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On meromorphic solutions of Malmquist type difference equations

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Abstract. Recently, the present authors used Nevanlinna theory to provide a classification for the Malmquist type difference equations of the form $f(z+1)^n = R(z,f)$ (†) that have transcendental meromorphic solutions, where $R(z,f)$ is rational in both arguments. In this paper, we first complete the classification for the case $\deg_f(R(z,f)) = n$ of (†) by identifying a new equation that was left out in our previous work. We will actually derive all the equations in this case based on some new observations on (†). Then, we study the relations between (†) and its differential counterpart $(f')^n = R(z,f)$. We show that most autonomous equations, singled out from (†) with $n=2$, have a natural continuum limit to either the differential Riccati equation $f' = a + f^2$ or the differential equation $(f')^2 = a(f^2-\tau_1^2)(f^2-\tau_2^2)$, where $a \neq 0$ and τ_i are constants such that $\tau_1^2 \neq \tau_2^2$. The latter second degree differential equation and the symmetric QRT map are derived from each other using the bilinear method and the continuum limit method.

Malmquist-tyyppisten differenssiyhtälöiden meromorfisista ratkaisuista

Tiivistelmä. Äskettäin sovelsimme Nevanlinnan teoriaa muotoa $f(z+1)^n = R(z,f)$ (†) oleviin Malmquist-tyyppisiin differenssiyhtälöihin, joilla on transkendenttisia meromorfisia ratkaisuja. Edellä $R(z,f)$ on rationaalinen molempien argumenttien suhteen. Tässä artikkelissa täydennämme ensin luokittelun tapaukselle $\deg_f(R(z,f)) = n$ (†), tunnistamalla uuden yhtälön, joka jäi huomiotta aiemmassa työssämme. Itse asiassa johdamme kaikki yhtälöt tässä tapauksessa perustuen joihinkin uusiin havaintoihin yhtälöstä (†). Tämän jälkeen tutkimme yhteyksiä yhtälön (†) ja sen differentiaalivastineen $(f')^n = R(z,f)$ välillä. Osoitamme, että useimmilla autonomisilla yhtälöillä, jotka erottuvat em. luokittelussa yhtälöstä (†), kun $n=2$, on luonnollinen jatkumoraja-arvo joko Riccati-yhtälöön $f' = a + f^2$ tai differentiaaliyhtälöön $(f')^2 = a(f^2-\tau_1^2)(f^2-\tau_2^2)$, missä $a \neq 0$ ja τ_i ovat vakioita siten, että $\tau_1^2 \neq \tau_2^2$. Johdamme jälkimmäisen differentiaaliyhtälön ja symmetrisen QRT yhtälön toisistaan bilineaarisen menetelmän ja jatkumoraja-arvomenetelmän avulla.

1. Introduction

The classical Malmquist theorem [18] states that: If the first order differential equation $f' = R(z,f)$, where $R(z,f)$ is rational in both arguments, has a transcendental meromorphic solution, then this equation reduces into the Riccati equation

$$f' = a_2 f^2 + a_1 f + a_0, \tag{1.1}$$

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where a_0, a_1 and a_2 are rational functions. Generalizations of Malmquist's theorem for the equation

$$(f')^n = R(z,f), \quad n \in \mathbb{N}, \tag{1.2}$$

have been given by Yosida [29] and Laine [16]. Steinmetz [24], and Bank and Kaufman [2] proved that if (1.2) has rational coefficients and a transcendental meromorphic solution, then by a suitable Möbius transformation, (1.2) can be either mapped to the Riccati equation (1.1), or to one of the equations in the following list:

$$(f')^2 = a(f-b)^2(f-\tau_1)(f-\tau_2), \tag{1.3}$$

$$(f')^2 = a(f-\tau_1)(f-\tau_2)(f-\tau_3)(f-\tau_4), \tag{1.4}$$

$$(f')^3 = a(f-\tau_1)^2(f-\tau_2)^2(f-\tau_3)^2, \tag{1.5}$$

$$(f')^4 = a(f-\tau_1)^2(f-\tau_2)^3(f-\tau_3)^3, \tag{1.6}$$

$$(f')^6 = a(f-\tau_1)^3(f-\tau_2)^4(f-\tau_3)^5, \tag{1.7}$$

where a and b are rational functions, and $\tau_1,\ldots,\tau_4$ are distinct constants. See [17, Chapter 10] for more information about Malmquist–Yosida–Steinmetz type theorems.

Recently, the present authors [15, 30] used Nevanlinna theory to provide a classification for a natural difference analogue of equation (1.2), i.e., the first-order difference equation

$$f(z+1)^n = R(z,f), \tag{1.8}$$

where $n \in \mathbb{N}$ and $R(z,f)$ is rational in both arguments. In particular, it is shown in [15] that if the difference equation (1.8) has a transcendental meromorphic solution f of hyper-order < 1, then either f satisfies a difference linear or Riccati equation

$$f(z+1) = a_1(z)f(z) + a_2(z), \tag{1.9}$$

$$f(z+1) = \frac{b_1(z)f(z)+b_2(z)}{f(z)+b_3(z)}, \tag{1.10}$$

where $a_i(z)$ and $b_j(z)$ are rational functions, or, by implementing a transformation $f \to \alpha f$ or $f \to 1/(\alpha f)$ with an algebraic function α of degree at most 2, (1.8) reduces into one of the following equations:

$$f(z+1)^2 = 1 - f(z)^2, \tag{1.11}$$

$$f(z+1)^2 = 1 - \left(\frac{\delta(z)f(z)-1}{f(z)-\delta(z)}\right)^2, \tag{1.12}$$

$$f(z+1)^2 = 1 - \left(\frac{f(z)+3}{f(z)-1}\right)^2, \tag{1.13}$$

$$f(z+1)^2 = \frac{f(z)^2-\kappa^2}{f(z)^2-1}, \tag{1.14}$$

$$f(z+1)^3 = 1 - f(z)^{-3}, \tag{1.15}$$

where $\delta(z) \not\equiv \pm 1$ is an algebraic function of degree 2 at most and $\kappa^2 \neq 0,1$ is a constant. Finite-order meromorphic solutions of the autonomous forms of the equations (1.9)–(1.15) are presented explicitly in [15]. These results provide a natural difference analogue of Steinmetz' generalization of Malmquist's theorem in the sense of Ablowitz, Halburd and Herbst [1], who suggested that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is a good candidate for a difference analogue of the Painlevé property [4]. It was shown that the finite-order condition of the proposed difference Painlevé property can be relaxed to hyper-order strictly less than one in [6], and recently to hyper-order equal to one limitedly in [14, 31]. Further, by discarding the assumption that the meromorphic solution is of hyper-order < 1 and considering transcendental meromorphic solutions of (1.8) with $\deg_f(R(z,f)) = n$, it was shown in [15] that either f satisfies (1.9) or (1.10), or (1.8) can be transformed into one of the equations (1.11)–(1.15), or one of the following equations:

$$f(z+1)^2 = \eta^2(f(z)^2 - 1), \tag{1.16}$$

$$f(z+1)^2 = 2(1 - f(z)^{-2}), \tag{1.17}$$

$$f(z+1)^2 = \frac{1 + f(z)^2}{1 - f(z)^2}, \tag{1.18}$$

$$f(z+1)^2 = \theta\frac{f(z)^2 - \kappa_1 f(z) + 1}{f(z)^2 + \kappa_1 f(z) + 1}, \tag{1.19}$$

$$f(z+1)^3 = 1 - f(z)^3, \tag{1.20}$$

where $\theta = \pm 1$, $\eta \neq 1$ is the cubic root of 1 and κ_1 is a constant such that $\kappa_1^2(\kappa_1^2 - 4) = 2(1-\theta)\kappa_1^2 - 8(1+\theta)$. Transcendental meromorphic solutions of the five equations above are elliptic functions composed with entire functions and have hyper order ≥ 1.

This paper has two purposes. The first one is to complete the classification for the case $\deg_f(R(z,f)) = n$ of (1.8). When classifying equation (1.8) for the case $\deg_f(R(z,f)) \neq n$ in [30], we made some new observations that also apply to equation (1.8) in the case where $\deg_f(R(z,f)) = n$. In the next Section 2, we will use these new observations to derive the ten equations (1.11)–(1.20) in a straightforward manner and, at the same time, identify the following equation which was omitted in [15]:

$$f(z+1)^2 = \frac{1}{2}\frac{(1+\delta)^2}{1+\delta^2}\frac{(f-1)(f-\delta^2)}{(f-\delta)^2}, \tag{1.21}$$

where $\delta \neq 0, \pm 1, \pm i$ is a constant such that

$$8\delta^5(\delta^2 + 1) - (\delta + 1)^4 = 0. \tag{1.22}$$

Transcendental meromorphic solutions of (1.21) are Jacobi elliptic functions composed with entire functions and have hyper-order at least 1, as is shown in Section 2 below. With this new equation (1.21), the results in [15, 30] can be summarised as: If equation (1.8), where $R(z,f)$ is rational in both arguments, has a transcendental meromorphic solution, then (1.8) can be reduced into one out of 30 equations. Moreover, the autonomous versions of all these 30 equations can be solved in terms of elliptic functions, exponential type functions or functions which are solutions to a certain autonomous first-order difference equation having meromorphic solutions with preassigned asymptotic behavior. We mention that Nakamura and Yanagihara [19] and Yanagihara [28] have already classified equation (1.8) in the case where $R(z,f)$ is a polynomial in f with constant coefficients.

The second purpose of this paper is to describe relations between meromorphic solutions of (1.8) and (1.2) when $n = 2$. In Section 3, we will consider relations between meromorphic solutions of the seven equations (1.9)–(1.14) and equations (1.1), (1.3) and (1.4) in the autonomous case. Equations (1.11) and (1.14) are, in fact, special cases of the symmetric Quispel–Roberts–Thompson (QRT) map [20, 21], which will be introduced in Section 3 below. The symmetric QRT map and (1.3) or (1.4) are derived from each other by using the *bilinear method* and the *continuum limit method*. The bilinear method was first used by Hirota [8, 9, 10] to find nonlinear partial difference equations that are difference analogues of some basic partial differential equations. The application of this method here implies that each of the four difference equations (1.11)–(1.14) has a natural continuum limit to equations (1.1), (1.3) or (1.4). Moreover, in Section 3, we also show that each of the five equations (1.16)–(1.19) and (1.21) can be mapped to the symmetric QRT map by doing suitable transformations and thus have a continuum limit to the differential equation (1.4). Finally, in Section 4, we will provide some comments on our results.

