Annales Fennici Mathematici
Volumen 48, 2023, 515–535

Boundedness properties of maximal operators on Lorentz spaces

Dariusz Kosz

Abstract. We study mapping properties of the centered Hardy–Littlewood maximal operator $\mathcal{M}$ acting on Lorentz spaces. Given $p \in (1,\infty)$ and a metric measure space $\mathfrak{X} = (X,\rho,\mu)$ we let $\Omega^p_{\rm HL}(\mathfrak{X}) \subset [0,1]^2$ be the set of all pairs $(\frac{1}{q},\frac{1}{r})$ such that $\mathcal{M}$ is bounded from $L^{p,q}(\mathfrak{X})$ to $L^{p,r}(\mathfrak{X})$. Under mild assumptions on μ, for each fixed p all possible shapes of $\Omega^p_{\rm HL}(\mathfrak{X})$ are characterized. Namely, we show that the boundary of $\Omega^p_{\rm HL}(\mathfrak{X})$ either is empty or takes the form

$$\{\delta\} \times [0, \lim_{u\to\delta} F(u)] \cup \{(u,F(u)) \colon u \in (\delta,1]\},$$

where $\delta \in [0,1]$ and $F\colon [\delta,1] \to [0,1]$ is concave, nondecreasing, and satisfies $F(u) \le u$. Conversely, for each such F we find $\mathfrak{X}$ such that $\mathcal{M}$ is bounded from $L^{p,q}(\mathfrak{X})$ to $L^{p,r}(\mathfrak{X})$ if and only if the point $(\frac{1}{q},\frac{1}{r})$ lies on or under the graph of F, that is, $\frac{1}{q} \ge \delta$ and $\frac{1}{r} \le F\big(\frac{1}{q}\big)$.

Maksimaalioperaattoreiden rajallisuus Lorentzin avaruuksissa

Tiivistelmä. Tutkimme keskitetyn Hardyn–Littlewoodin maksimaalioperaattorin $\mathcal{M}$ kuvausominaisuuksia Lorentzin avaruuksien välillä. Jos $p \in (1,\infty)$ ja $\mathfrak{X} = (X,\rho,\mu)$ on metrinen mitta-avaruus, olkoon $\Omega^p_{\rm HL}(\mathfrak{X}) \subset [0,1]^2$ niiden parien $(\frac{1}{q},\frac{1}{r})$ joukko, joilla $\mathcal{M}$ kuvaa avaruuden $L^{p,q}(\mathfrak{X})$ rajoitetusti avaruuteen $L^{p,r}(\mathfrak{X})$. Lievillä oletuksilla mitasta μ selvitämme joukon $\Omega^p_{\rm HL}(\mathfrak{X})$ kaikki mahdolliset muodot jokaisella kiinteällä eksponentilla p. Osoitamme, että joukon $\Omega^p_{\rm HL}(\mathfrak{X})$ reuna on joko tyhjä tai muotoa

$$\{\delta\} \times [0, \lim_{u\to\delta} F(u)] \cup \{(u,F(u)) \colon u \in (\delta,1]\},$$

missä $\delta \in [0,1]$ ja $F\colon [\delta,1] \to [0,1]$ on konkaavi, ei-vähenevä ja toteuttaa ehdon $F(u) \le u$. Vastaavasti jokaista tällaista funktiota F kohti löydämme sellaisen avaruuden $\mathfrak{X}$, että $\mathcal{M}$ kuvaa avaruuden $L^{p,q}(\mathfrak{X})$ rajoitetusti avaruuteen $L^{p,r}(\mathfrak{X})$, jos ja vain jos piste $(\frac{1}{q},\frac{1}{r})$ on funktion F kuvaajan alapuolella ts. $\frac{1}{q} \ge \delta$ ja $\frac{1}{r} \le F\big(\frac{1}{q}\big)$.

1. Introduction

Consider an arbitrary metric measure space $\mathfrak{X}$, that is, a triple (X,ρ,μ), where X is a set, ρ is a metric, and μ is a Borel measure. For any $x \in X$ and $s \in (0,\infty)$ let

$$B(x,s) \coloneqq B_\rho(x,s) \coloneqq \{y \in X \colon \rho(x,y) < s\}$$

denote the open ball centered at x and of radius s. Then the associated *centered Hardy–Littlewood maximal operator* $\mathcal{M}_\mathfrak{X}$ is defined by

$$\mathcal{M}_\mathfrak{X} f(x) \coloneqq \sup_{s\in(0,\infty)} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f| \,\mathrm{d}\mu, \quad x \in X,$$

where $f\colon X \to \mathbb{C}$ is any Borel function. For balls B satisfying $\mu(B) \in \{0,\infty\}$ we use the convection $\frac{1}{\mu(B)}\int_B |f|\,\mathrm{d}\mu = 0$.

https://doi.org/10.54330/afm.131758
2020 Mathematics Subject Classification: Primary 42B25, 46E30.
Key words: Centered Hardy–Littlewood maximal operator, Lorentz space, nondoubling metric measure space.

The aim of this article is to study mapping properties of $\mathcal{M}_{\mathfrak{X}}$ for various $\mathfrak{X}$. The only constraint on $\mathfrak{X}$ that appears a few times later on, including our main result Theorem 1, is that we additionally assume $\mu(X \setminus \operatorname{supp}(\mu)) = 0$, where

$$\operatorname{supp}(\mu) := \big\{x \in X \colon \mu(B(x,s)) > 0 \text{ for all } s \in (0,\infty)\big\}$$

is the *support* of μ. Loosely speaking, this condition allows us to avoid pathological cases in which it is not *a priori* obvious that $\mathcal{M}_{\mathfrak{X}} f$ is essentially larger than f. We note that the equality $\mu(X \setminus \operatorname{supp}(\mu)) = 0$ holds trivially if $\mathfrak{X}$ is separable. It is true also when $\mathfrak{X}$ is locally compact and μ is a Radon measure (cf. [4, p. 218]).

Let us recall that an operator $\mathcal{H}$ is said to be *of strong type* (p,p) (resp. *of weak type* (p,p)) for some $p \in [1,\infty]$ if $\mathcal{H}$ is bounded on $L^p(\mathfrak{X})$ (resp. from $L^p(\mathfrak{X})$ to $L^{p,\infty}(\mathfrak{X})$). Thus, for example, $\mathcal{M}_{\mathfrak{X}}$ is of strong type (∞,∞) no matter what the exact structure of $\mathfrak{X}$ is. Moreover, if $\mathfrak{X}$ is *doubling* (that is, $\mu(B(x,2s)) \le C\mu(B(x,s))$ holds with some $C \in (0,\infty)$ independent of x and s), then $\mathcal{M}_{\mathfrak{X}}$ is also of weak type $(1,1)$ and hence, by interpolation, of strong type (p,p) for each $p \in (1,\infty)$. For arbitrary (nondoubling) $\mathfrak{X}$ it may happen that the weak type $(1,1)$ inequality for $\mathcal{M}_{\mathfrak{X}}$ fails to occur. For example, Sjögren [16] showed that this is the case for the two-dimensional Gaussian measure $\mathrm{d}\mu(x,y) = e^{-(x^2+y^2)/2}\,\mathrm{d}x\,\mathrm{d}y$ and the uncentered Hardy–Littlewood maximal operator (by "uncentered" we mean that the supremum in the definition is taken over the family of balls containing x, not only those centered at x).

