + k_1 - 1$ be the unique index such that $p_{j_1} q_{k_1}$ appears in ρ_{ℓ_1}. In particular, $\ell_1 \in L_1$. Note that $j_1 \neq k_1$, otherwise the coefficient of $p_{j_1} q_{k_1}$ would be zero.

For $\ell \in L_1$, we write

$$\rho_\ell - c_\ell = \rho_\ell - \frac{c_\ell}{c_{\ell_1}} \rho_{\ell_1} + \frac{c_\ell}{c_{\ell_1}} (\rho_{\ell_1} - c_{\ell_1}), \quad \ell \in L_1.$$

Then (19) can be written as

$$\begin{aligned} Y &= \sum_{\ell \notin L_1} A_\ell \rho_\ell + \sum_{\ell \in L_1} A_\ell \frac{c_\ell}{c_{\ell_1}} (\rho_{\ell_1} - c_{\ell_1}) + \sum_{\ell \in L_1} A_\ell \left(\rho_\ell - \frac{c_\ell}{c_{\ell_1}} \rho_{\ell_1} \right) \\ &= \sum_{\ell \notin L_1} A_\ell \rho_\ell + \widehat{A}_{\ell_1} (\rho_{\ell_1} - c_{\ell_1}) + \sum_{\ell \in L_1 \setminus \{\ell_1\}} A_\ell \left(\rho_\ell - \frac{c_\ell}{c_{\ell_1}} \rho_{\ell_1} \right), \end{aligned} \tag{20}$$

where $\widehat{A}_{\ell_1} := \sum_{\ell \in L_1} A_\ell \frac{c_\ell}{c_{\ell_1}}$. We will construct a polynomial Y for which this equation has no solutions.

If p_{j_1} appears in some of the terms in ρ_ℓ, then it appears only in the term $p_{j_1} q_{k(\ell)}$, where $k(\ell) := \ell + 1 - j_1$. We define

$$\begin{aligned} L_2 &:= \{\ell \in \{0, 1, \dots, 2n\} \colon p_{j_1} \text{ appears in } \rho_\ell\} \\ &= \{\ell \in \{0, 1, \dots, 2n\} \colon \operatorname{coeff}(p_{j_1} q_{k(\ell)}, \rho_\ell) \neq 0\}. \end{aligned}$$

Substituting 0 for each variable in $\{p_0, p_1, \dots, p_{n+1}\} \setminus \{p_{j_1}\}$, we obtain (after this substitution) $\rho_\ell = 0$ for $\ell \notin L_2$, and $\rho_\ell = (j_1 - k(\ell)) p_{j_1} q_{k(\ell)}$ for $\ell \in L_2$. Denoting by B_ℓ the polynomial obtained from A_ℓ through the substitution, (20) becomes

$$\begin{aligned} Y = &\sum_{\ell \in L_2 \setminus L_1} B_\ell \, (j_1 - k(\ell)) p_{j_1} q_{k(\ell)} + B_{\ell_1} ((j_1 - k_1) p_{j_1} q_{k_1} - c_{\ell_1}) \\ &+ \sum_{\ell \in L_2 \cap L_1 \setminus \{\ell_1\}} B_\ell \left((j_1 - k(\ell)) p_{j_1} q_{k(\ell)} - \frac{c_\ell}{c_{\ell_1}} (j_1 - k_1) p_{j_1} q_{k_1} \right). \end{aligned} \tag{21}$$

If $L_2 \cap L_1 = \{\ell_1\}$, we set $Y = 1$. Then (21) becomes

$$1 = \sum_{\ell \in L_2 \setminus L_1} B_\ell \, (j_1 - k(\ell)) p_{j_1} q_{k(\ell)} + B_{\ell_1} ((j_1 - k_1) p_{j_1} q_{k_1} - c_{\ell_1}).$$

Substituting $q_{k_1} = 1$ and all other $q_k = 0$, leads to a polynomial C_{ℓ_1} obtained from B_{ℓ_1} by this substitution and $1 = C_{\ell_1}((j_1 - k_1) p_{j_1} - c_{\ell_1})$, which cannot hold for all $p_{j_1} \in \mathbf{C}$, so we reached a contradiction.