2. Derivations of the equations (1.11)–(1.21)

In this section, we use some new observations on equation (1.8) to derive the eleven equations (1.11)–(1.21). We shall assume that the reader is familiar with the standard notation and fundamental results of Nevanlinna theory (see, e.g., [7]). For a nonconstant meromorphic function $f(z)$, recall that a value $a \in \mathbb{C} \cup \{\infty\}$ is said to be a *completely ramified value* of $f(z)$ when $f(z) - a = 0$ has no simple roots. A direct consequence of Nevanlinna's second main theorem is that a transcendental meromorphic function can have at most four completely ramified values in $\mathbb{C} \cup \{\infty\}$.

2.1. Derivations of the eleven equations (1.11)–(1.21). To prove the theorems of [15, 30], we have actually used Yamanoi's second main theorem for small functions as targets [25, 26]. Denote the field of rational functions by $\mathcal{R}$ and set $\hat{\mathcal{R}} = \mathcal{R} \cup \{\infty\}$. Throughout this section, we say that $c(z) \in \hat{\mathcal{R}}$ is a *completely ramified rational function* of a transcendental meromorphic function $f(z)$ when the equation $f(z) = c(z)$ has at most finitely many simple roots and that $c(z)$ is a *Picard exceptional rational function* of $f(z)$ when $N(r, c, f) = O(\log r)$. We also say that $c(z)$ has multiplicity m if all the roots of $f(z) = c(z)$ have multiplicity at least m with at most finitely many exceptions. Yamanoi's second main theorem yields that a transcendental meromorphic function can have at most two Picard exceptional rational functions and also that the inequality

$$\sum_{i=1}^{q} \Theta(c_i, f) \leq 2 \tag{2.1}$$

holds for any collection of $c_1, \cdots, c_q \in \hat{\mathcal{R}}$ when f is transcendental. Moreover, we have

Theorem 2.1. *A non-constant transcendental meromorphic function $f(z)$ can have at most four completely ramified rational functions.*

All the above statements hold when the field $\mathcal{R}$ is extended slightly to include algebraic functions. For simplicity, in the following we will not distinguish algebraic functions and rational functions. For example, we always use the terms 'completely ramified rational function' and 'Picard exceptional rational function' of f even though sometimes they may actually refer to algebraic functions.

We will restrict ourselves to consider equation (1.8) with $n = \deg_f(R(z, f)) \geq 2$. Moreover, from now on we use the suppressed notation $\overline{f} = f(z+1)$ and $\underline{f} = f(z-1)$. We write equation (1.8) as

$$\overline{f}^n = R(z, f) = \frac{P(z, f)}{Q(z, f)}, \tag{2.2}$$

where $P(z,f)$ and $Q(z,f)$ are two polynomials in f with polynomial coefficients having no common factors and of degrees p and q, respectively. Then $\deg_f(R(z,f)) = \max\{p,q\} = n$. For simplicity, we may also write $P(z,f)$ and $Q(z,f)$ as

$$P(z,f) = a_p(f-\alpha_1)^{k_1}\cdots(f-\alpha_\mu)^{k_\mu} \tag{2.3}$$

and

$$Q(z,f) = (f-\beta_1)^{l_1}\cdots(f-\beta_\nu)^{l_\nu}, \tag{2.4}$$

where a_p now denotes a rational function, α_i and β_j are in general algebraic functions, distinct from each other, and $k_i, l_j \in \mathbb{N}$ denote the orders of the roots α_i and β_j, respectively. We may suppose that the *greatest common divisor* of $k_1,\ldots,k_\mu,l_1,\ldots,l_\nu$, denoted by $k=(k_1,\ldots,k_\mu,l_1,\ldots,l_\nu)$, is 1. Denote N_c to be the total number of α_i with $k_i<n$ and β_j with $l_j<n$. As in [30], the classification for equation (2.2) will be according to the number N_c of the roots α_i in (2.3) and β_j in (2.4) and whether some of them is zero.

First, we summarize the analysis on the roots α_i of $P(z,f)$ and β_j of $Q(z,f)$ in the proof of [15, Theorem 2] and formulate the following Lemmas 2.2 and 2.3.

Lemma 2.2. *Let f be a transcendental meromorphic solution of equation* (2.2). *Then α_i is either a Picard exceptional rational function of f or a completely ramified rational function of f with multiplicity $n/(n,k_i)$ and β_j is either a Picard exceptional rational function of f or a completely ramified rational function of f with multiplicity $n/(n,l_j)$. Moreover, if $q=0$, then $N_c\in\{2,3\}$; if $q\geq 1$, then $N_c\in\{2,3,4\}$.*

Proof. The first two assertions are direct results from the proof of [15]. Now the inequality (2.1) implies that $N_c\leq 4$. In particular, when $q=0$, if $N_c=4$, then by the inequality (2.1) we see that the roots α_1, α_2, α_3 and α_4 are all completely ramified functions of f with multiplicity 2, implying that the order $k_i=n/2$ for all α_i, which is impossible. Therefore, when $q=0$ we must have $N_c=2$ or $N_c=3$. □

Lemma 2.3. [15] *Let f be a transcendental meromorphic solution of equation* (2.2). *Then none of α_i in* (2.3) *such that $k_i<n$ is* 0. *Moreover, if $q\geq 1$, then after doing a bilinear transformation $f\to 1/f$, if necessary, we may suppose that $p=q=n$.*

By Lemma 2.3, we may only consider the two cases that $p=n$, $q=0$ or that $p=q=n$ below. Moreover, if $P(z,f)$ has two or more distinct roots, then none of them vanishes identically. We use the idea in the proof of [30, Lemma 3] to prove the following

Lemma 2.4. *Let f be a transcendental meromorphic solution of equation* (2.2). *If γ is a nonzero rational function, then γ cannot be a Picard exceptional rational function of f. Moreover, if $\gamma\not\equiv 0$ is a completely ramified rational function of f with multiplicity m, then $\omega\gamma$ is a completely ramified rational function of f with multiplicity m, where ω is the n-th root of 1. In particular, if* 0 *is a root of $Q(z,f)$ of order less than n, then* 0 *is not a Picard exceptional rational function of f.*

Proof. First, we suppose that $\gamma\not\equiv 0$ is a Picard exceptional rational function of f. Under our assumptions on equation (2.2), we see that at least one of α_i and β_j in (2.3) and (2.4) is non-zero and of order less than n. Denote this α_i or β_j by δ and the order of this root by t_1. As in the proof of [30, Lemma 3], we put

$$u=\frac{\overline{f}}{f-\delta},\quad v=\frac{1}{f-\delta}. \tag{2.5}$$

Then u and v are two functions meromorphic apart from at most finitely branch points and we have

$$\overline{f} = \frac{u}{v}, \quad f = \frac{1}{v} + \delta, \tag{2.6}$$

and it follows that (2.2) becomes

$$u^n = \frac{P_1(z, v)}{Q_1(z, v)} v^{n_1}, \tag{2.7}$$

where $n_1 \in \mathbb{Z}$, $P_1(z, v)$ and $Q_1(z, v)$ are two polynomials in v having no common factors and none of the roots of $P_1(z, v)$ or $Q_1(z, v)$ is zero. Denote by $p_1 = \deg_v(P_1(z, v))$ the degree of $P_1(z, v)$ in v and by $q_1 = \deg_v(Q_1(z, v))$ the degree of $Q_1(z, v)$ in v, respectively. By simple calculations, when $p = n$, $q = 0$ we get $n_1 = 0$, $p_1 = p - t_1$ and $q_1 = 0$; when $p = q = n$ and $\delta = \alpha_i$ we get $n_1 = n$, $p_1 = p - t_1$ and $q_1 = n$; when $p = q = n$ and $\delta = \beta_j$ we get $n_1 = n$, $p_1 = n$ and $q_1 = n - t_1$. Therefore, we always have $p_1 - q_1 + n_1 \neq n$. Then by the same arguments as in the proof of [30, Lemma 3], we may consider the roots of $\overline{f} - \overline{\gamma} = 0$ and also $\overline{f} - \omega\overline{\gamma} = 0$, where ω is the n-th root of 1, and finally obtain that the equation $\overline{f} - \omega\overline{\gamma} = 0$ can have at most finitely many roots, i.e., $\omega\gamma$ is a Picard exceptional rational function of f. This implies that $n = 2$. Moreover, $N_c = 2$ and the two roots of $P(z, f)$ or $Q(z, f)$ are $\pm\gamma$ for otherwise by Lemma 2.2 f would have 3 Picard exceptional rational functions or 2 Picard exceptional rational functions with one more completely ramified rational function, a contradiction to the inequality (2.1). But then it follows from (2.2) that either 0 or ∞ is a Picard exceptional rational function of f, a contradiction. Therefore, γ cannot be a Picard exceptional rational function of f.

Next, we suppose that $\gamma \not\equiv 0$ is a completely ramified rational function of f with multiplicity m. We also do the transformations in (2.5) and get the equation in (2.7). Then by the same arguments as in the proof of [30, Lemma 3], we obtain that the equation $\overline{f} - \omega\overline{\gamma} = 0$ can have at most finitely many roots with multiplicities less than m, i.e., $\omega\gamma$ is also a completely ramified rational function of f with multiplicity m.