There are several articles devoted to studying various mapping properties of the Hardy–Littlewood maximal operators (or their modifications) for nondoubling spaces (see, e.g., [1, 14, 15, 17]). It is particularly interesting to find spaces for which such properties are very specific. Li wrote a series of papers [10, 11, 12] in which the so-called cusp spaces have been introduced for this purpose. For example, in [11] it is shown that for each fixed $p_0 \in (1,\infty)$ there exists $\mathfrak{X}$ such that, given $p \in [1,\infty]$, the associated centered maximal operator is of strong type (p,p) if and only if $p > p_0$.

Recently, the author also contributed to the development of this field [7, 8, 9]. In particular, in [9] certain mapping properties of $\mathcal{M}_{\mathfrak{X}}$ acting on Lorentz spaces $L^{p,q}(\mathfrak{X})$ have been studied. More precisely, it is proven there that for each $p_0, q_0, r_0 \in (1,\infty)$ with $r_0 \ge q_0$ it is possible to construct

(a) a (nondoubling) space $\mathfrak{X}$ such that $\mathcal{M}_{\mathfrak{X}}$ is bounded from $L^{p_0,q_0}(\mathfrak{X})$ to $L^{p_0,r_0}(\mathfrak{X})$, but is not bounded from $L^{p_0,q_0}(\mathfrak{X})$ to $L^{p_0,r}(\mathfrak{X})$ for all $r \in [1,r_0)$,

(b) a (nondoubling) space $\mathfrak{X}$ such that $\mathcal{M}_{\mathfrak{X}}$ is bounded from $L^{p_0,1}(\mathfrak{X})$ to $L^{p_0,r_0}(\mathfrak{X})$, but is not bounded from $L^{p_0,q_0}(\mathfrak{X})$ to $L^{p_0,r_0}(\mathfrak{X})$.

In both cases above the boundedness of $\mathcal{M}_{\mathfrak{X}}$ acting from $L^{p,q}(\mathfrak{X})$ to $L^{p,r}(\mathfrak{X})$ is studied and only one of the parameters q, r is varying, while both, the remaining parameter and p, are fixed. In Theorem 1 below we strengthen the results of [9] by providing a detailed analysis of a more complex problem in which p is the only fixed parameter and all pairs (q,r) are studied simultaneously. Namely, we characterize all possible shapes of the sets

$$\Omega^p_{\mathrm{HL}}(\mathfrak{X}) := \big\{(\tfrac{1}{q},\tfrac{1}{r}) \in [0,1]^2 \colon \mathcal{M}_{\mathfrak{X}} \text{ is bounded from } L^{p,q}(\mathfrak{X}) \text{ to } L^{p,r}(\mathfrak{X})\big\} \subset [0,1]^2$$

(the shapes of these sets are described in terms of their topological boundaries, where the underlying space is the square $[0,1]^2$ with its natural topology).

Theorem 1. *Fix $p \in (1,\infty)$. Then for each $\mathfrak{X}$ satisfying $\mu(X \setminus \operatorname{supp}(\mu)) = 0$ one of the following two possibilities holds:*

- *the boundary of $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ is empty, that is, $\Omega^p_{\mathrm{HL}}(\mathfrak{X}) = \emptyset$ or $\Omega^p_{\mathrm{HL}}(\mathfrak{X}) = [0,1]^2$,*

- *the boundary of* $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ *is of the form*
$$\{\delta\} \times [0, \lim_{u\to\delta} F(u)] \cup \{(u, F(u)) \colon u \in (\delta, 1]\},$$
where $\delta \in [0,1]$ *and* $F\colon [\delta,1] \to [0,1]$ *is concave, nondecreasing, and satisfies* $F(u) \le u$.

Conversely, for each F *as above there exists* $\mathfrak{X}$ *such that* $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ *is the set of points lying on or under the graph of* F, *that is, points* $(\frac{1}{q}, \frac{1}{r}) \in [0,1]^2$ *with* $\frac{1}{q} \ge \delta$ *and* $\frac{1}{r} \le F\big(\frac{1}{q}\big)$.

Although Theorem 1 is stated for the centered Hardy–Littlewood maximal operator, it is possible to obtain its analogous version with the uncentered operator used instead.

To prove Theorem 1 we should focus on two separate tasks. First we want to indicate some conditions that the sets $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ must satisfy in general, in order to ensure that no situation other than those listed in Theorem 1 is possible. This problem is treated in Section 6 (see Remark 1, Remark 2, and Theorem 3). The second goal, which turns out to be the more challenging one, is to introduce a special class of spaces for which we can precisely control the behavior of the maximal operator and, at the same time, this behavior is very peculiar. This problem is covered by Theorem 2 stated below. We note that, in fact, Theorem 2 is slightly more general and it consists of four similar results which have been collected together for the sake of completeness. In what follows, for each $p \in (1,\infty)$ and $q, r \in [1,\infty]$ by $\mathbf{c}(p,q,r,\mathfrak{X})$ we mean the smallest constant C for which
$$\|\mathcal{M}_{\mathfrak{X}} f\|_{p,r} \le C \|f\|_{p,q}, \quad f \in L^{p,q}(\mathfrak{X}),$$
holds (we put $\mathbf{c}(p,q,r,\mathfrak{X}) = \infty$ if no such constant exists).

Theorem 2. *Fix* $p \in (1,\infty)$ *and* $\delta \in [0,1]$ *(resp.* $\delta \in [0,1)$*). Let* $F\colon [\delta,1] \to [0,1]$ *(resp.* $F\colon (\delta,1] \to [0,1]$*) be concave, nondecreasing, and satisfy* $F(u) \le u$ *for each* $u \in [\delta,1]$ *(resp.* $u \in (\delta,1]$*). Then*

- *there exists a (nondoubling) metric measure space* $\mathfrak{Y}$ *such that* $\mathbf{c}(p,q,r,\mathfrak{Y}) < \infty$ *if and only if* $\frac{1}{q} \ge \delta$ *(resp.* $\frac{1}{q} > \delta$*) and* $\frac{1}{r} \le F\big(\frac{1}{q}\big)$,
- *there exists a (nondoubling) metric measure space* $\mathfrak{Z}$ *such that* $\mathbf{c}(p,q,r,\mathfrak{Z}) < \infty$ *if and only if* $\frac{1}{q} \ge \delta$ *(resp.* $\frac{1}{q} > \delta$*) and* $\frac{1}{r} < F\big(\frac{1}{q}\big)$.