If $|L_2 \cap L_1| > 1$, let $\ell_2 \in (L_2 \cap L_1) \setminus \{\ell_1\}$, then $p_{j_1} q_{k(\ell_2)}$ has nonzero coefficient in ρ_{ℓ_2}, and we set $Y = q_{k(\ell_2)}$. Substitute $q_k = 0$ for $k \notin \{k(\ell) \colon \ell \in L_1 \cap L_2\}$, and denote by D_ℓ the polynomial obtained from B_ℓ by this substitution. Then (21) becomes

$$q_{k(\ell_2)} = D_{\ell_1}((j_1 - k_1)p_{j_1}q_{k_1} - c_{\ell_1}) + \sum_{\ell \in L_2 \cap L_1 \setminus \{\ell_1\}} D_\ell \left((j_1 - k(\ell))p_{j_1}q_{k(\ell)} - \frac{c_\ell}{c_{\ell_1}}(j_1 - k_1)p_{j_1}q_{k_1} \right).$$

Finally, we substitute $p_{j_1} = 1$ and $q_{k_1} = c_{\ell_1}/(j_1 - k_1)$ and $q_{k(\ell)} = \frac{c_\ell}{j_1 - k(\ell)}$ for $\ell \in L_2 \cap L_1 \setminus \{\ell_1, \ell_2\}$, which, denoting by E_{ℓ_2} the polynomial obtained from D_{ℓ_2} by the substitution, yields

$$q_{k(\ell_2)} = E_{\ell_2}((j_1 - k(\ell_2))q_{k(\ell_2)} - c_{\ell_2}).$$

This is impossible since $c_{\ell_2} \neq 0$ since $\ell_2 \in L_1$, and $j_1 - k(\ell_2) \neq 0$ since $\ell_2 \in L_2$. This concludes the proof of existence.

Step 2: Number of normalized solutions. By Remark 5.2, the solutions can be normalized by (13). By the correspondence established in Proposition 5.3, the number of (normalized) solutions in Theorems 5.1 and 1.3 are the same, and we show the bound for the latter. If F is a rational function and $\varphi(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ a Möbius transformation, then F and $\varphi \circ F$ have the same critical points. Following Goldberg [12], we call two rational functions F, F_1 equivalent if $F_1 = \varphi \circ F$ with a Möbius transformation φ. Goldberg [12, Thm. 1.3] showed that the number of equivalence classes (with respect to this equivalence relation) of rational functions with $2n$ prescribed critical points is bounded by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Given a rational function of degree $n+1$ with the critical points $\zeta_1, \ldots, \zeta_{2n}$, there is an equivalent rational function of type $(n+1, n)$ and, moreover, there is one that is normalized by $F(z) = z + O(1/z)$ at infinity. We show that this is the only normalized function in its equivalence class. Indeed, if $F_1 = \varphi \circ F$ with a Möbius transformation φ and $F_1(z) = z + O(1/z)$, then $\infty = F_1(\infty) = \varphi(F(\infty)) = \varphi(\infty)$, hence φ has the form $\varphi(z) = az + b$ with $a \in \mathbf{C} \setminus \{0\}$ and $b \in \mathbf{C}$. Then, at infinity, $F_1(z) = z + O(1/z) = \varphi(F(z)) = az + b + O(1/z)$, which implies $a = 1$, $b = 0$, i.e., $\varphi(z) = z$ and $F_1 = F$. Thus, there is only one rational function normalized by $z + O(1/z)$ at infinity in each equivalence class, which establishes the bound on the number of normalized solutions in Theorem 1.3 and Theorem 5.1. □

Remark 5.4. In [12, Thm. 1.3], it is also shown that the bound in Theorems 1.3 and 5.1 is attained in the generic case, though not always. For $n = 2$, the exact number is characterized in [12, Thm. 1.4]: There is exactly one rational function if the cross ratio of $(\zeta_1, \ldots, \zeta_4)$ is equal to $(1 + i\sqrt{3})/2$, and there are exactly $C_2 = 2$ such rational functions otherwise.

6. Solutions for $n = 1$ and symmetry considerations

If $n = 1$, the rational functions in Theorems 1.1, 1.2 and 1.3 are uniquely determined when normalized by $F(z) = z + O(1/z)$ at infinity, and can be determined explicitly.

Proposition 6.1.