Finally, we suppose that 0 is a root of $Q(z, f)$ of order less than n. Then there is a β_j such that $\omega\beta_j$ is a completely ramified rational function of f with multiplicity $m \geq 2$, where ω is the n-th root of 1. If 0 is a Picard exceptional rational function of f, then it follows from (2.2) that the roots of $P(z, f)$ are all Picard exceptional rational functions of f, which together with previous discussions shows that f has at least 3 Picard exceptional rational functions, which is impossible. Thus 0 cannot be a Picard exceptional rational function of f. The proof is complete. □

Corresponding to [30, Lemma 4] in the case $\deg_f(R(z, f)) \neq n$ of equation (2.2), we have the following

Lemma 2.5. *Let f be a transcendental meromorphic solution of equation* (2.2). *Then $n = 2$ or $n = 3$. Moreover, α_i in* (2.3) *with $k_i < n$ and β_j in* (2.4) *with $l_j < n$ are all simple.*

Proof. We consider the cases $q = 0$ and $p = q = n$, respectively. If $n \geq 4$, then by Lemma 2.4 a nonzero rational function γ cannot be Picard exceptional rational function of f. When $q = 0$, if $n \geq 4$, then at least one of α_i in (2.3) has order k_i coprime with n, which with Lemma 2.2 shows that α_i is a completely ramified rational function of f with multiplicity n. However, since $\alpha_i \not\equiv 0$, by Lemma 2.4 f would have 4 completely ramified rational functions with multiplicity n, a contradiction to the inequality (2.1). Therefore, when $q = 0$ we have $n = 2$ or $n = 3$. When $p = q = n$, we suppose that $n \geq 4$. Recall that $N_c \leq 4$. If some α_i in (2.3) has order k_i such that $(n, k_i) < n/2$, then we get a similar contradiction as in the case $q = 0$. Since $n \geq 4$, this implies that either $P(z, f)$ has only one root or that $P(z, f)$ has two distinct α_i with two orders k_i satisfying $k_i = n/2$. In the first case, $Q(z, f)$ has at least two distinct roots and none of β_j in (2.4) is zero for otherwise by Lemma 2.4 it follows that f has 5 completely ramified rational functions, a contradiction to Theorem 2.1; but then we also have a contradiction as in the case $q = 0$ since at least one β_j is a completely ramified rational function with multiplicity n. In the latter case, $Q(z, f)$ must have two distinct roots and none of β_j in (2.4) is zero for otherwise f would have 5 completely ramified rational function of f, a contradiction to Theorem 2.1; but then we also have a contradiction as in the case $q = 0$ since at least one β_j has order l_j co-prime with n and thus is a completely ramified rational function with multiplicity n. Therefore, when $p = q = n$, we also have $n = 2$ or $n = 3$.

Clearly, when $n = 2$, α_i in (2.3) with $k_i < n$ and β_j in (2.4) with $l_j < n$ are all simple. We claim that α_i in (2.3) with $k_i < n$ and β_j in (2.4) with $l_j < n$ are also simple when $n = 3$. In fact, when $n = 3$, since f has 3 non-zero completely ramified rational functions with multiplicities 3 it follows by Yamanoi's second main theorem that 0 and ∞ are both not completely ramified rational functions of f. If one α_i or β_j in (2.3) and (2.4) is not simple, then by a simple analysis as in the proof of [30, Lemma 4] we conclude that there are at least $T(r, f) + o(T(r, f))$ many points z_0 such that $f(z_0 + 1) = 0$ or $f(z_0 + 1) = \infty$ with multiplicity $m_0 \geq 2$ and then by computing $\overline{N}(r, 1/\overline{f})$ or $\overline{N}(r, \overline{f})$ as in the proof of [30, Lemma 4] we will get a contradiction. We omit those details. □

Let $\gamma \not\equiv 0$ be a completely ramified rational function of f with multiplicity $m \geq 2$. We further consider the roots of the equation $\overline{f}^n - \overline{\gamma}^n = 0$. In particular, we may choose $\gamma = \alpha_i$ or $\gamma = \beta_j$. By Lemma 2.4, $\omega\gamma$ is a completely ramified rational function of f with multiplicity m, where ω is an n-th root of 1. By (2.2), when $q = 0$, we have

$$\overline{f}^n - \overline{\gamma}^n = P(z, f) - \overline{\gamma}^n = a_p(f - \gamma_1)^{t_1} \cdots (f - \gamma_\tau)^{t_\tau}, \tag{2.8}$$

or, when $q \geq 1$, we have

$$\overline{f}^n - \overline{\gamma}^n = \frac{P(z, f) - \overline{\gamma}^n Q(z, f)}{Q(z, f)} = \frac{a_{p_\tau}(f - \gamma_1)^{t_1} \cdots (f - \gamma_\tau)^{t_\tau}}{Q(z, f)}, \tag{2.9}$$

where $\gamma_1, \cdots, \gamma_\tau$ are in general algebraic functions distinct from each other and $t_1, \cdots, t_\tau \in \mathbb{N}$ denote the orders of the roots $\gamma_1, \ldots, \gamma_\tau$, respectively, and $t_1 + \cdots + t_\tau = p_\tau \in \mathbb{N}$. In (2.8) we have $p_\tau = n$ and in (2.9) we have either $p_\tau < n$ when $p = q = n$ and $a_p = \overline{\gamma}^n$ or $p_\tau = n$ otherwise. We apply the analysis in the proof of [30, Lemma 3] to equations (2.8) and (2.9), respectively, and get the following

Lemma 2.6. *Let f be a transcendental meromorphic solution of equation* (2.2)*. Suppose that $\gamma \not\equiv 0$ is a completely ramified rational function of f with multiplicity $m \geq 2$. If some γ_i in* (2.8) *or* (2.9) *has order $t_i < m$, then γ_i is a completely ramified rational function of f. In particular, in* (2.9) *if $0 < q - p_\tau < m$, then ∞ is a completely ramified rational function of f. Further, suppose that $\zeta_i, \ldots, \zeta_t$ are completely ramified rational functions of f such that $\sum_{i=1}^{t} \Theta(\zeta_i, f) = 2$. Then, for each γ_i in* (2.8) *or* (2.9)*, if γ_i is not a completely ramified rational function of f, then $t_i = m$; if γ_i is a completely ramified rational function of f with multiplicity $m_i \geq 2$, then $t_i m_i = m$. In particular, for* (2.9)*, when $1 \leq p_\tau < q$, if ∞ is not a completely* *ramified rational function of f, then $q-p_\tau = m$; if ∞ is a completely ramified rational function of f with multiplicity $m_\infty \geq 2$, then $(q-p_\tau)m_\infty = m$.*

With the above five lemmas, we are ready to derive the eleven equations (1.11)–(1.21) from (2.2). We make two remarks. First, when $p = q = n = 3$, if $P(z,f)$ has only one root α and $Q(z,f)$ has three distinct roots β_1, β_2 and β_3, then by Lemma 2.4 and the inequality (2.1) we see that none of β_j is zero, for otherwise f would have at least four completely ramified rational functions with multiplicity 3, which is impossible. Then the transformation $f \to 1/f$ leads (2.2) to the case that $p = n = 3$, $q = 0$ when $\alpha \equiv 0$ or to the case that $p = q = n = 3$ and $Q(z,f)$ has only one root when $\alpha \not\equiv 0$. Second, when $p = q = n = 2$, if $P(z,f)$ has only one root α and $Q(z,f)$ has two distinct nonzero roots β_1 and β_2, then the transformation $f \to 1/f$ leads (2.2) to the case that $p = n = 2$, $q = 0$ when $\alpha \equiv 0$ or to the case that $p = q = n = 2$ and $Q(z,f)$ has only one root when $\alpha \not\equiv 0$. On the other hand, if one of the two roots β_1 and β_2 is zero, then the transformation $f \to 1/f$ leads (2.2) to the case that $p = 1$, $q = 2$ and $Q(z,f)$ has only one root. Therefore, by Lemmas 2.2, 2.3 and 2.5, we see that we only need to consider the following six cases of (2.2):

(1) $p = n = 3$, $q = 0$ and $P(z,f)$ has three distinct non-zero roots α_1, α_2 and α_3;
(2) $p = q = n = 3$, $P(z,f)$ has three distinct non-zero roots α_1, α_2 and α_3 and $Q(z,f)$ has only one root β;
(3) $p = n = 2$, $q = 0$ and $P(z,f)$ has two distinct non-zero roots α_1 and α_2;
(4) $p = q = n = 2$, $P(z,f)$ has two distinct non-zero roots α_1 and α_2 and $Q(z,f)$ has only one root β;
(5) $p = 1$, $q = n = 2$, $P(z,f)$ has only one non-zero root α and $Q(z,f)$ has only one root β;
(6) $p = q = n = 2$, $P(z,f)$ has two distinct non-zero roots α_1 and α_2 and $Q(z,f)$ has two distinct roots β_1 and β_2.

For each of the above six cases, we shall use Lemma 2.6 together with Theorem 2.1 or the inequality (2.1) to consider (2.8) and (2.9) as in [30] to determine the coefficients of $R(z,f)$ of (2.2), which then yield the 11 equations (1.11)–(1.21) after doing a bilinear transformation $f \to \alpha f$ with a suitable algebraic function α. Below we apply this strategy to each of the above six cases respectively.

Consider first case (1). By Lemma 2.4, for each α_i, $\omega\alpha_i$ is a completely ramified rational function of f with multiplicity 3 for any ω such that $\omega^3 = 1$. By the inequality (2.1) we may suppose that $\alpha_2 = \eta\alpha_1$ and $\alpha_3 = \eta^2\alpha_1$ for an η such that $\eta^2 + \eta + 1 = 0$. Thus, by doing a linear transformation $f \to \alpha_1 f$, we may rewrite equation (2.2) as

$$\overline{f}^3 = c(1 - f^3), \tag{2.10}$$

where $c = -a_p\alpha_1^3/\overline{\alpha}_1^3$ is a rational function. By (2.10), we consider

$$\overline{f}^3 - 1 = c(1 - f^3) - 1 = -c\left(f^3 - \frac{c-1}{c}\right). \tag{2.11}$$

Note that f now has three distinct completely ramified rational functions $1, \eta, \eta^2$ with multiplicity 3 and has no other completely ramified rational functions. By Lemma 2.6, we must have $c - 1 = 0$. This gives the equation (1.20).