Two comments regarding Theorem 2 are in order. Firstly, although the word "exists" is used above, the spaces $\mathfrak{Y}$ and $\mathfrak{Z}$ are constructed explicitly. In fact, our construction process originates in a beautiful idea of Stempak [18], who provided interesting examples of spaces while dealing with some related problem regarding modified maximal operators. Secondly, Theorem 2 does not cover the extreme cases $\Omega^p_{\mathrm{HL}}(\mathfrak{X}) = \emptyset$ and $\Omega^p_{\mathrm{HL}}(\mathfrak{X}) = [0,1]^2$. For the latter one, it suffices to take a single point with a trivial metric and counting measure. For the former one, see the comment after Lemma 6.

The rest of the paper is organized as follows. In Section 2 we recall basic properties of Lorentz spaces and the so-called *space combining technique* introduced in [9]. Next, in Section 3 and Section 4 we study the behavior of the maximal operator for the so-called *test spaces* and *composite test spaces*. The latter class will be used in Section 5 to prove Theorem 2. Section 6 is devoted to indicating properties of $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$, which allow us to deduce the first part of Theorem 1. In particular, we formulate a suitable interpolation theorem for Lorentz spaces $L^{p,q}(\mathfrak{X})$ with the first parameter fixed and the second varying (see Theorem 3). This result follows from a more general fact [3, Theorem 5.3.1] using advanced interpolation methods. However, in Appendix we give its elementary proof, which, to the author's best knowledge, has never appeared in the literature so far.

To avoid misunderstandings, we note that several times in the paper we identify $1/\infty$ and $1/0$ with 0 and ∞, respectively, when dealing with $q, r \in [1,\infty]$ and $u, F(u) \in [0,1]$. Also, for $\delta = 1$ the conventions $[\delta, 1] = \{1\}$, $(\delta, 1] = \emptyset$, and $\lim_{u\to\delta} F(u) = F(1)$ are used. For each $n \in \mathbb{N} := \{1, 2, \dots\}$ the symbol $[n]$ stands for the set $\{1, \dots, n\}$. Also, by $\mathbb{1}_E$ we mean the indicator function of a given Borel set E. Finally, we emphasize that, in view of the equality $\mathcal{M}_{\mathfrak{X}} f = \mathcal{M}_{\mathfrak{X}} |f|$, we can and will focus only on functions $f \geq 0$.

Acknowledgments. This article constitutes a part of PhD Thesis written in the Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, under the supervision of Professor Krzysztof Stempak. I would like to express my deep gratitude to Professor Stempak for his valuable remarks and continuous help during the preparation of the paper.

I am also indebted to Professor Lech Maligranda who explained to me how Theorem 3 in Section 6 can be derived from the general theory of interpolation.

I am thankful to the anonymous referees for their insightful comments that allowed me to improve the presentation of the results.

I was supported by the National Science Centre of Poland (2016/21/N/ST1/01496) and, while preparing the final version of the article, by the Basque Government (BERC 2022-2025), by the Spanish State Research Agency (CEX2021-001142-S and RYC2021-031981-I), and by the Foundation for Polish Science (START 032.2022).

2. Preliminaries

We begin with some basic facts about Lorentz spaces $L^{p,q}(\mathfrak{X})$ (for more detailed information see, e.g., [2]). For any Borel function $f\colon X \to \mathbb{C}$ we define the *distribution function* $d_f\colon [0,\infty) \to [0,\infty]$ by

$$d_f(t) := \mu(\{x \in X : |f(x)| \geq t\}).$$

Then for any $p \in [1,\infty)$ and $q \in [1,\infty]$ the space $L^{p,q}(\mathfrak{X})$ consists of those functions f for which the following quasinorm

$$\|f\|_{p,q} := \begin{cases} p^{1/q} \Big(\int_0^\infty \big(t\, d_f(t)^{1/p}\big)^q \frac{\mathrm{d}t}{t} \Big)^{1/q} & \text{if } q \in [1,\infty), \\ \sup_{t\in(0,\infty)} t\, d_f(t)^{1/p} & \text{if } q = \infty, \end{cases}$$

is finite. Recall that if $p = q$, then $(L^{p,q}(\mathfrak{X}), \|\cdot\|_{p,q})$ coincides with the standard Lebesgue space $(L^p(\mathfrak{X}), \|\cdot\|_p)$. Now we present several properties of $L^{p,q}(\mathfrak{X})$ spaces. The metric measure space is arbitrary here, except for the condition $\mu(X) < \infty$ assumed in Fact 2.

Fact 1. Let $p \in (1,\infty)$ and $q \in [1,\infty]$. Then there exists a numerical constant $\mathbf{C}_\triangle(p,q)$ independent of $\mathfrak{X}$ such that

$$\Big\| \sum_{n\in\mathbb{N}} f_n \Big\|_{p,q} \leq \mathbf{C}_\triangle(p,q) \sum_{n\in\mathbb{N}} \|f_n\|_{p,q}, \quad f_n \in L^{p,q}(\mathfrak{X}),\ n \in \mathbb{N}.$$

Fact 2. Let $p \in (1, \infty)$ and $q \in [1, \infty]$, and assume that $\mu(X) < \infty$. Then there exists a numerical constant $\mathbf{C}_{\text{avg}}(p, q)$ independent of $\mathfrak{X}$ such that

$$\|f_{\text{avg}}\|_{p,q} \le \mathbf{C}_{\text{avg}}(p, q)\|f\|_{p,q}, \quad f \in L^{p,q}(\mathfrak{X}),$$

where $f_{\text{avg}} := \|f\|_1/\mu(\mathfrak{X})$ is a constant function.

Fact 3. Let $p \in (1, \infty)$ and $1 \le q \le r \le \infty$. Then $L^{p,q}(\mathfrak{X}) \subset L^{p,r}(\mathfrak{X})$ and there exists a numerical constant $\mathbf{C}_{\hookrightarrow}(p, q, r)$ independent of $\mathfrak{X}$ such that

$$\|f\|_{p,r} \le \mathbf{C}_{\hookrightarrow}(p, q, r)\|f\|_{p,q}, \quad f \in L^{p,q}(\mathfrak{X}).$$

These facts are rather well known. For the proof of Fact 3 see, e.g., [2, Proposition 4.2]. Fact 1 and Fact 2 are in turn easy consequences of [2, Lemma 4.5 and Theorem 4.6].