1. *Given distinct $\zeta_1, \zeta_2 \in \mathbf{C}$, then $F(z) = z + \frac{r}{z-p}$ with $r = ((\zeta_1 - \zeta_2)/2)^2$ and $p = (\zeta_1 + \zeta_2)/2$ is the unique rational function in Theorem 1.3 with the normalization $F(z) = z + O(1/z)$ at infinity. Its critical values are $\eta_1 = (3\zeta_1 - \zeta_2)/2$, $\eta_2 = (3\zeta_2 - \zeta_1)/2$.*

2. *Given distinct* $\eta_1, \eta_2 \in \mathbf{C}$*, then* $F(z) = z + \frac{r}{z-p}$ *with* $r = ((\eta_1 - \eta_2)/4)^2$ *and* $p = (\eta_1 + \eta_2)/2$ *is the unique rational function in Theorem 1.2 with the normalization* $F(z) = z + O(1/z)$*. Its critical points are* $\zeta_1 = (3\eta_1 + \eta_2)/4$*,* $\zeta_2 = (\eta_1 + 3\eta_2)/4$*.*
3. *Let* γ *be a Jordan arc with distinct endpoints* η_1*,* η_2*. Then* F *from 2. is the unique open up mapping from a Jordan domain* G *to* $\mathbf{C}_\infty \setminus \gamma$*. The boundary of* G *consists of the two branches of* $F^{-1}(\gamma)$*.*

Proof. Let $F(z) = z + \frac{r}{z-p}$ and assume that $F(\zeta_j) = \eta_j$ and $F'(\zeta_j) = 0$ for $j = 1, 2$. Then $F'(\zeta_j) = 0$ is equivalent to $r = (\zeta_j - p)^2$, for $j = 1, 2$. This implies $(\zeta_1 - p)^2 = (\zeta_2 - p)^2$ and $\zeta_1 - p = -(\zeta_2 - p)$ (because $\zeta_1 - p = \zeta_2 - p$ is impossible, since $\zeta_1 \neq \zeta_2$ in 1., and $\zeta_1 = \zeta_2$ implies $\eta_1 = \eta_2$, which is not possible in 2.). We thus obtain $p = (\zeta_1 + \zeta_2)/2$ and $r = ((\zeta_1 - \zeta_2)/2)^2$. To complete 1., the critical values are $\eta_1 = f(\zeta_1) = (3\zeta_1 - \zeta_2)/2$ and $\eta_2 = f(\zeta_2) = (3\zeta_2 - \zeta_1)/2$. For 2., we compute ζ_1, ζ_2 from η_1, η_2, and obtain successively $\zeta_1 = (3\eta_1 + \eta_2)/4$, $\zeta_2 = (\eta_1 + 3\eta_2)/4$, $r = ((\eta_1 - \eta_2)/4)^2$ and $p = (\eta_1 + \eta_2)/2$. Finally, if γ is a Jordan arc with endpoints η_1, η_2, then F in 2. is the open up mapping, and the domain G is bounded by the two branches of $F^{-1}(\gamma)$. □

An open up mapping is also a solution of the prescribed critical values problem; see Proposition 3.2. Conversely, a solution of the critical value problem is in general not an open up mapping when $n \geq 2$, and we give two examples below. The reason becomes apparent from Theorem 3.3: The open up mapping in Theorem 1.1 depends on the endpoints of the arcs (critical values) and the topology of $\mathbf{C} \setminus \bigcup_{j=1}^n \gamma_j$. In contrast, only the information about the critical values is present in Theorem 1.2. For $n = 1$, a rational function of type $(2, 1)$ has critical values η_1, η_2 if and only if it is an open up mapping for any Jordan arc connecting η_1, η_2.

We reformulated Theorems 1.2 and 1.3 in terms of polynomials. Since the number of normalized solutions is finite, one can apply several existing methods and algorithms to solve the polynomial systems symbolically. We just mention [7, Sect. 10.4], [9] and the references therein. For many of the methods, it is crucial that the number of solutions is finite. Also, there are several numerical algorithms and solvers available, see e.g. [2] and [40]. Finally, if the finitely many solutions of the critical value problem have been obtained, one can go through these solutions and verify which one is the open up mapping for a given set of arcs. We use this approach in Section 7.

The examples in the following section show that it is advantageous to simplify the polynomial equations, e.g. by reducing the number of equations and unknowns. Therefore we investigate some symmetric settings and how one can simplify the polynomial equations. We collect three useful results on the open up mapping when E has some symmetry.