Consider next case (2). By the same arguments as in case (1), we may suppose that $\alpha_2 = \eta\alpha_1$ and $\alpha_3 = \eta^2\alpha_1$ for an η such that $\eta^2 + \eta + 1 = 0$. Thus, by doing a linear transformation $f \to \alpha_1 f$, we may rewrite equation (2.2) as

$$\overline{f}^3 = \frac{c(f^3-1)}{(f-\delta)^3}, \tag{2.12}$$

where $c = a_p/\overline{\alpha}_1^3$ and $\delta = \beta/\alpha_1$ are in general algebraic functions. By (2.12), we consider

$$\overline{f}^3 - 1 = \frac{c(f^3-1)}{(f-\delta)^3} - 1 = \frac{c(f^3-1)-(f-\delta)^3}{(f-\delta)^3}. \tag{2.13}$$

Note that f now has three distinct completely ramified rational functions $1, \eta, \eta^2$ with multiplicity 3 and has no other completely ramified rational functions. If $c \neq 1$, then by Lemma 2.6 we must have $c(f^3-1)-(f-\delta)^3 = (c-1)(f-\gamma)^3$ for some algebraic function γ distinct from δ. However, a simple comparison on the terms of degrees 1 and 2 on both sides of this equation yields $\delta = \gamma$, a contradiction. If $c = 1$, since the terms of degree 3 cancel out, then by Lemma 2.6 we must have $\delta = 0$ so that $c(f^3-1)-(f-\delta)^3$ reduces to be an algebraic function. This gives the equation (1.15).

Consider next case (3). We claim that $\alpha_1 + \alpha_2 = 0$. Otherwise, by Lemma 2.4, f has four completely ramified rational functions with multiplicities 2, namely $\pm\alpha_1$ and $\pm\alpha_2$. Now we consider

$$\overline{f}^2 - \overline{\alpha}_1^2 = a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_1^2. \tag{2.14}$$

By Lemma 2.6, if some root of the polynomial $a_p(f-\alpha_1)(f-\alpha_2)-\overline{\alpha}_1^2$ is not equal to $-\alpha_1$ or $-\alpha_2$, then this root has order two. This implies that $-\alpha_1$ and $-\alpha_2$ are either both simple roots of the polynomial $a_p(f-\alpha_1)(f-\alpha_2)-\overline{\alpha}_1^2$, or neither of them are. Note that $a_p(f-\alpha_1)(f-\alpha_2)-\overline{\alpha}_i^2$ cannot be a square of some polynomial in f for both $i = 1, 2$. By considering $\overline{f}^2 - \overline{\alpha}_2^2$ again, then, in the first case we conclude by Lemma 2.6 that the polynomial $a_p(f-\alpha_1)(f-\alpha_2)-\overline{\alpha}_2^2$ is a square of some polynomial in f and in the latter case we conclude by Lemma 2.6 that $-\alpha_1$ and $-\alpha_2$ are both simple roots of the polynomial $a_p(f-\alpha_1)(f-\alpha_2)-\overline{\alpha}_2^2$. We only need to consider the first case. Now, by doing a linear transformation $f \to \alpha_1 f$, we have the system of two equations:

$$\begin{aligned} c(f-1)(f-\kappa) - 1 &= c(f+1)(f+\kappa), \\ c(f-1)(f-\kappa) - \overline{\kappa}^2 &= c(f-\delta)^2, \end{aligned} \tag{2.15}$$

where $c = a_p\alpha_1^2/\overline{\alpha}_1^2$, $\kappa = \alpha_2/\alpha_1$ and δ are in general algebraic functions. However, by comparing the coefficients on both sides of the two equations in (2.15), we deduce from the resulting coefficient relations that $1+\kappa = 0$, a contradiction. Therefore, $\alpha_1 + \alpha_2 = 0$. Then, by doing a linear transformation $f \to \alpha_1 f$, we may rewrite equation (2.2) as

$$\overline{f}^2 = c(1-f^2), \tag{2.16}$$

where $c = -a_p\alpha_1^2/\overline{\alpha}_1^2$ is a rational function. By (2.16), we consider

$$\overline{f}^2 - 1 = c(1-f^2) - 1 = -c\left(f^2 - \frac{c-1}{c}\right). \tag{2.17}$$

Note that f now has two distinct completely ramified rational functions ± 1. If $c - 1 = 0$, then $c = 1$ and we get the equation (1.11). Otherwise, $c - 1 \neq 0$, then by

Lemma 2.6 we see that $\pm\sqrt{(c-1)/c}$ are both completely ramified rational functions of f. Again, we consider

$$\overline{f}^2 - \frac{\overline{c}-1}{\overline{c}} = c(1-f^2) - \frac{\overline{c}-1}{\overline{c}} = -c\left(f^2 - \frac{\overline{c}c-\overline{c}+1}{\overline{c}c}\right). \tag{2.18}$$

Since now f has four completely ramified rational functions $\pm 1, \pm\sqrt{(c-1/)c}$, then by Lemma 2.6 we must have $\overline{c}c - \overline{c} + 1 = 0$, i.e., $c = -\eta^2$, where η is a constant such that $\eta^2 + \eta + 1 = 0$. This gives the equation (1.16).

Consider next case (4). In this case, if $\alpha_1 + \alpha_2 \neq 0$, then by Lemma 2.4, f has four completely ramified rational functions with multiplicities 2, namely $\pm\alpha_1$ and $\pm\alpha_2$. Suppose first that $a_p \neq \overline{\alpha}_1^2, \overline{\alpha}_2^2$. We consider

$$\overline{f}^2 - \overline{\alpha}_1^2 = \frac{a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_1^2(f-\beta)^2}{(f-\beta)^2}. \tag{2.19}$$

By the same arguments as in case (3), $-\alpha_1$ and $-\alpha_2$ are either both simple roots of the polynomial $a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_1^2(f-\beta)^2$, or neither of them are. If $a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_i^2(f-\beta)^2$ is a square of some polynomial in f for both $i=1,2$, then by computing the two discriminants $\Delta_i := [a_p(\alpha_1+\alpha_2) - 2\overline{\alpha}_i^2\beta]^2 - 4(a_p - \overline{\alpha}_i^2)(a_p\alpha_1\alpha_2 - \overline{\alpha}_i^2\beta^2) = 0$, $i = 1,2$, we deduce that $(\beta-\alpha_1)(\beta-\alpha_2) = 0$, which is impossible. By considering $\overline{f}^2 - \overline{\alpha}_2^2$ again, then, in the first case we conclude by Lemma 2.6 that the polynomial $a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_2^2(f-\beta)^2$ is a square of some polynomial in f and in the latter case we conclude by Lemma 2.6 that $-\alpha_1$ and $-\alpha_2$ are both simple roots of the polynomial $a_p(f-\alpha_1)(f-\alpha_2) - \overline{\alpha}_2^2(f-\beta)^2$. We only need to consider the first case. Now, by doing a linear transformation $f \to \alpha_1 f$, we have the system of two equations:

$$\begin{aligned} c(f-1)(f-\kappa) - (f-\delta)^2 &= (c-1)(f+1)(f+\kappa), \\ c(f-1)(f-\kappa) - \overline{\kappa}^2(f-\delta)^2 &= (c-\overline{\kappa}^2)(f-\gamma)^2, \end{aligned} \tag{2.20}$$

where $c = a_p/\overline{\alpha}_1^2$, $\kappa = \alpha_2/\alpha_1$, $\delta = \beta/\alpha_1$ and γ are in general algebraic functions. By comparing the coefficients on both sides of the two equations in (2.20), we deduce from the resulting coefficient relations that $\kappa = \delta^2$, $\gamma = -\delta$, $c = \frac{1}{2}\frac{(1+\delta)^2}{1+\delta^2}$ and $\delta \neq 0, \pm 1, \pm i$ satisfies $8\overline{\delta}^4(\delta^2+1)\delta = (\delta+1)^4$. Note that $\delta \equiv 1$ solves this equation. We see that δ is a constant and thus α_1 and α_2 are both rational functions. This gives the equation (1.21). Now, if $a_p = \overline{\alpha}_1^2$ or $a_p = \overline{\alpha}_2^2$, then by similar discussions as above, we have the system of two equations:

$$\begin{aligned} (f-1)(f-\kappa) - (f-\delta)^2 &= c_1, \\ (f-1)(f-\kappa) - \overline{\kappa}^2(f-\delta)^2 &= (1-\overline{\kappa}^2)(f-\gamma)^2, \end{aligned} \tag{2.21}$$

or

$$\begin{aligned} \overline{\kappa}^2(f-1)(f-\kappa) - (f-\delta)^2 &= (\overline{\kappa}^2-1)(f+1)(f+\kappa), \\ \overline{\kappa}^2(f-1)(f-\kappa) - \overline{\kappa}^2(f-\delta)^2 &= c_2, \end{aligned} \tag{2.22}$$

where $\delta = \beta/\alpha_1$, c_1, c_2 and γ are in general algebraic functions. However, by comparing the coefficients on both sides, we deduce from the resulting coefficient relations obtained from the system of two equations in (2.21) that $\delta = \gamma$ and the resulting coefficient relations obtained from the system of two equations in (2.22) that $1 + \kappa = 0$, both of which are impossible. On the other hand, if $\alpha_1 + \alpha_2 = 0$, then by doing a linear transformation $f \to \alpha_1 f$, we may rewrite equation (2.2) as

$$\overline{f}^2 = \frac{c(1-f^2)}{(f-\delta)^2}, \tag{2.23}$$

where $c = -a_p/\overline{\alpha}_1^2$ and $\delta = \beta/\alpha_1$ are in general algebraic functions. By (2.23), we consider

$$\overline{f}^2 - 1 = \frac{c(1-f^2)}{(f-\delta)^2} - 1 = \frac{c(1-f^2)-(f-\delta)^2}{(f-\delta)^2}. \tag{2.24}$$

When $c=-1$, if $\delta \neq 0$, then by Lemma 2.6 we see that $2\delta/(\delta^2+1)$ and ∞ are both completely ramified rational functions of f. However, by Lemma 2.4 f would have 5 completely ramified rational functions, a contradiction to Theorem 2.1. Therefore, we must have $\delta = 0$ when $c=-1$. When $c \neq -1$, if $c(1-f^2)-(f-\delta)^2$ has only one root, i.e., the discriminant $\Delta := 4\delta^2 + 4(c+1)(c-\delta^2) = 0$, then $c = \delta^2 - 1$. The above two cases give the equation (1.12). Otherwise, we have $c \neq -1$ and $c(1-f^2)-(f-\delta)^2$ has two distinct roots, say δ_1 and δ_2, which are both completely ramified rational functions of f by Lemma 2.6. By Lemma 2.4, $\pm\delta_1$ and $\pm\delta_2$ are all completely ramified rational functions of f. By Theorem 2.1 we must have $\delta_1 + \delta_2 = 0$. It follows that $\delta = 0$ and $\delta_1^2 = c/(c+1)$. Again, we consider

$$\overline{f}^2 - \frac{\overline{c}}{\overline{c}+1} = \frac{c(1-f^2)}{f^2} - \frac{\overline{c}}{\overline{c}+1} = \frac{c - [(\overline{c}c + c + \overline{c})/(\overline{c}+1)]f^2}{f^2}. \tag{2.25}$$

If $\overline{c}c + c + \overline{c} \neq 0$, then by Lemma 2.6 we see that $\pm\sqrt{c(\overline{c}+1)/(\overline{c}c+c+\overline{c})}$ are both completely ramified rational functions of f, a contradiction to Theorem 2.1. Therefore, we must have $\overline{c}c + c + \overline{c} = 0$ and thus $c = -2$. This gives the equation (1.17).