We also need the following auxiliary lemma.

Lemma 1. *Fix $p \in (1, \infty)$ and $q \in [1, \infty]$, and consider a sequence $(f_n)_{n\in\mathbb{N}}$ of functions with disjoint supports A_n. Assume that for each $n \in \mathbb{N}$ and $t \in (0, \infty)$ we have either $d_{f_n}(t) \ge \mu(A_{n+1} \cup A_{n+2} \cup \cdots)$ or $d_{f_n}(t) = 0$. Then for some numerical constant $\mathbf{C}_1 = \mathbf{C}_1(p, q)$ independent of $\mathfrak{X}$ and $(f_n)_{n\in\mathbb{N}}$ we have: if $q \in [1, \infty)$,*

$$\mathbf{C}_1^{-1} \left(\sum_{n\in\mathbb{N}} \|f_n\|_{p,q}^q\right)^{1/q} \le \left\|\sum_{n\in\mathbb{N}} f_n\right\|_{p,q} \le \mathbf{C}_1 \left(\sum_{n\in\mathbb{N}} \|f_n\|_{p,q}^q\right)^{1/q},$$

or, if $q = \infty$,

$$\mathbf{C}_1^{-1} \sup_{n\in\mathbb{N}} \|f_n\|_{p,\infty} \le \left\|\sum_{n\in\mathbb{N}} f_n\right\|_{p,\infty} \le \mathbf{C}_1 \sup_{n\in\mathbb{N}} \|f_n\|_{p,\infty}.$$

Proof. Let $f = \sum_{n\in\mathbb{N}} f_n$ and consider $q \in [1, \infty)$ (the case $q = \infty$ is similar). Then, under the specified assumptions, the quantities $d_f(t)^{q/p}$ and $\sum_{n\in\mathbb{N}} d_{f_n}(t)^{q/p}$ are comparable to each other with multiplicative constants possibly depending on p and q but independent of $t \in (0, \infty)$. Integrating both quantities against the weight t^{q-1} completes the proof. □

The main tool used in the proof of Theorem 2 is the following space combining technique which was introduced in [9].

Proposition 1. (cf. [9, Proposition 1]) Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a given sequence of metric measure spaces $\mathfrak{X}_n = (X_n, \rho_n, \mu_n)$. Assume that each of them consists of finitely many (not zero) elements and every singleton is of positive (finite and not zero) measure. Let $\mathfrak{X} = (X, \rho, \mu)$ be the space constructed with the aid of $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ by using the method described below.

Step 1. Introduce ρ'_n and μ'_n by rescaling (if necessary) ρ_n and μ_n, respectively, so that the following conditions are satisfied:

- the diameter of X_n with respect to ρ'_n is strictly smaller than 1,
- $0 < 2\mu'_{n+1}(X_{n+1}) \le \mu'_n(\{x\})$ for every $x \in X_n$ and $n \in \mathbb{N}$.

Step 2. Denote $\mathfrak{X}'_n = (X_n, \rho'_n, \mu'_n)$ and notice that $\mathbf{c}(p, q, r, \mathfrak{X}_n) = \mathbf{c}(p, q, r, \mathfrak{X}'_n)$ for all $n \in \mathbb{N}$, $p \in (1, \infty)$, and $q, r \in [1, \infty]$.

Step 3. Set $X \coloneqq \bigcup_{n\in\mathbb{N}} X_n$, assuming that $X_{n_1} \cap X_{n_2} = \emptyset$ when $n_1 \neq n_2$. Finally, define the metric ρ on X by

$$\rho(x,y) \coloneqq \begin{cases} \rho'_n(x,y) & \text{if } \{x,y\} \subset X_n \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise,} \end{cases}$$

and the measure μ on X by

$$\mu(E) \coloneqq \sum_{n\in\mathbb{N}} \mu'_n(E \cap X_n), \quad E \subset X.$$

Then for each $p \in (1,\infty)$ and $1 \le q \le r \le \infty$ we have

$$\mathbf{C}^{-1} \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n) \le \mathbf{c}(p,q,r,\mathfrak{X}) \le \mathbf{C} \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n), \tag{1}$$

where $\mathbf{C} = \mathbf{C}(p,q,r)$ is a numerical constant independent of $(\mathfrak{X}_n)_{n\in\mathbb{N}}$.

Proof. We present the proof for the sake of completeness. To this end, it will be convenient to deal with the local and global versions of $\mathcal{M}_{\mathfrak{X}}$:

$$\mathcal{M}_{\mathrm{loc}} f(x) \coloneqq \sup_{s\in(0,1]} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f| \,\mathrm{d}\mu, \quad x \in X,$$

and

$$\mathcal{M}_{\mathrm{glob}} f(x) \coloneqq \sup_{s\in(1,\infty)} \frac{1}{\mu(B(x,s))} \int_{B(x,s)} |f| \,\mathrm{d}\mu, \quad x \in X.$$

Let us first show

$$\sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n) \le \mathbf{c}(p,q,r,\mathfrak{X}),$$

assuming that $\mathbf{c}(p,q,r,\mathfrak{X}) < \infty$ holds. For fixed $n \in \mathbb{N}$ take $f \in L^{p,q}(\mathfrak{X}'_n)$ and extend it to $F \in L^{p,q}(\mathfrak{X})$ by setting $F(x) \coloneqq 0$ for $x \in X \setminus X_n$. Then $\|F\|_{p,q} = \|f\|_{p,q}$ (here the symbol $\|\cdot\|_{p,q}$ refers to function spaces over different measure spaces). Moreover, by the definition of ρ, we have $\mathcal{M}_{\mathfrak{X}} F(x) \ge \mathcal{M}_{\mathrm{loc}} F(x) = \mathcal{M}_{\mathfrak{X}'_n} f(x)$ for any $x \in X_n$. Consequently, if $\|\mathcal{M}_{\mathfrak{X}} F\|_{p,r} \le \mathbf{c}(p,q,r,\mathfrak{X}) \|F\|_{p,q}$, then also $\|\mathcal{M}_{\mathfrak{X}'_n} f\|_{p,r} \le \mathbf{c}(p,q,r,\mathfrak{X}) \|f\|_{p,q}$. This justifies the first inequality in (1).