Lemma 6.2. *Let* $E = \gamma_1 \cup \ldots \cup \gamma_n$ *be the union of the disjoint Jordan arcs* $\gamma_1, \ldots, \gamma_n$*, and let* F *be the open up mapping of type* $(n+1, n)$ *with* $F(z) = z + O(1/z)$ *at infinity.*

1. *Suppose that* $z \in E$ *if and only if* $-z \in E$*. Then* F *is odd, i.e.,* $F(-z) = -F(z)$*.*
2. *Suppose that* $z \in E$ *if and only if* $\overline{z} \in E$*. Then* F *is real, i.e.,* $F(z) = \overline{F(\overline{z})}$*.*

Proof. Let $F \colon \mathbf{C}_\infty \setminus K \to \mathbf{C}_\infty \setminus E$ with $F(z) = z + O(1/z)$ be the open up mapping from Theorem 1.1. Then

$$G : \mathbf{C}_\infty \setminus (-K) \to \mathbf{C}_\infty \setminus E, \quad G(z) = -F(-z),$$

is also an open up mapping of type $(n+1,n)$ with $G(z) = z + O(1/z)$ for $z \to \infty$, hence $F = G$ by Theorem 1.1. The proof of 2. is similar. $\square$

Lemma 6.3. *Let $F(z) = z + \sum_{j=1}^n \frac{r_j}{z-p_j}$ with $r_1, \dots, r_n \in \mathbf{C} \setminus \{0\}$ and distinct poles $p_1, \dots, p_n \in \mathbf{C}$. Then F is odd if and only if the poles appear in pairs $\pm p$ with equal residues. In particular, if F and n are odd, then one pole is at the origin.*

Proof. Let F be odd. Since the partial fraction decomposition is unique and

$$F(z) = -F(-z) = z + \sum_{j=1}^n \frac{r_j}{z+p_j},$$

the poles appear in pairs $\pm p$ with equal residues. The converse is obvious. $\square$

Corollary 6.4. *Let $\eta_1, \dots, \eta_{2n} \in \mathbf{C}$ be distinct such that $\eta \in \{\eta_1, \dots, \eta_{2n}\}$ implies $-\eta \in \{\eta_1, \dots, \eta_{2n}\}$. Then there is an odd rational function that solves the critical value problem in Theorem 1.2.*

Proof. Connect the critical values $\eta_1, \dots, \eta_{2n}$ by arcs $\gamma_1, \dots, \gamma_n$, such that $E = \bigcup_{j=1}^n \gamma_j$ has the property $z \in E$ if and only if $-z \in E$. Note that the arcs themselves need not be symmetric with respect to the origin, but their union E needs to be symmetric. The arcs can be constructed as follows. There exists a line through the origin that divides the plane in two half-planes, each containing n of the points $\eta_1, \dots, \eta_{2n}$ (by symmetry). If $n \geq 2$, choose two points in one half-plane and connect them by a Jordan arc γ_1 in that half-plane, such that γ_1 contains no other critical value. Then $\gamma_2 = -\gamma_1$ also connects two critical values in the other half-plane. This way, we construct disjoint Jordan arcs connecting the critical values. If there is only one critical point left in each half-plane (which happens if n is odd), we connect these two by a Jordan arc γ_n that is symmetric with respect to the origin (and disjoint from all previous arcs). Then the open up mapping of E is odd by Lemma 6.2 and a solution of the critical value problem. $\square$

7. Two examples and further comments

We give two examples when $n = 2$, i.e., in the case of two disjoint Jordan arcs γ_1, γ_2 with endpoints $\eta_1, \eta_2, \eta_3, \eta_4$. The open up mapping in this case has the form

$$F(z) = z + \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} \tag{22}$$

with $r_1, r_2 \in \mathbf{C} \setminus \{0\}$ and distinct $p_1, p_2 \in \mathbf{C}$. Without loss of generality, we assume here and in the following that the rational functions are normalized by $z + O(1/z)$ at infinity; see Theorems 1.1 and 1.2 and also Remark 1.5.

We also determine all rational functions of the form (22) with critical values $\eta_1, \eta_2, \eta_3, \eta_4$. The problem then is to find r_1, r_2, p_1, p_2 and $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbf{C}$ with

$$F(\zeta_j) = \eta_j, \quad F'(\zeta_j) = 0, \quad j = 1, 2, 3, 4. \tag{23}$$

By Theorem 1.2, there are $3H_2 = 12$ rational functions of the form (22) satisfying (23). By Proposition 3.2, the open up mapping is among these.