Consider next case (5). In this case, by doing a linear transformation $f \to -\alpha f$, we may rewrite equation (2.2) as

$$\overline{f}^2 = \frac{c(f+1)}{(f-\delta)^2}, \tag{2.26}$$

where $c = -a_p/\alpha\overline{\alpha}^2$ and $\delta = -\beta/\alpha$ are rational functions. Then from previous discussions and Lemmas 2.2 and 2.4, we see that ∞, ± 1 and $\pm\delta$ are all completely ramified rational functions of f. By Theorem 2.1, we must have $\delta = 1$. By (2.26), we consider

$$\overline{f}^2 - 1 = \frac{c(f+1)}{(f-1)^2} - 1 = \frac{c(f+1)-(f-1)^2}{(f-1)^2}. \tag{2.27}$$

If $c(f+1)-(f-1)^2$ has two distinct roots, then by Lemma 2.6 these two roots are both completely ramified rational functions, a contradiction to Theorem 2.1. Thus $c(f+1)-(f-1)^2$ can have only one root, which implies that the discriminant $\Delta := (c+2)^2 + 4(c-1) = 0$, i.e., $c = -8$. This gives the equation (1.13).

Consider finally case (6). By Lemma 2.2, $\pm\alpha_1$, $\pm\alpha_2$, $\pm\beta_1$ and $\pm\beta_2$ are all completely ramified rational functions of f. By Theorem 2.1, we must have either $\alpha_1 + \alpha_2 = 0$ and $\beta_1 + \beta_2 = 0$ or $\alpha_1 + \beta_1 = 0$ and $\alpha_2 + \beta_2 = 0$. When $\alpha_1 + \alpha_2 = 0$ and $\beta_1 + \beta_2 = 0$, by doing a linear transformation $f \to \beta_1 f$, we may rewrite equation (2.2) as

$$\overline{f}^2 = \frac{c(f^2-\kappa^2)}{f^2-1}, \tag{2.28}$$

where $c = a_p/\overline{\beta}_1^2$ and $\kappa = \alpha_1/\beta_1$ are in general algebraic functions. We see that $\kappa^2 \neq 0, 1$. By (2.28), we consider

$$\overline{f}^2 - 1 = \frac{c(f^2-\kappa^2)}{f^2-1} - 1 = \frac{(c-1)f^2 - (c\kappa^2-1)}{f^2-1}. \tag{2.29}$$

If $c \neq 1$ and $c \neq 1/\kappa^2$, then by Lemma 2.6, $\pm\sqrt{(c\kappa^2-1)/(c-1)}$ are both completely ramified rational functions of f and thus f would have 6 completely ramified rational functions, a contradiction to Theorem 2.1. Therefore, $c = 1$ or $c = 1/\kappa^2$. If $c = 1$, then we get the equation (1.14); if $c = 1/\kappa^2$, then we consider the equation $\overline{f}^2 - \overline{\kappa}^2$ and by the same arguments as above to obtain that $1/\kappa^2 = \overline{\kappa}^2$, i.e., $\kappa^2 = -1$ and $c = -1$ and thus we get the equation (1.18). On the other hand, when $\alpha_1 + \beta_1 = 0$ and $\alpha_2 + \beta_2 = 0$, by doing a linear transformation $f \to \sqrt{\alpha_1\alpha_2}f$, we may rewrite equation (2.2) as

$$\overline{f}^2 = c\frac{(f-\delta)(f-\delta^{-1})}{(f+\delta)(f+\delta^{-1})}, \tag{2.30}$$

where $c = a_p/\overline{\alpha}_1\overline{\alpha}_2$ and $\delta = \sqrt{\alpha_1/\alpha_2}$ are in general algebraic functions. By the same arguments as for the equation (2.19), we may consider $\overline{f}^2 - \overline{\delta}^2$ and conclude by Lemma 2.6 that the polynomial $c(f-\delta)(f-\delta^{-1}) - \overline{\delta}^2(f+\delta)(f+\delta^{-1})$ is a square of some polynomial in f when $c \neq \overline{\delta}^2$ or reduces to be an algebraic function c_1 when $c = \overline{\delta}^2$. Since $\delta^2 \neq 0, \pm 1, \pm i$, a straightforward comparison shows that the latter case is impossible. Similarly, we may consider $\overline{f}^2 - \overline{\delta}^{-2}$ and conclude that the polynomial $c(f-\delta)(f-\delta^{-1}) - \overline{\delta}^{-2}(f+\delta)(f+\delta^{-1})$ is a square of some polynomial in f. Thus we have the system of two equations:

$$\begin{aligned} c(f-\delta)(f-\delta^{-1}) - \overline{\delta}^2(f+\delta)(f+\delta^{-1}) &= (c-\overline{\delta}^2)(f-\gamma_1)^2, \\ c(f-\delta)(f-\delta^{-1}) - \overline{\delta}^{-2}(f+\delta)(f+\delta^{-1}) &= (c-\overline{\delta}^{-2})(f-\gamma_2)^2, \end{aligned} \tag{2.31}$$

where γ_1 and γ_2 are in general algebraic functions. By comparing the coefficients on both sides of the two equations in (2.31), we deduce from the resulting coefficient relations that $\gamma_1^2 = \gamma_2^2 = c^2 = 1$ and $d = \delta + \delta^{-1}$ satisfies $\overline{d}^2(d^2-4) = 2(1-c)d^2 - 8(1+c)$. We see that d is a constant. Writing $c = \theta$ and $d = \kappa_1$, we have the equation (1.19) and also complete the classification for equation (2.2).

2.2. Growth of meromorphic solutions of equation (1.21). We show that all transcendental meromorphic solutions f of equation (1.21) have hyper-order ≥ 1. Note that f is twofold ramified over each of $\pm 1, \pm\delta^2$. Then there exists an entire function $\varphi(z)$ such that f is written as $f(z) = \mathrm{sn}(\varphi(z))$, where $\mathrm{sn}(\varphi) = \mathrm{sn}(\varphi, 1/\delta^2)$ is the Jacobi elliptic function with the modulus $1/\delta^2$ and satisfies the first order differential equation $\mathrm{sn}'(\varphi)^2 = (1-\mathrm{sn}(\varphi)^2)(1-\mathrm{sn}(\varphi)^2/\delta^4)$. Moreover, by the second main theorem we have $T(r,f) = N(r, 1/(f-1)) + O(\log r)$. Let z_0 be a point such that $f(z_0) = \mathrm{sn}(\varphi(z_0)) = 1$. It follows from (1.21) that $f(z_0+1) = \mathrm{sn}(\varphi(z_0+1)) = 0$. Computing the Maclaurin series for $\mathrm{sn}(\varphi)$ and $\mathrm{sn}(\overline{\varphi})$ around the point z_0, respectively, we get

$$\mathrm{sn}(\varphi(z)) = 1 - \frac{\delta^4-1}{\delta^4}(\varphi(z)-\varphi(z_0)) + \cdots = 1 - \frac{\delta^4-1}{\delta^4}\varphi'(z_0)(z-z_0) + \cdots, \tag{2.32}$$

and

$$\mathrm{sn}(\varphi(z+1)) = \varphi(z+1) - \varphi(z_0+1) + \cdots = \varphi'(z_0+1)(z-z_0) + \cdots. \tag{2.33}$$

By substituting the above two expressions into (1.21) and then comparing the second-degree terms on both sides of the resulting equation, we find

$$\varphi'(z_0+1)^2 = \frac{1}{2}\frac{(1+\delta)^4}{\delta^4}\varphi'(z_0)^2. \tag{2.34}$$

A simple computation together with equation (1.22) shows that $(1+\delta)^4 \neq 2\delta^4$. Define $G(z) := \varphi'(z+1)^2 - \frac{1}{2}\frac{(1+\delta)^4}{\delta^4}\varphi'(z)^2$. From the discussions in [15] we know that $T(r,\overline{f}) = T(r,f) + O(\log r)$. Since $\mathrm{sn}(z)$ has positive order of growth, then by [7, p. 50] we have $T(r,\varphi) = o(T(r,f))$ and $T(r,\overline{\varphi}) = o(T(r,\overline{f}))$, where $r \to \infty$. If $G \not\equiv 0$, then $T(r,G) \le o(T(r,f))$, $r \to \infty$, which is impossible since G has $T(r,f) + O(\log r)$ many zeros. Thus $G(z) \equiv 0$. Now, $\varphi'(z+1) = \pm\frac{1}{\sqrt{2}}\frac{(1+\delta)^2}{\sigma^2}\varphi'(z)$ and by integration we see that φ is an entire function such that $T(r,\varphi) \ge Kr$ for some $K > 0$ and all $r \ge r_0$ with some $r_0 \ge 0$. Since $\mathrm{sn}(z)$ has positive exponent of convergence of zeros and $f(z) = \mathrm{sn}(\varphi(z))$, then the fact that f is of hyper-order at least one is a consequence of Lemma 2.7 below, which is a slightly modified version of [17, Lemma 5.20].