Conversely, let us show

$$\mathbf{c}(p,q,r,\mathfrak{X}) \le \mathbf{C} \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n).$$

Assume that $r < \infty$ (the case $r = \infty$ is similar) and let $F \in L^{p,q}(\mathfrak{X})$. By Fact 1 we have

$$\|\mathcal{M}_{\mathfrak{X}} F\|_{p,r} \le \mathbf{C}_{\triangle}(p,r) \big(\|\mathcal{M}_{\mathrm{loc}} F\|_{p,r} + \|\mathcal{M}_{\mathrm{glob}} F\|_{p,r} \big).$$

Define $f_n \in L^{p,q}(\mathfrak{X}'_n)$, $n \in \mathbb{N}$, by restricting F to X_n. Using Lemma 1, together with the definitions of ρ and μ, we see that

$$\begin{aligned} \|\mathcal{M}_{\mathrm{loc}} F\|_{p,r} \le \mathbf{C}_1(p,r) \Big(\sum_{n\in\mathbb{N}} \|\mathcal{M}_{\mathrm{loc}} F \cdot \mathbb{1}_{X_n}\|_{p,r}^r \Big)^{1/r} &= \mathbf{C}_1(p,r) \Big(\sum_{n\in\mathbb{N}} \|\mathcal{M}_{\mathfrak{X}'_n} f_n\|_{p,r}^r \Big)^{1/r} \\ &\le \mathbf{C}_1(p,r) \Big(\sum_{n\in\mathbb{N}} \big(\mathbf{c}(p,q,r,\mathfrak{X}_n) \|f_n\|_{p,q} \big)^r \Big)^{1/r} \\ &\le \mathbf{C}_1(p,r) \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n) \Big(\sum_{n\in\mathbb{N}} \|f_n\|_{p,q}^r \Big)^{1/r}. \end{aligned}$$

Using Lemma 1 again, we obtain

$$\Big(\sum_{n\in\mathbb{N}} \|f_n\|_{p,q}^r\Big)^{1/r} \le \Big(\sum_{n\in\mathbb{N}} \|f_n\|_{p,q}^q\Big)^{1/q} = \Big(\sum_{n\in\mathbb{N}} \|F\cdot \mathbb{1}_{X_n}\|_{p,q}^q\Big)^{1/q} \le \mathbf{C}_1(p,q)\|F\|_{p,q}.$$

Now we estimate $\|\mathcal{M}_{\mathrm{glob}}F\|_{p,r}$. Note that $\mathcal{M}_{\mathrm{glob}}F \equiv \|F\|_1/\mu(X)$ is constant by the definition of ρ. Thus, Fact 2 and Fact 3 imply

$$\|\mathcal{M}_{\mathrm{glob}}F\|_{p,r} \le \mathbf{C}_{\mathrm{avg}}(p,r)\|F\|_{p,r} \le \mathbf{C}_{\mathrm{avg}}(p,q)\mathbf{C}_{\hookrightarrow}(p,q,r)\|F\|_{p,q}.$$

Consequently,

$$\mathbf{c}(p,q,r,\mathfrak{X}) \le \mathbf{C}_{\triangle}(p,r)\big(\mathbf{C}_1(p,r)\mathbf{C}_1(p,q)\sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n) + \mathbf{C}_{\mathrm{avg}}(p,q)\mathbf{C}_{\hookrightarrow}(p,q,r)\big)$$

and it remains to notice that $\sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n)$ cannot be arbitrarily small. To this end, we take $g \coloneqq \mathbb{1}_{X_1} \in L^{p,q}(\mathfrak{X}_1')$ and observe that

$$\|\mathcal{M}_{\mathfrak{X}_1'}g\|_{p,r} = \|g\|_{p,r} = \frac{(p/r)^{1/r}}{(p/q)^{1/q}}\|g\|_{p,q}$$

(here for $r=\infty$ we use the convention $\infty^{1/\infty}=1$). Hence,

$$\sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{X}_n) \ge \mathbf{c}(p,q,r,\mathfrak{X}_1) \ge \frac{(p/r)^{1/r}}{(p/q)^{1/q}}$$

and the proof is complete. □

Two comments are in order. Firstly, whenever we want to apply Proposition 1 later on, we omit the details related to the proper indexing of the component spaces. The only important fact is that each time we use countably many spaces. Secondly, we indicate that each space $\mathfrak{X}$ obtained by using Proposition 1 is nondoubling. Indeed, fix $\epsilon \in (0,\infty)$ and let $n_0 = n_0(\epsilon) \in \mathbb{N}$ be such that $\mu(X_{n_0}) < \epsilon\mu(X_1)$. Then for any $x \in X_{n_0}$ we have $B(x,1) = X_{n_0}$ which implies $\mu(B(x,1)) < \epsilon\mu(X_1)$, while $\mu(B(x,2)) = \mu(X) \ge \mu(X_1)$.

3. Test spaces

We now introduce and analyze certain auxiliary structures which we call *test spaces* later on. We emphasize here that each test space may be used as a component space in Proposition 1, because it consists of finitely many elements.

Informally, our goal is to find a space such that the maximal operator splits into two parts, one of them having trivial mapping properties and the other one looking like

$$\ell^q([N]) \ni (x_1,\dots,x_N) \mapsto (C(x_1+\dots+x_N),\dots,C(x_1+\dots+x_N)) \in \ell^r([M]),$$

which has norm $CN^{1-1/q}M^{1/r}$ (here $\ell^q([N])$ or $\ell^r([M])$ is the usual Lebesque space introduced with respect to counting measure on $[N]$ or $[M]$). Analyzing the last expression and taking suitable triples (C_n,N_n,M_n) in Proposition 1, we recover Theorem 2.

Below we work with fixed parameters $p\in(1,\infty)$ and $N,M,L\in\mathbb{N}$, and associate with each quadruple (p,N,M,L) four sequences $(m_i)_{i\in[N]}, (h_i)_{i\in[N]}, (\alpha_k)_{k\in[M]}, (\beta_k)_{k\in[M]}$ of positive integers, satisfying the following properties:

(i) $h_N/h_i \in \mathbb{N}$,
(ii) $m_{i+1} \ge 2m_ih_i$,
(iii) $1 \le m_i^{1-p}h_i < 2$,

(iv) $\alpha_1 \geq 2m_N h_N$,
(v) $\alpha_{k+1} \geq 2\alpha_k L \beta_k h_N$,
(vi) $1 \leq \alpha_k^{1-p} \beta_k h_N < 2$.

Let us check that the properties (i)–(vi) can be met simultaneously. Set $m_1 = h_1 = 1$. Then we specify m_{i+1} and h_{i+1} for some $i \in [N-1]$, assuming that $m_1, \dots, m_i, h_1, \dots, h_i$ have already been chosen. We take $m_{i+1} \geq 2m_i h_i$ such that $\{h \in \mathbb{N} \colon 1 \leq m_{i+1}^{1-p} h < 2\}$ contains at least h_i elements. Then we choose h_{i+1} satisfying both $h_{i+1}/h_i \in \mathbb{N}$ and $1 \leq m_{i+1}^{1-p} h_{i+1} < 2$. Thus, the properties (i)–(iii) are satisfied. Next we take α_1 such that $\alpha_1 \geq 2m_N h_N$ and $\alpha_1^{1-p} h_N < 2$ hold, and choose β_1 satisfying $1 \leq \alpha_1^{1-p} \beta_1 h_N < 2$. Then we specify α_{k+1} and β_{k+1} for some $k \in [M-1]$, assuming that $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ have already been chosen. We take $\alpha_{k+1} \geq 2\alpha_k L \beta_k h_N$. Since $\alpha_{k+1}^{1-p} h_N \leq \alpha_1^{1-p} h_N < 2$, we can choose β_{k+1} satisfying $1 \leq \alpha_{k+1}^{1-p} \beta_{k+1} h_N < 2$. Thus, the properties (iv)–(vi) are satisfied.