7.1. An example with symmetry. As first example, we determine the open up mapping for

$$E = [-2, -1] \cup [1, 2] \tag{24}$$

with Jordan arcs $\gamma_1 = [1,2]$ and $\gamma_2 = [-2,-1]$ as well as all rational functions of the form (22) with critical values

$$\eta_1 = 1, \quad \eta_2 = 2, \quad \eta_3 = -1, \quad \eta_4 = -2. \tag{25}$$

Since E is symmetric with respect to the origin, the open up mapping (22) has the simpler form

$$F(z) = z + \frac{r}{z-p} + \frac{r}{z+p} \quad \text{with } r, p \in \mathbf{C} \setminus \{0\} \tag{26}$$

by Lemma 6.2 and Lemma 6.3.

Accordingly, we simplify also the critical value problem and ask only for odd solutions F of the form (26) of

$$F(\pm\zeta_1) = \pm\eta_1 = \pm 1, \quad F(\pm\zeta_2) = \pm\eta_2 = \pm 2, \quad F'(\pm\zeta_1) = F'(\pm\zeta_2) = 0. \tag{27}$$

One can obtain four distinct symbolical solutions of (27) (the following numbers are rounded to 4 digits)

$$F_1(z) = z + \frac{0.1272}{z - 0.0658i} + \frac{0.1272}{z + 0.0658i}, \tag{28}$$

$$F_2(z) = z + \frac{0.0630}{z - 1.4786} + \frac{0.0630}{z + 1.4786}, \tag{29}$$

$$F_3(z) = z + \frac{0.4605 + 0.1279i}{z - (0.3958 - 0.3693i)} + \frac{0.4605 + 0.1279i}{z + (0.3958 - 0.3693i)}, \tag{30}$$

$$F_4(z) = z + \frac{0.4605 - 0.1279i}{z - (0.3958 + 0.3693i)} + \frac{0.4605 - 0.1279i}{z + (0.3958 + 0.3693i)}. \tag{31}$$

By Lemma 6.2, the open up map of $E = [-2,-1] \cup [1,2]$ is odd and real (i.e., $F(z) = \overline{F(\overline{z})}$), and hence must be F_2. By computing the pre-images of γ_1 and γ_2, we find that F_2 is indeed the (unique) open up mapping for γ_1 and γ_2, see Figure 2, while the other functions are not open up mappings for γ_1 and γ_2. This confirms that a function with critical values at the endpoints of the arcs in Theorem 1.2 is in general not an open up mapping for the arcs.

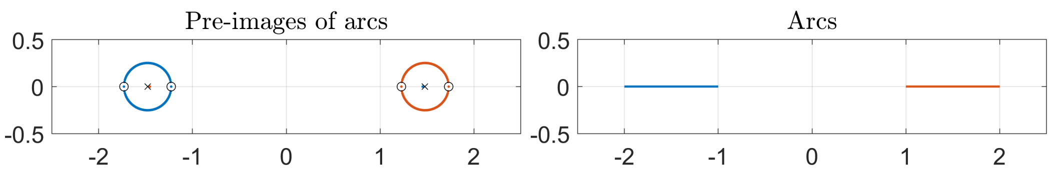

Figure 2. The function in (29) is the minimal degree rational open up mapping for $[-2,-1] \cup [1,2]$. Right: Arcs. Left: Pre-images of the arcs, critical points (black circles) and poles (crosses) of F_2.

However, each of the functions F_1, F_2, F_3, F_4 is an open up mapping for a suitable set of disjoint Jordan arcs, where each Jordan arc connects two of the critical values. Examples of configurations opened up by F_1, F_3, F_4 are displayed in Figure 3. This leads to the following conjecture.

Conjecture. Let F be a rational function of type $(n+1, n)$ with distinct critical values $\eta_1, \ldots, \eta_{2n} \in \mathbf{C}$. Then there exists a set of disjoint Jordan arcs $\gamma_1, \ldots, \gamma_n$, each arc connecting two points in $\{\eta_1, \ldots, \eta_{2n}\}$, such that F is the open up mapping in Theorem 1.1 for the arcs $\gamma_1, \ldots, \gamma_n$.

A deformation (fixed endpoint homotopy) of the arcs as described in Theorem 3.3 yields homotopic configurations that are also opened up by the same function. Note that for some functions there are also other (non-homotopic) configurations that are opened up. This shows that the open up configuration for a given function in Theorem 1.2 is not unique up to fixed endpoint homotopy.