Lemma 2.7. *Let g be a meromorphic function such that the exponent of convergence of zeros $\lambda = \lambda(g) > 0$, and let $\varphi = \varphi(z)$ be an entire function such that $T(r,\varphi) \ge cr$ for some $c > 0$ and all $r \ge r_0$ with some $r_0 \ge 0$. Then the hyper order of $g \circ \varphi$ is at least one.*

Proof. We consider the zeros of $g\circ\varphi$. Since φ is an entire function, then φ has at most one finite Picard's exceptional value. Thus we may choose a constant $r_1 \ge r_0$ such that φ takes in $|z| < t$ every value w in the annulus $r_1 < |w| < M(t,\varphi)$, provided that t is large enough. Let g have $\mu(t)$ zeros in this annulus, counted according to their multiplicity. Then by the definition of λ, we have

$$\limsup_{r\to\infty}\frac{\log n(r)}{\log r} = \limsup_{t\to\infty}\frac{\log\mu(t)}{\log M(t,\varphi)} = \lambda > 0. \tag{2.35}$$

Hence, for some $\tau > 0$, there exists a sequence (t_n) tending to $+\infty$ such that

$$\mu(t_n) > (M(t_n,\varphi))^\tau \ge \left(e^{ct_n}\right)^\tau, \tag{2.36}$$

where c is a positive constant. The second inequality above follows by our assumption since $\log M(t,\varphi) \ge T(t,\varphi)$ for all large t. Now, $g\circ\varphi$ has at least $\mu(t)$ zeros in $|z| < t$. Making using of (2.36), we have

$$\limsup_{r\to\infty}\frac{\log\log n(r,1/g\circ\varphi)}{\log r} \ge \limsup_{t_n\to\infty}\frac{\log\log\mu(t_n,1/g\circ\varphi)}{\log t_n} \ge 1. \tag{2.37}$$

By the fact that $T(r,1/g\circ\varphi) \ge N(r,1/g\circ\varphi)$, we conclude that the hyper order of $g\circ\varphi$ is at least one. Thus our assertion follows. □

We also note that the fact that all meromorphic solutions of each equation in the list (1.16)–(1.20) can be proved using the above method since all solutions of them are elliptic functions composed with entire functions.

3. Relations between equations (1.2) and (1.8) when $n = 2$

In this section, we use the bilinear method and the continuum limit method to study the relations between equations (1.2) and (1.8) for the case $n = 2$. For the description of the bilinear method, see [8] or [11]. Here we provide a brief overview of the continuum limit method: Let k be a positive integer, and ε be a complex number. We set a pair of relations:

$$\mu(z,t,\varepsilon)=0, \quad \nu(f(z),w(t,\varepsilon),\varepsilon)=0. \tag{3.1}$$

According to this, we transform a difference equation

$$\Omega_0(z,f(z+1),\cdots,f(z+k))=0 \tag{3.2}$$

to a certain difference equation

$$\Omega_1(t,w(t,\varepsilon),\cdots,w(t+k\varepsilon,\varepsilon))=0. \tag{3.3}$$

Letting $\varepsilon\to 0$, with some conditions on coefficients of Ω_1, we derive a differential equation:

$$\Omega_1(t,w'(t,0),\cdots,w^{(k)}(t,0))=0. \tag{3.4}$$

It is clear that the first order linear difference equation has a continuum limit to the first order linear differential equation in the autonomous case. In the two subsections below, we describe the relations between the difference equations (1.11)–(1.14) and the differential equations (1.1), (1.3) or (1.4). We also show how to take the continuum limit for the five equations (1.16)–(1.19) and (1.21).

3.1. Relations between (1.12) and (1.13) and the Riccati equation (1.1). In [12], Ishizaki discussed the relation between a differential Riccati equation and a difference Riccati equation. We first recall Ishizaki's results below. For the differential Riccati equation (1.1), we assume that $a_2\not\equiv 0$ from now on. It is elementary to show that a suitable linear transformation on f leads equation (1.1) to

$$f'=f^2+A(z), \tag{3.5}$$

where $A(z)$ is a rational function formulated in terms of a_j and their derivatives; see [17, chapter 9]. Ishizaki used the bilinear method to derive a difference Riccati equation from (3.5) in the following way: Setting $f(z)=u(z)/v(z)$, then equation (3.5) becomes

$$u'(z)v(z)-u(z)v'(z)=u(z)^2+A(z)v(z)^2, \tag{3.6}$$

which is gauge invariant. In other words, for any $h(z)$, $\tilde{u}(z)=u(z)h(z)$ and $\tilde{v}(z)=v(z)h(z)$ also satisfy the differential equation (3.6) in place of $u(z)$ and $v(z)$, respectively. Corresponding to equation (3.6), we choose a difference equation

$$u(z+1)v(z)-u(z)v(z+1)=u(z)u(z+1)+A(z)v(z)v(z+1), \tag{3.7}$$

having the property of being gauge invariant. Setting $f(z)=u(z)/v(z)$ in the difference equation above, then we obtain

$$f(z+1)-f(z)=f(z+1)f(z)+A(z), \tag{3.8}$$

i.e.,

$$f(z+1)=\frac{f(z)+A(z)}{1-f(z)}. \tag{3.9}$$

On the other hand, for the difference Riccati equation (1.10), Ishizaki showed that if $b_1(z)\neq -b_3(z+1)$, by doing the transformation $f(z)\to[(-b_3-\underline{b}_1)f+(-b_3+\underline{b}_1)]/2$ we obtain the difference equation (3.9) with

$$A(z)=\frac{-4b_2-b_1\underline{b}_1+3b_1b_3-\underline{b}_1\overline{b}_3-b_3\overline{b}_3}{(b_3+\underline{b}_1)(\overline{b}_3+b_1)}. \tag{3.10}$$

Set

$$t = \varepsilon z, \quad f = \varepsilon w(t, \varepsilon), \tag{3.11}$$

with the condition $A(z) = \varepsilon^2 \tilde{A}(t, \varepsilon)$ and $\lim_{\varepsilon \to 0} \tilde{A}(t, \varepsilon) = \tilde{A}(t, 0)$. Since $f(z+1) = \varepsilon w(\varepsilon(z+1), \varepsilon) = \varepsilon w(t + \varepsilon, \varepsilon)$, we have

$$w(t+\varepsilon, \varepsilon) - w(t, \varepsilon) = \varepsilon w(t+\varepsilon, \varepsilon) w(t, \varepsilon) + \varepsilon \tilde{A}(t, \varepsilon). \tag{3.12}$$

By letting $\varepsilon \to 0$, we have

$$w'(t, 0) = w(t, 0)^2 + \tilde{A}(t, 0), \tag{3.13}$$

which is equation (3.5). In particular, if A is a constant, then we replace A with $\varepsilon^2 \tilde{A}$ with a constant $\tilde{A}$.

With the introduction above, let's look at the two equations (1.12) and (1.13), respectively. For equation (1.12), if we put $f = (\gamma + \gamma^{-1})/2$, then it follows that

$$\overline{\gamma}^2 \pm 2i \frac{\delta \gamma^2 - 2\gamma + \delta}{\gamma^2 - 2\delta\gamma + 1} \overline{\gamma} - 1 = 0.$$

Solving the equation above, we get four different difference Riccati equations:

$$\overline{\gamma} = \left\{ -\theta \frac{(\pm i\delta - \sqrt{1-\delta^2})\gamma \pm i}{\gamma - \delta \pm i\sqrt{1-\delta^2}} \right\}^{\theta}, \quad \theta = \pm 1.$$

It is easy to see that these four difference equations do not have any common solutions. Take the following difference Riccati equation as an example:

$$\overline{\gamma} = \frac{(-i\delta + \sqrt{1-\delta^2})\gamma - i}{\gamma + (-\delta + i\sqrt{1-\delta^2})}, \tag{3.14}$$

consider the case where δ is a constant. If $2\delta^2 \neq 1$, then by doing the transformation

$$\gamma \to \frac{(1+i)(\delta - \sqrt{1-\delta^2})}{2} \gamma + \frac{(1-i)(\delta + \sqrt{1-\delta^2})}{2},$$

we obtain from the equation above that

$$\overline{\gamma} - \gamma = \overline{\gamma}\gamma + A, \tag{3.15}$$

where A has the form in (3.10) with $b_1 = (-i\delta + \sqrt{1-\delta^2})$, $b_2 = -i$ and $b_3 = -\delta + i\sqrt{1-\delta^2}$. Therefore, for the solutions of (1.12) such that (3.14) hold, we set

$$\begin{aligned} &t = \varepsilon z, \\ &f = \frac{1}{2}\left(\gamma + \frac{1}{\gamma}\right), \\ &\gamma = \frac{(1+i)(\delta - \sqrt{1-\delta^2})}{2} \varepsilon w(t, \varepsilon) + \frac{(1-i)(\delta + \sqrt{1-\delta^2})}{2}, \end{aligned} \tag{3.16}$$

and replace A with $\varepsilon^2 A$. Then we have the equation in (3.12) and, by letting $\varepsilon \to 0$, we finally obtain the equation in (3.13). Equation (1.13) is dealt with in a similar way. From the results in [15], if we put $f = \frac{1-u^2}{u^2}$, $\sqrt{2}u = \frac{1}{2}(\gamma + \gamma^{-1})$, then we also get four different difference Riccati equations which do not have any common solutions. Consider the following case:

$$f = \frac{1-u^2}{u^2} = \frac{8\gamma^2 - (\gamma^2+1)^2}{(\gamma^2+1)^2}, \quad \overline{\gamma} = -\frac{-(1+\sqrt{2})\gamma + i}{\gamma - i + i\sqrt{2}}.$$

By making similar substitutions as in (3.16), we may obtain the equation in (3.12) and, by letting $\varepsilon \to 0$, we obtain the equation in (3.13).