The four sequences will determine the structure of the test space constructed below. Here we formulate a few thoughts that one should keep in mind later on:

- our space consists of two levels (lower and upper) with points of N and M types, respectively (see Figure 1 below in this section),
- the sequences $(m_i)_{i \in [N]}$ and $(\alpha_k)_{k \in [M]}$ are used to define the associated measure, while $(h_k)_{i \in [N]}$ and $(\beta_k)_{k \in [M]}$ determine the number of elements of each type,
- the property (i) makes the set of points of a given type divisible into a suitable number of equinumerous subsets,
- the properties (ii) and (v) say that the sequences $(m_i)_{i \in [N]}$ and $(\alpha_k)_{k \in [M]}$ grow very fast; huge differences between the masses of points of different types allow one to use Lemma 1 frequently,
- the properties (iv) and (v) say that the values α_i are large compared with m_k and h_k; the points from the upper level have much greater masses than the ones from the lower level and Lemma 1 can be applied also in this context,
- the properties (iii) and (vi) are of technical nature; they keep the right balance between the number of points of a given type and the mass of each one of them,
- the property (v) is the only one where the parameter L is used.

We fix $K \in [1, \infty)$ and define a test space $\mathfrak{S} = \mathfrak{S}_{p,N,M,K,L} = (S, \rho, \mu)$ as follows. Set

$$S := \{x_{i,j}, x^\circ_{k,l} \colon i \in [N], j \in [h_i], k \in [M], l \in [L\beta_k h_N]\},$$

where all elements are different. We use auxiliary symbols for some subsets of S:

$$S^\circ := \{x^\circ_{k,l} \colon k \in [M], l \in [L\beta_k h_N]\};$$

for $i \in [N]$ and $k \in [M]$,

$$S_i := \{x_{i,j} \colon j \in [h_i]\}, \ S^\circ_k := \{x^\circ_{k,l} \colon l \in [L\beta_k h_N]\};$$

for $i \in [N]$, $j \in [h_i]$, and $k \in [M]$,

$$S^\circ_{i,j,k} := \Big\{x^\circ_{k,l} \colon l \in \Big[\frac{j}{h_i} L\beta_k h_N\Big] \setminus \Big[\frac{j-1}{h_i} L\beta_k h_N\Big]\Big\}.$$

Observe that the sets $S^\circ_{i,j,k}$, $j \in [h_i]$, are disjoint and, in view of (i), each of them contains exactly $L\beta_k h_N / h_i$ elements. Moreover, $\bigcup_{j \in [h_i]} S^\circ_{i,j,k} = S^\circ_k$ holds for each $i \in [N]$.

We introduce μ by letting $\mu(\{x_{i,j}\}) := m_i$ and $\mu(\{x^\circ_{k,l}\}) := K\alpha_k$. Then the following inequalities hold: for each $x \in S^\circ$ by (iv),

$$\mu(\{x\}) > \mu(S \setminus S^\circ), \tag{2}$$

for each $i \in [N] \setminus \{1\}$ and $x \in S_i$ by (ii),

$$\mu(\{x\}) > \mu(S_1 \cup \cdots \cup S_{i-1}), \tag{3}$$

and for each $k \in [M] \setminus \{1\}$ and $x^\circ \in S^\circ_k$ by (v),

$$\mu(\{x^\circ\}) > \mu(S^\circ_1 \cup \cdots \cup S^\circ_{k-1}). \tag{4}$$

Finally, we define the distance function ρ by the formula

$$\rho(x,y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } \{x,y\} = \{x_{i,j}, x^\circ_{k,l}\} \text{ and } x^\circ_{k,l} \in S^\circ_{i,j,k}, \\ 2 & \text{otherwise.} \end{cases}$$

It is worth noting here that for each $i \in [N]$, $k \in [M]$, and $x^\circ \in S^\circ_k$, there is exactly one point $x \in S_i$ such that $\rho(x, x^\circ) = 1$. For this point we will use the symbol $\Gamma_i(x^\circ)$.

Figure 1 shows a model of the space (S, ρ) for $N = 3$ and $M = 2$. The solid line between two points indicates that the distance between them equals 1. Otherwise the distance equals 2.

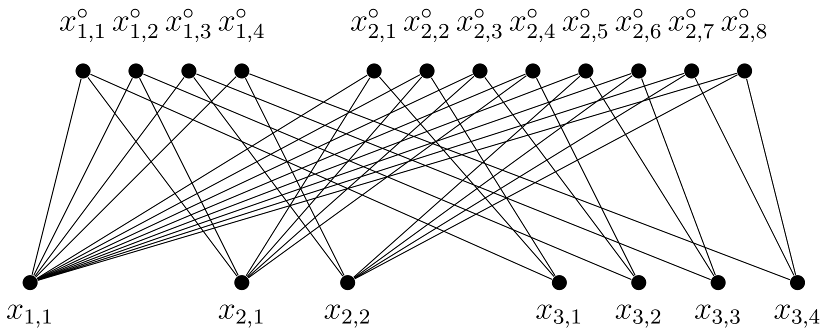

Figure 1. The model of the space (S, ρ) for $N = 3$ and $M = 2$.

For the reader's convenience we explicitly describe any ball $B \subset S$. Thus we have: for $i \in [N]$, $j \in [h_i]$,

$$B(x_{i,j}, s) = \begin{cases} \{x_{i,j}\} & \text{for } 0 < s \le 1, \\ \{x_{i,j}\} \cup \{x^\circ \in S^\circ \colon \Gamma_i(x^\circ) = x_{i,j}\} & \text{for } 1 < s \le 2, \\ S & \text{for } 2 < s, \end{cases}$$

and, for $k \in [M]$, $l \in [L\beta_k h_N]$,

$$B(x^\circ_{k,l}, s) = \begin{cases} \{x^\circ_{k,l}\} & \text{for } 0 < s \le 1, \\ \{x^\circ_{k,l}\} \cup \{\Gamma_i(x^\circ_{k,l}) \colon i \in [N]\} & \text{for } 1 < s \le 2, \\ S & \text{for } 2 < s. \end{cases}$$

Now, for each fixed $i \in [N]$ and $k \in [M]$, we introduce a linear operator $\mathcal{A}_{k,i} = \mathcal{A}_{k,i,\mathfrak{S}}$ given by the formula

$$\mathcal{A}_{k,i} f(x) := \begin{cases} \frac{f(\Gamma_i(x))\,\mu(\{\Gamma_i(x)\})}{\mu(\{x\})} & \text{if } x \in S^\circ_k, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $\Gamma_i(x)$, we see that $\mathcal{A}_{k,i} f$ depends only on the values of f on S_i.