Next, we consider the prescribed critical value problem in general form, i.e., (22) and (23). We multiply (23) with the denominators and solve the obtained polynomial equations symbolically (with Singular and also with Magma), The result of this computation is the uniquely determined Gröbner basis consisting of 41 polynomials. This new set of polynomial equations is then solved. Some of the coefficients (which are rational numbers) have numerators and denominators in reduced form of magnitude 10^{40}.

The computation yields 12 distinct rational functions, i.e., all solutions, including the odd functions $F_1, \ldots, F_4$ above, and 8 solutions that are not of the form (26). In particular, the symmetry of the critical values (endpoints of the arcs) does not imply symmetry of the rational function. The symmetric problem of prescribed critical values (with values ± 1 and ± 2, see (22) and (25)) has a nonsymmetric solution (e.g., $r_1 = -0.0005$, $r_2 = 0.0630$, $p_1 = -1.4786$, $p_2 = -1.4998$ where the values are rounded to 4 digits).

In this first example, we computed the open up mapping of $[-2,-1] \cup [1,2]$ by solving the simplified prescribed critical value problem (27), and also computed all rational functions with prescribed critical values $1, 2, -1, -2$. If the aim is to compute the open up mapping, then (27) is better suited than (23), since it involves approximately half the number of unknowns and equations and is thus easier and faster to solve.

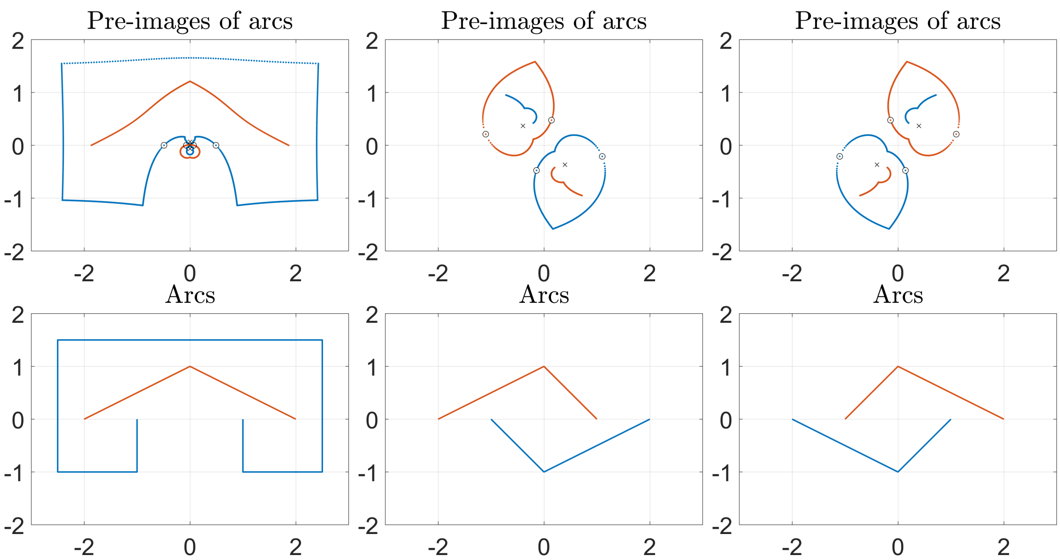

Figure 3. Arcs (bottom) and pre-images (top) that are opened up under F_1, F_3, F_4 (left to right).

7.2. An example without symmetry. In the second example let $\gamma_1 = [1,2]$ and $\gamma_2 = [i, 2i] = \{(1-t)i + t2i \colon 0 \le t \le 1\}$. Therefore we consider the critical value problem (23) with

$$\eta_1 = 1, \quad \eta_2 = 2, \quad \eta_3 = i, \quad \eta_4 = 2i. \tag{32}$$

We determined the corresponding Gröbner basis with Singular and Magma. The Gröbner basis consists of 77 polynomials and some of the rational coefficients have numerators and denominators of size 10^{75}.

In total, we obtain 12 distinct rational functions of the form (22) with critical values $1, 2, i, 2i$, i.e., all solutions have been computed; see Theorem 1.2. Finally, we determine which solution is the open up mapping for $[1, 2] \cup [i, 2i]$ by computing the pre-images of $\gamma_1 = [1, 2]$ and $\gamma_2 = [i, 2i]$, and obtain that the open up mapping is (coefficients rounded to four digits)

$$F(z) = z + \frac{-0.0625 - 0.0009i}{z - (0.0214 + 1.5203i)} + \frac{0.0625 - 0.0009i}{z - (1.5203 + 0.0214i)}. \tag{33}$$

Figure 4 visualizes the open up mapping. The left panel shows the arcs (blue and red) with a grid (black dots). The right panel shows the pre-image domain $\mathbf{C}_\infty \setminus K$ bounded by Jordan curves (blue and red) and the pre-image of the grid under F (black dots).