3.2. Relations between (1.11), (1.14), (1.16)–(1.19) and (1.21) and the differential equations (1.3) or (1.4). The autonomous versions of the seven difference equations (1.11), (1.14), (1.16)–(1.19) and (1.21) are closely related to the QRT map [20, 21], which is defined by the system of two equations:

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}, \tag{3.17}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}, \tag{3.18}$$

where

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \delta_0 & \varepsilon_0 & \zeta_0 \\ \kappa_0 & \lambda_0 & \mu_0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \delta_1 & \varepsilon_1 & \zeta_1 \\ \kappa_1 & \lambda_1 & \mu_1 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \tag{3.19}$$

$$\begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \delta_0 & \kappa_0 \\ \beta_0 & \varepsilon_0 & \lambda_0 \\ \gamma_0 & \zeta_0 & \mu_0 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \times \begin{pmatrix} \alpha_1 & \delta_1 & \kappa_1 \\ \beta_1 & \varepsilon_1 & \lambda_1 \\ \gamma_1 & \zeta_1 & \mu_1 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \tag{3.20}$$

where '$\times$' denotes the cross product of two vectors. In the symmetric case, i.e.,

$$\begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \varepsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix} = \begin{pmatrix} \alpha_i & \delta_i & \kappa_i \\ \beta_i & \varepsilon_i & \lambda_i \\ \gamma_i & \zeta_i & \mu_i \end{pmatrix}, \quad i = 0, 1, \tag{3.21}$$

the QRT family reduces into a single equation

$$w_{n+1} = \frac{f_1(w_n) - w_{n-1} f_2(w_n)}{f_2(w_n) - w_{n-1} f_3(w_n)}, \tag{3.22}$$

by taking $x_n = w_{2n}$ and $y_n = w_{2n+1}$ in (3.17) and (3.18). The symmetric QRT family possesses an invariant:

$$\begin{aligned} &(\alpha_0 + K\alpha_1)x_{n+1}^2 x_n^2 + (\beta_0 + K\beta_1)(x_{n+1}^2 x_n + x_{n+1} x_n^2) + (\gamma_0 + K\gamma_1)(x_{n+1}^2 + x_n^2) \\ &+ (\varepsilon_0 + K\varepsilon_1)x_{n+1}x_n + (\zeta_0 + K\zeta_1)(x_{n+1} + x_n) + (\mu_0 + K\mu_1) = 0, \end{aligned} \tag{3.23}$$

where K is a constant. By doing a Möbius transformation $x_n \to \frac{\alpha_1 x_n + \beta_1}{\alpha_2 x_n + \beta_2}$ with suitable constants α_i and β_j, the symmetric QRT map in (3.23) takes the form:

$$\alpha x_{n+1}^2 x_n^2 + \beta(x_{n+1}^2 + x_n^2) + \gamma x_{n+1} x_n + \delta = 0, \tag{3.24}$$

where α, β, γ and δ are constants; see [22]. By reinterpreting discrete equations as difference equations (see [1]), we see later that (1.14) reduces to a special case of the symmetric QRT map in the generic case (i.e., $\alpha\delta \neq 0$) and equation (1.11) is the symmetric QRT map in the degenerate case (i.e., $\alpha\delta = 0$). Moreover, the five equations (1.16)–(1.19) and (1.21) can also be mapped to the symmetric QRT map, as is shown below.

Suppose that a and b in (1.3) and (1.4) are both constants. For simplicity, we treat equation (1.3) as a special case of equation (1.4) with $\tau_1 = \tau_3$. By doing a Möbius transformation $f \to \frac{\alpha_1 f + \beta_1}{\alpha_2 f + \beta_2}$ with suitable constants α_i and β_j, if necessary, we may assume that $\tau_1 + \tau_3 = 0$ and $\tau_2 + \tau_4 = 0$. Thus we may write (1.4) as

$$(f')^2 = a(f^2 - \tau_1^2)(f^2 - \tau_2^2), \tag{3.25}$$

where $\tau_1^2 \neq \tau_2^2$ and it is possible that $\tau_1^2 = 0$. Now the bilinear method applies. Following Ishizaki, we set $f(z) = u(z)/v(z)$ and obtain from equation (3.25) that

$$[u'(z)v(z) - u(z)v'(z)]^2 = a(z)[u(z)^2 - \tau_1^2 v(z)^2][u(z)^2 - \tau_2^2 v(z)^2], \tag{3.26}$$

which is gauge invariant. Corresponding to this equation, we choose a difference equation

$$\begin{aligned} &[u(z+1)v(z) - u(z)v(z+1)]^2 \\ &= a(z)[u(z)u(z+1) - \tau_1^2 v(z)v(z+1)][u(z)u(z+1) - \tau_2^2 v(z)v(z+1)], \end{aligned} \tag{3.27}$$

having the property of being gauge invariant. Setting $f(z) = u(z)/v(z)$ in the difference equation above, we have

$$[f(z+1) - f(z)]^2 = a[f(z+1)f(z) - \tau_1^2][f(z+1)f(z) - \tau_2^2]. \tag{3.28}$$

which is a special case of the symmetric QRT map (3.23) after expansion.

When $\tau_1^2 = 0$, we do the transformation $f \to 1/f$ and obtain from (3.28) that

$$[f(z+1) - f(z)]^2 = -a\tau_2^2 \left(f(z+1)f(z) - 1/\tau_2^2 \right). \tag{3.29}$$

We see that equation (1.11) is included in (3.29). By setting $t = \varepsilon z$, $f(z) = w(t, \varepsilon)$ and giving $\varepsilon^2(-a\tau_2^2)$ in place of $-a\tau_2^2$, then equation (3.29) has a continuum limit to the equation $(f')^2 = -a(\tau_2^2 f^2 - 1)$; see [13]. The equation $(f')^2 = -a(\tau_2^2 f^2 - 1)$ can be obtained from equation (3.25) by doing the transformation $f \to 1/f$.

When $\tau_1^2 \neq 0$, we re-scale f by $f \to (\tau_1\tau_2)^{1/2} f$ and then expand (3.28) to obtain the canonical form of the symmetric QRT map:

$$\overline{f}^2 f^2 + A(\overline{f}^2 + f^2) + 2B\overline{f}f + 1 = 0, \tag{3.30}$$

where $A = -1/(a\tau_1\tau_2)$ and $B = [2 - a(\tau_1^2 + \tau_2^2)]/(2a\tau_1\tau_2)$. In particular, for equation (1.14), we may re-scale f by $f \to f/\kappa_1$ with a constant κ_1 first to obtain the equation

$$\overline{f}^2 f^2 - \kappa_1^2(\overline{f}^2 + f^2) + \kappa_2^2 = 0, \tag{3.31}$$

where $\kappa_2^2 = \kappa_1^4 \kappa^2$. By doing the transformation $f \to \alpha\frac{f-\beta}{f+\beta}$ with constants α, β satisfying $\alpha^4 = \kappa_2^2$, we obtain from (3.31) the canonical form in (3.30) and the corresponding coefficients A and B in (3.30) are

$$\begin{aligned} A &= \beta^2, \\ B &= 2\frac{\alpha^4 + 2\kappa_1^2\alpha^2 + \kappa_2^2}{\alpha^4 - 2\kappa_1^2\alpha^2 + \kappa_2^2}\beta^2, \end{aligned}$$

respectively. It is well-known that equation (3.30) is parameterized by elliptic functions; see [3, 22] or [5]. Here we incorporate the process of parametrization from [5]. Define the parameters k and ε such that

$$\begin{aligned} A &= -\frac{1}{k\mathrm{sn}^2\,\varepsilon}, \\ B &= \frac{\mathrm{cn}\,\varepsilon\ \mathrm{dn}\,\varepsilon}{k\mathrm{sn}^2\,\varepsilon}, \end{aligned} \tag{3.32}$$

respectively. These choices of A and B imply that

$$k + k^{-1} = (B^2 - A^2 - 1)A^{-1}. \tag{3.33}$$

Therefore, considering equation (3.30) as a quadratic equation for $\overline{f}$, and using the transformation $f = k^{1/2}\text{sn}\ u$, where sn u denotes the Jacobi elliptic sn function with argument u and modulus k, we have

$$\text{sn}\ \overline{u} = \frac{\text{cn}\ \varepsilon\ \text{dn}\ \varepsilon\ \text{sn}\ u \pm \text{sn}\ \varepsilon\ \text{cn}\ u\ \text{dn}\ u}{1 - k^2\text{sn}^2\ \varepsilon\ \text{sn}^2\ u}. \tag{3.34}$$

This is solved by $u = \varepsilon z + C$, where C is a free parameter. Using the expressions of A and B in (3.32), we rewrite equation (3.30) as

$$(\overline{f} - f)^2 = (k\text{sn}^2\ \varepsilon)\overline{f}^2 f^2 + 2\left(\text{cn}\ \varepsilon\ \text{dn}\ \varepsilon - 1\right)\overline{f}f + k\text{sn}^2\ \varepsilon. \tag{3.35}$$

By the above process, if we set

$$t = \varepsilon z, \quad f = k^{1/2}w(t, \varepsilon), \tag{3.36}$$

then, since $f(z+1) = k^{1/2}w(t+\varepsilon, \varepsilon)$, by dividing $k\text{sn}^2\ \varepsilon$ on both sides of equation (3.35) we get

$$\begin{aligned} &\frac{[w(t+\varepsilon,\varepsilon) - w(t,\varepsilon)]^2}{\text{sn}^2\ \varepsilon} \\ &= k^2 w(t+\varepsilon,\varepsilon)^2 w(t,\varepsilon)^2 + \left(\frac{2\text{cn}\ \varepsilon\ \text{dn}\ \varepsilon - 2}{\text{sn}^2\ \varepsilon}\right) w(t+\varepsilon,\varepsilon)w(t,\varepsilon) + 1. \end{aligned} \tag{3.37}$$

Recall the Maclaurin series for sn ε, cn ε and dn ε, respectively:

$$\begin{aligned} \text{sn}\ \varepsilon &= \varepsilon - (1+k^2)\frac{\varepsilon^3}{3!} + (1 + 14k^2 + k^4)\frac{\varepsilon^5}{5!} + \cdots, \\ \text{cn}\ \varepsilon &= 1 - \frac{\varepsilon^2}{2!} + (1+k^4)\frac{\varepsilon^4}{4!} + \cdots, \\ \text{dn}\ \varepsilon &= 1 - k^2\frac{\varepsilon^2}{2!} + k^2(4+k^2)\frac{\varepsilon^4}{4!} + \cdots. \end{aligned} \tag{3.38}$$

By substituting the above series into (3.37) and then letting $\varepsilon \to 0$, we obtain the following differential equation:

$$[w'(t,0)]^2 = (k^2 w(t,0)^2 - 1)(w(t,0)^2 - 1), \tag{3.39}$$

which is equation (3.25). In particular, for equation (3.31), we see that this process yields $2\frac{\alpha^4+2\kappa_1^2\alpha^2+\kappa_2^2}{\alpha^4-2\kappa_1^2\alpha^2+\kappa_2^2} \to -1$ as $\varepsilon \to 0$. Recalling that $\alpha^4 = \kappa_2^2$, this implies that $\kappa_2 \to \pm\frac{\kappa_1}{3}$ as $\varepsilon \to 0$. By combining the results above together, we conclude that (1.14) has a continuum limit to the differential equation (1.4).