In the following lemma we estimate the norm of $\mathcal{A}_{k,i}$ acting from $L^{p,q}(\mathfrak{S})$ to $L^{p,r}(\mathfrak{S})$.

Lemma 2. *Let $\mathfrak{S} = \mathfrak{S}_{p,N,M,K,L}$ be the metric measure space defined as above. Fix $1 \le q \le r \le \infty$, $i \in [N]$, and $k \in [M]$, and consider the operator $\mathcal{A}_{k,i}$. Then there exists a numerical constant $\mathbf{C}_2 = \mathbf{C}_2(p,q,r)$ independent of N, M, K, L, i, and k such that*

$$\mathbf{C}_2^{-1}\, K^{-1+1/p} L^{1/p} \le \|\mathcal{A}_{k,i}\|_{L^{p,q}(\mathfrak{S}) \to L^{p,r}(\mathfrak{S})} \le \mathbf{C}_2\, K^{-1+1/p} L^{1/p}.$$

Proof. First we estimate $\|\mathcal{A}_{k,i}\|_{L^{p,q}(\mathfrak{S}) \to L^{p,r}(\mathfrak{S})}$ from above. Take $f \in L^{p,q}(\mathfrak{S})$. Since $\mathcal{A}_{k,i} f \equiv \mathcal{A}_{k,i}(f \cdot \mathbb{1}_{S_i})$, we can assume that the support of f is contained in S_i. If this is the case, then for each $t \in (0,\infty)$ we have the equality

$$d_{\mathcal{A}_{k,i} f}(t) = \frac{K L \alpha_k \beta_k h_N}{m_i h_i} d_f(t K \alpha_k / m_i)$$

because every point with mass m_i where f equals t determines $L\beta_k h_N / h_i$ points with masses $K\alpha_k$ where $\mathcal{A}_{k,i} f$ equals $t m_i / K\alpha_k$. Then a simple calculation gives

$$\|\mathcal{A}_{k,i} f\|_{p,r} = K^{-1+1/p} L^{1/p} m_i^{1-1/p} h_i^{-1/p} \alpha_k^{-1+1/p} \beta_k^{1/p} h_N^{1/p} \|f\|_{p,r}.$$

Thus, in view of (iii), (vi), and Fact 3, we obtain

$$\|\mathcal{A}_{k,i} f\|_{p,r} \le 2\, \mathbf{C}_{\hookrightarrow}(p,q,r)\, K^{-1+1/p} L^{1/p} \|f\|_{p,q}.$$

Finally, consider $g := \mathbb{1}_{S_i}$. Then we have $\mathcal{A}_{k,i} g \equiv \frac{m_i}{K\alpha_k} \mathbb{1}_{S_k^\circ}$ and hence

$$\begin{aligned} \frac{\|\mathcal{A}_{k,i} g\|_{p,r}}{\|g\|_{p,q}} &= \frac{(p/r)^{1/r}}{(p/q)^{1/q}} K^{-1+1/p} L^{1/p} m_i^{1-1/p} h_i^{-1/p} \alpha_k^{-1+1/p} \beta_k^{1/p} h_N^{1/p} \\ &\ge \frac{1}{2} \frac{(p/r)^{1/r}}{(p/q)^{1/q}} K^{-1+1/p} L^{1/p}, \end{aligned}$$

where in the last inequality we again used (iii) and (vi). □

Next we introduce a linear operator $\mathcal{A} = \mathcal{A}_{\mathfrak{S}}$ given by the formula

$$\mathcal{A} f(x) := \sum_{i \in [N]} \sum_{k \in [M]} \mathcal{A}_{k,i} f(x).$$

We see that $\mathcal{A} f$ depends only on the values of f on $S \setminus S^\circ$.

As before, we estimate the norm of $\mathcal{A}$ acting from $L^{p,q}(\mathfrak{S})$ to $L^{p,r}(\mathfrak{S})$.

Lemma 3. *Let $\mathfrak{S} = \mathfrak{S}_{p,N,M,K,L}$ be the metric measure space defined as above. Fix $1 \le q \le r \le \infty$ and consider the operator $\mathcal{A}$. Then there exists a numerical constant $\mathbf{C}_3 = \mathbf{C}_3(p,q,r)$ independent of N, M, K, and L such that*

$$\mathbf{C}_3^{-1}\, K^{-1+1/p} L^{1/p} M^{1/r} N^{1-1/q} \le \|\mathcal{A}\|_{L^{p,q}(\mathfrak{S}) \to L^{p,r}(\mathfrak{S})} \le \mathbf{C}_3\, K^{-1+1/p} L^{1/p} M^{1/r} N^{1-1/q}.$$

Proof. First we estimate $\|\mathcal{A}\|_{L^{p,q}(\mathfrak{S}) \to L^{p,r}(\mathfrak{S})}$ from above. Take $f \in L^{p,q}(\mathfrak{S})$. Since $\mathcal{A} f \equiv \mathcal{A}(f \cdot \mathbb{1}_{S \setminus S^\circ})$, we can assume that the support of f is contained in $S \setminus S^\circ$. We decompose $f = \sum_{i \in [N]} f_i$, where $f_i := f \cdot \mathbb{1}_{S_i}$. Then we have, by Lemma 1 in view of (3),

$$\mathbf{C}_1(p,q)\, \|f\|_{p,q} \ge \Big(\sum_{i \in [N]} \|f_i\|_{p,q}^q \Big)^{1/q}$$

and, by Lemma 1 in view of (4),

$$\|\mathcal{A}f\|_{p,r} \leq \mathbf{C}_1(p,r) \left(\sum_{k\in[M]} \left\|\mathcal{A}f \cdot \mathbb{1}_{S_k^\circ}\right\|_{p,r}^r \right)^{1/r}.$$