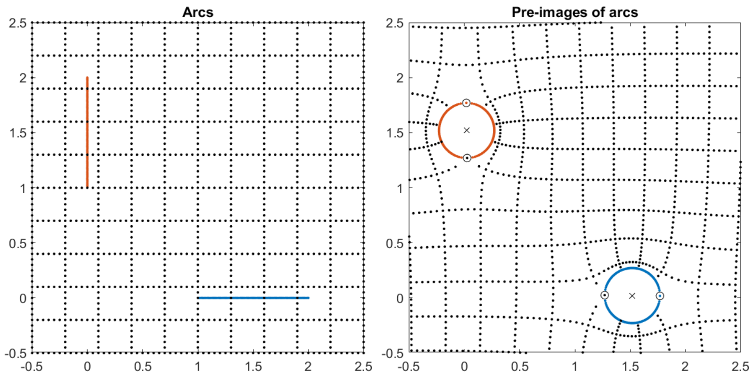

Figure 4. Open up mapping for $[1, 2] \cup [i, 2i]$. Left: arcs and a grid (black). Right: Domain $\mathbf{C}_\infty \setminus K$ bounded by Jordan curves and the pre-image of the grid under the open up mapping.

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References

[1] Adams, W. W. and P. Loustaunau: An introduction to Gröbner bases. - Grad. Stud. Math. 3, Amer. Math. Soc., Providence, RI, 1994.

[2] Allgower, E. L. and K. Georg: Introduction to numerical continuation methods. - Classics Appl. Math. 45, SIAM, Philadelphia, PA, 2003.

[3] Aptekarev, A. I., V. A. Kalyagin, V. G. Lysov, and D. N. Toulyakov: Equilibrium of vector potentials and uniformization of the algebraic curves of genus 0. - J. Comput. Appl. Math. 233:3, 2009, 602–616.

[4] Aptekarev, A. I., D. N. Toulyakov, and W. Van Assche: Hyperelliptic uniformization of algebraic curves of the third order. - J. Comput. Appl. Math. 284, 2015, 38–49.

[5] Aptekarev, A. I., D. N. Tulyakov, and M. Yattselev: On the parametrization of a complex algebraic curve of genus 2. - Mat. Zametki 98:5, 2015, 782–785.

[6] Beardon, A. F., T. K. Carne, and T. W. Ng: The critical values of a polynomial. - Constr. Approx. 18:3, 2002, 343–354.

[7] Cox, D. A., J. Little, and D. O'Shea: Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. - Undergrad. Texts Math., 4th edition, Springer, Cham, 2015.

[8] Crescimanno, M., and W. Taylor: Large N phases of chiral QCD_2. - Nuclear Phys. B 437:1, 1995, 3–24.

[9] Eder, C., and J.-C. Faugère: A survey on signature-based algorithms for computing Gröbner bases. - J. Symbolic Comput. 80:3, 2017, 719–784.

[10] Eremenko, A., and A. Gabrielov: Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. - Ann. of Math. (2) 155:1, 2002, 105–129.

[11] Forster, O.: Lectures on Riemann surfaces. - Grad. Texts in Math. 81, Springer-Verlag, New York, 1991.

[12] Goldberg, L. R.: Catalan numbers and branched coverings by the Riemann sphere. - Adv. Math. 85:2, 1991, 129–144.

[13] Goluzin, G. M.: Geometric theory of functions of a complex variable. - Transl. Math. Monogr. 26, Amer. Math. Soc., Providence, R.I., 1969.

[14] Henrici, P.: Applied and computational complex analysis. - Wiley-Interscience, John Wiley & Sons, New York-London-Sydney, 1974.

[15] Hidalgo, R. A., and M. Seppälä: Numerical Schottky uniformizations: Myrberg's opening process. - In: Computational approach to Riemann surfaces, Lecture Notes in Math. 2013, Springer, Heidelberg, 2011, 195–209.

[16] Hurwitz, A.: Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten. - Math. Ann. 39, 1891, 1–61 (in German).

[17] Hurwitz, A., and R. Courant: Funktionentheorie. - Nauka, Moscow, 1968 (in Russian).