We now consider the five equations (1.16)–(1.19) and (1.21). We take the equation (1.16) as an example to show that this equation can be transformed into the symmetric form, which is included in the QRT family, and have a continuum limit to (3.25). It is easy to see that solutions of equation (1.16) also satisfy the following two equations:

$$\begin{aligned} \overline{f}^2 + \eta^2 &= \eta^2 f^2, \\ \overline{f}^2 - 1 &= \eta^2(f^2 + \eta^2). \end{aligned} \tag{3.40}$$

Since all roots of $f(z) \pm i\eta = 0$ have even multiplicities, we see that $\frac{f+i\eta}{f-i\eta} = h^2$ for some meromorphic function h. It follows that $f = i\eta\frac{h^2+1}{h^2-1}$. Denote $H = \frac{h^2+1}{2h}$. By dividing the first equation in (3.40) by the second equation in (3.40) on both sides, we obtain

$$\frac{\overline{f}^2+\eta^2}{\overline{f}^2-1}=\frac{f^2}{f^2+\eta^2}=\left(\frac{h^2+1}{2h}\right)^2=H^2, \tag{3.41}$$

i.e.,

$$\overline{f}^2H^2-(\overline{f}^2+H^2)-\eta^2=0, \tag{3.42}$$

which is a biquadratic equation with respect to $\overline{f}$ and H. Instead of considering equation (3.42) directly, we may first re-scale f and H simultaneously by $f\to f/\kappa_1$ and $H\to H/\kappa_1$ with a constant κ_1 to obtain the equation $\overline{f}^2H^2-\kappa_1^2(\overline{f}^2+H^2)-\eta^2\kappa_1^4=0$, where κ_1 is a constant. Then we do the Möbius transformations:

$$\overline{f}\to\alpha\frac{\overline{f}-\beta}{\overline{f}+\beta},\quad H\to\alpha\frac{H-\beta}{H+\beta}, \tag{3.43}$$

with suitable constants α and β, and obtain the canonical form of the symmetric QRT map:

$$\overline{f}^2H^2+A(\overline{f}^2+H^2)+2B\overline{f}H+1=0, \tag{3.44}$$

where A and B are both nonzero constants dependent on α, β and κ_1. The process of solving (3.30) shows that equation (3.44) is parameterized by elliptic functions and $\overline{f}=H(\overline{\varphi})$ for an entire function φ. In fact, if we define the parameters k and ε as in (3.32) and consider equation (3.44) as a quadratic equation for $\overline{f}$ with respect to H, then using the transformation $H=k^{1/2}\text{sn }\varphi$ and $\overline{f}=k^{1/2}\text{sn }\overline{\varphi}$, where sn φ denotes the Jacobi elliptic sn function with argument φ and modulus k, we have

$$\text{sn }\overline{\varphi}=\frac{\text{cn }\varepsilon\text{ dn }\varepsilon\text{ sn }\varphi\pm\text{sn }\varepsilon\text{ cn }\varphi\text{ dn }\varphi}{1-k^2\text{sn}^2\ \varepsilon\ \text{sn}^2\ \varphi}, \tag{3.45}$$

which is solved by $\varphi=\varepsilon\phi+C$ such that $\varphi=\varphi(z)$ is an entire function satisfying $\varphi(z+1)=\varphi(z)+\varepsilon$, where C is a free parameter. It follows that $\phi=\phi(z)$ is an entire function satisfying $\phi(z+1)=\phi(z)+1$. Thus $\phi(z)=\pi(z)+z$, where $\pi(z)$ is an arbitrary non-constant periodic function of period 1. We may suppose that $\pi(z)$ has a zero, say $\pi(z_0)=0$. Then $z_m=z_0+m$ is a zero of $\pi(z)$ for all integers $m\geq 0$. It follows that the infinite sequence $\{z_m\}$ satisfies $z_m\to\infty$ as $m\to\infty$ and $\phi_m=\phi(z_m)=z_m$ for all m. Therefore, if we set

$$t=\varepsilon_m z_m,\quad H=k^{1/2}w(t,\varepsilon_m),\quad \overline{f}=k^{1/2}w(t+\varepsilon_m,\varepsilon_m), \tag{3.46}$$

then we have from (3.44) that

$$\begin{aligned}&\frac{[w(t+\varepsilon_m,\varepsilon_m)-w(t,\varepsilon_m)]^2}{\text{sn}^2\ \varepsilon_m}\\&=k^2w(t+\varepsilon_m,\varepsilon_m)^2w(t,\varepsilon_m)^2+\left(\frac{2\text{cn}\,\varepsilon_m\,\text{dn}\,\varepsilon_m-2}{\text{sn}^2\,\varepsilon_m}\right)w(t+\varepsilon_m,\varepsilon_m)w(t,\varepsilon_m)+1.\end{aligned} \tag{3.47}$$

For a fixed t, we choose $\varepsilon_m=\frac{t}{z_m}$. By using the Maclaurin series for $\text{sn}\,\varepsilon_m$, $\text{cn}\,\varepsilon_m$ and $\text{dn}\,\varepsilon_m$, respectively, in (3.38), and then letting $\varepsilon_m\to 0$, we obtain exactly the differential equation in (3.39). For each of the four equations (1.17), (1.18) and (1.19) and (1.21), by using the same method as above we may obtain an equation of the form in (3.42) with a certain meromorphic function H and then also obtain the differential equation in (3.39) after taking a continuum limit. We omit those details.

4. Concluding remarks

The Malmquist type difference equations (1.8) with $\deg_f(R(z,f)) = n$ are revisited in this paper. In Section 2, we first complete the classification for equation (1.8) with $\deg_f(R(z,f)) = n$ by identifying one new equation (1.21) left out in our previous work. We have actually derived the eleven equations (1.11)–(1.21) using some recent observations on equation (1.8) in [30]. In Section 3, we study the relations between the Malmquist type differential and difference equations in the case $n = 2$. The seven equations (1.9)–(1.15) singled out from (1.8) have finite order meromorphic solutions and appear to be integrable from the viewpoint of the proposed difference analogue of the Painlevé property suggested by Ablowitz, Halburd and Herbst [1]. We point out that each of the equations (1.9)–(1.14) has a natural continuum limit to equations (1.1), (1.3) or (1.4). The process of taking a continuum limit from equation (1.14) to equation (1.4) also applies to some more equations singled out from (1.8) in the case $n = 2$, namely the five equations (1.16)–(1.19) and (1.21). These equations only have infinite order transcendental meromorphic solutions. However, they can also be mapped to the symmetric QRT map with respect to $\overline{f}$ and a meromorphic function H dependent on f, so that $\overline{f}$ and H are written in the form $\overline{f} = H(\varphi + \varepsilon)$ and $H = H(\varphi)$ with an argument φ which is an entire function of z. In fact, by looking at the proof of the main theorems in [30] and the discussions in the last section of [30], we see that most equations singled out from (1.8) with $n = 2$ in the autonomous case in [30] can be written in the form

$$\overline{f}^2 + R_1^2 = 1,$$

or the form

$$\overline{f}^2 R_2^2 - (\overline{f}^2 + R_2) + \kappa^2 = 0,$$

where R_1 and R_2 are rational functions in f or in a certain meromorphic function g such that $f = g^2 - 1$ or $f = a\frac{g^2-b}{g^2-c}$ for some constants a, b, c such that $abc \neq 0$, and thus are included in the QRT family defined in (3.17)–(3.22).

Recall from [22] that the QRT family defined in (3.17)–(3.22) possesses an invariant which is biquadratic in x_n and y_n:

$$\begin{aligned}&(\alpha_0 + K\alpha_1)x_n^2y_n^2 + (\beta_0 + K\beta_1)x_n^2y_n + (\gamma_0 + K\gamma_1)x_n^2 + (\delta_0 + K\delta_1)x_ny_n^2\\ &+ (\varepsilon_0 + K\varepsilon_1)x_ny_n + (\zeta_0 + K\zeta_1)x_n + (\kappa_0 + K\kappa_1)y_n^2\\ &+ (\lambda_0 + K\lambda_1)y_n + (\mu_0 + K\mu_1) = 0,\end{aligned} \tag{4.1}$$

where K plays the role of the integration constant. In the symmetric case, the invariant in (4.1) becomes just (3.23). In the generic case, it is shown in [22] that, by doing two Möbius transformations $x_n \to \frac{\alpha_1 x_n + \alpha_2}{\alpha_3 x_n + \alpha_4}$ and $y_n \to \frac{\beta_1 y_n + \beta_2}{\beta_3 y_n + \beta_4}$ with suitable constants α_i and β_j, respectively, (4.1) can also be mapped into the symmetric form:

$$x_n^2y_n^2 + A(x_n^2 + y_n^2) + 2Bx_ny_n + 1 = 0, \tag{4.2}$$

where A and B are constants. The process of solving (3.44) in Section 3 shows that x_n and y_n in (4.2) are parameterized by elliptic functions and $x_n = y_n(\overline{\varphi})$ with some entire function φ. Combining this fact and the process of taking continuum limit from equations (3.30) and (3.44) to the differential equation (1.4) in Section 3, we conclude that the QRT family defined in (3.17) and (3.18) always has a continuum limit to the first order differential equation (1.4) in the generic case.

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