[18] Kalajdzievski, S.: An illustrated introduction to topology and homotopy. - CRC Press, Boca Raton, FL, 2015.

[19] Kalmykov, S. I., and B. Nagy: Polynomial and rational inequalities on analytic Jordan arcs and domains. - J. Math. Anal. Appl. 430:2, 2015, 874–894.

[20] Kalmykov, S., B. Nagy, and V. Totik: Bernstein- and Markov-type inequalities for rational functions. - Acta Math. 219:1, 2017, 21–63.

[21] Kammerer, W. J.: Polynomial approximations to finitely oscillating functions. - Math. Comp. 15, 1961, 115–119.

[22] Kraus, D., and O. Roth: Critical points of inner functions, nonlinear partial differential equations, and an extension of Liouville's theorem. - J. Lond. Math. Soc. (2) 77:1, 2008, 183–202.

[23] Kristiansen, G. K.: Characterization of polynomials by means of their stationary values. - Arch. Math. (Basel) 43:1, 1984, 44–48.

[24] Kuhn, H.: Interpolation vorgeschriebener Extremwerte. - J. Reine Angew. Math. 238, 1969, 24–31.

[25] Lando, S. K., and A. K. Zvonkin: Graphs on surfaces and their applications. - Encyclopaedia Math. Sci. 141, Springer-Verlag, Berlin, 2004.

[26] Liesen, J., O. Sète, and M. S. Nasser: Fast and accurate computation of the logarithmic capacity of compact sets. - Comput. Methods Funct. Theory 17:4, 2017, 689–713.

[27] López Lagomasino, G., D. Pestana, J. M. Rodríguez, and D. Yakubovich: Computation of conformal representations of compact Riemann surfaces. - Math. Comp. 79:269, 2010, 365–381.

[28] Mednykh, A. D.: Nonequivalent coverings of Riemann surfaces with a prescribed ramification type. - Sibirsk. Mat. Zh. 25:4, 1984, 120–142.

[29] Mycielski, J., and S. Paszkowski: A generalization of Chebyshev polynomials. - Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8, 1960, 433–438.

[30] Myrberg, P. J.: Über die numerische Ausführung der Uniformisierung. - Acta Soc. Sc. Fennicae 7, 1920, 1–53 (in German).

[31] Nasser, M. M. S., J. Liesen, and O. Sète: Numerical computation of the conformal map onto lemniscatic domains. - Comput. Methods Funct. Theory 16:4, 2016, 609–635.

[32] Nasyrov, S. R.: Geometric problems of the theory of ramified coverings of Riemann surfaces. - Magarif, Kazan, 2008 (in Russian).

[33] Newman, M. H. A.: Elements of the topology of plane sets of points. Second edition. - Cambridge Univ. Press, Cambridge, 1964.

[34] Peltola, E., and Y. Wang: Large deviations of multichordal SLE_{0+}, real rational functions, and zeta-regularized determinants of Laplacians. - J. Eur. Math. Soc. (JEMS), 2023, doi:10.4171/JEMS/1274.

[35] Pommerenke, Ch.: Boundary behaviour of conformal maps. - Grundlehren Math. Wiss. 299, Springer-Verlag, Berlin, 1992.

[36] Rao, M., H. Stetkær, S. Fournais, and J. S. Møller: Complex analysis. An invitation. Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[37] Schlag, W.: A course in complex analysis and Riemann surfaces. - Grad. Stud. Math. 154, Amer. Math. Soc., Providence, RI, 2014.

[38] Semmler, G., and E. Wegert: Finite Blaschke products with prescribed critical points, Stieltjes polynomials, and moment problems. - Anal. Math. Phys. 9:1, 2019, 221–249.

[39] Seppälä, M.: Myrberg's numerical uniformization of hyperelliptic curves. - Ann. Acad. Sci. Fenn. Math. 29:1, 2004, 3–20.

[40] Sommese, A. J., and C. W. Wampler, II: The numerical solution of systems of polynomials. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[41] Thom, R.: L'équivalence d'une fonction différentiable et d'un polynome. - Topology 3:2, 1965, 297–307.

[42] Wegert, E.: Visual complex functions. An introduction with phase portraits. - Birkhäuser/Springer Basel AG, Basel, 2012.

[43] Widom, H.: Extremal polynomials associated with a system of curves in the complex plane. - Adv. Math. 3, 1969, 127–232.

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