Next, by using the definition of $\mathcal{A}$, Fact 1, and Lemma 2, we obtain

$$\begin{aligned}\|\mathcal{A}f \cdot \mathbb{1}_{S_k^\circ}\|_{p,r} &\leq \mathbf{C}_\triangle(p,r) \sum_{i\in[N]} \|\mathcal{A}_{k,i}f_i\|_{p,r} \\ &\leq \mathbf{C}_\triangle(p,r)\mathbf{C}_2(p,q,r)K^{-1+1/p}L^{1/p} \sum_{i\in[N]} \|f_i\|_{p,q}\end{aligned}$$

for each $k \in [M]$. Therefore,

$$\|\mathcal{A}f\|_{p,r} \leq \mathbf{C}_1(p,r)\mathbf{C}_\triangle(p,r)\mathbf{C}_2(p,q,r)K^{-1+1/p}L^{1/p}M^{1/r} \sum_{i\in[N]} \|f_i\|_{p,q}.$$

On the other hand, Hölder's inequality applied with respect to i gives

$$\left(\sum_{i\in[N]} \|f_i\|_{p,q}^q \right)^{1/q} \geq N^{-1+1/q} \sum_{i\in[N]} \|f_i\|_{p,q}.$$

Combining all these estimates we conclude that

$$\|\mathcal{A}f\|_{p,r} \leq \mathbf{C}_1(p,q)\mathbf{C}_1(p,r)\mathbf{C}_\triangle(p,r)\mathbf{C}_2(p,q,r)K^{-1+1/p}L^{1/p}M^{1/r}N^{1-1/q}\|f\|_{p,q}.$$

Finally, consider $g \coloneqq \sum_{i\in[N]}(h_i m_i)^{-1/p} \cdot \mathbb{1}_{S_i}$. Then, by using (iii), we have

$$\mathcal{A}g \geq \sum_{k\in[M]} \frac{N}{2^{1/p}K\alpha_k} \cdot \mathbb{1}_{S_k^\circ}$$

and thus

$$\begin{aligned}\frac{\|\mathcal{A}g\|_{p,r}}{\|g\|_{p,q}} &\geq \frac{(p/r)^{1/r}}{(p/q)^{1/q}} \frac{\left(\sum_{k\in[M]} \left(K^{-1+1/p}L^{1/p}N\alpha_k^{-1+1/p}\beta_k^{1/p}h_N^{1/p}\right)^r\right)^{1/r}}{2^{1/p}\,\mathbf{C}_1(p,q)\mathbf{C}_1(p,r)N^{1/q}} \\ &\geq \frac{(p/r)^{1/r}}{(p/q)^{1/q}} \frac{K^{-1+1/p}L^{1/p}M^{1/r}N^{1-1/q}}{2^{1/p}\,\mathbf{C}_1(p,q)\mathbf{C}_1(p,r)},\end{aligned}$$

where in the first inequality we used (3) and (4) in order to apply Lemma 1 to g and $\mathcal{A}g$, respectively, and in the second inequality we used (vi). □

In the following lemma we estimate the norm of the maximal operator $\mathcal{M}_{\mathfrak{S}}$ acting from $L^{p,q}(\mathfrak{S})$ to $L^{p,r}(\mathfrak{S})$. This is the main result of this section.

Lemma 4. *Let $\mathfrak{S} = \mathfrak{S}_{p,N,M,K,L}$ be the metric measure space defined as above. Fix $1 \leq q \leq r \leq \infty$ and consider the associated operator $\mathcal{M}_{\mathfrak{S}}$. Then there exists a numerical constant $\mathbf{C}_4 = \mathbf{C}_4(p,q,r)$ independent of N, M, K, and L such that*

$$\mathbf{C}_4^{-1}\left(1+K^{-1+1/p}L^{1/p}M^{1/r}N^{1-1/q}\right) \leq \mathbf{c}(p,q,r,\mathfrak{S}) \leq \mathbf{C}_4\left(1+K^{-1+1/p}L^{1/p}M^{1/r}N^{1-1/q}\right).$$

Proof. First we estimate $\mathbf{c}(p,q,r,\mathfrak{S})$ from above. Take nonnegative $f \in L^{p,q}(\mathfrak{S})$ such that $\|f\|_{p,q} = 1$. One can check that

$$\mathcal{M}_{\mathfrak{S}}f \leq \max\{f, 4\mathcal{A}f, 2\widetilde{\mathcal{M}}f, f_{\mathrm{avg}}\},$$

where $\widetilde{\mathcal{M}}f \coloneqq \mathbb{1}_{S \setminus S^\circ} \cdot \max_{x^\circ \in S^\circ} f(x^\circ)$. Therefore, by Fact 1, we have

$$\|\mathcal{M}_{\mathfrak{S}} f\|_{p,r} \leq 4\, \mathbf{C}_{\triangle}(p,r) \Big(\|f\|_{p,r} + \|\mathcal{A} f\|_{p,r} + \|\widetilde{\mathcal{M}} f\|_{p,r} + \|f_{\mathrm{avg}}\|_{p,r} \Big).$$

The inequalities $\|\widetilde{\mathcal{M}} f\|_{p,r} \leq \|f\|_{p,r}$ and $\|f_{\mathrm{avg}}\|_{p,r} \leq \mathbf{C}_{\mathrm{avg}}(p,r) \|f\|_{p,r}$ follows respectively from (2) and Fact 2. Combining all these estimates with Fact 3 and Lemma 3 we conclude that $\|\mathcal{M}_{\mathfrak{S}} f\|_{p,r}$ is controlled by

$$4\, \mathbf{C}_{\triangle}(p,r) \Big(\mathbf{C}_{\hookrightarrow}(p,q,r) \big(2 + \mathbf{C}_{\mathrm{avg}}(p,r)\big) + \mathbf{C}_3(p,q,r) \big(K^{-1+1/p} L^{1/p} M^{1/r} N^{1-1/q} \big) \Big).$$

Now we estimate $\mathbf{c}(p,q,r,\mathfrak{S})$ from below. First, arguing as in the last part of the proof of Proposition 1, we obtain $\mathbf{c}(p,q,r,\mathfrak{S}) \geq (p/r)^{1/r} (p/q)^{-1/q}$. Finally, the inequality

$$\mathbf{c}(p,q,r,\mathfrak{S}) \geq \frac{K^{-1+1/p} L^{1/p} M^{1/r} N^{1-1/q}}{2 \mathbf{C}_3(p,q,r)}$$

follows from Lemma 3, since by (2) we have $\mathcal{M}_{\mathfrak{S}} f \geq \mathcal{A} f / 2$ for each $f \in L^{p,q}(\mathfrak{S})$. □

At the end of this section we reformulate the result of the previous lemma in a way that makes it easier to use later on.

Corollary 1. *Fix $p \in (1,\infty)$, $\lambda \in (0,\infty)$, and $a, b, \kappa \in \mathbb{N}$. Let $\mathfrak{S}_{(p,\lambda,a,b,\kappa)}$ be the test space $\mathfrak{S}_{p,N,M,K,L}$ with p as above, $N = \kappa^b$, $M = \kappa^a$, and some K, L satisfying $K^{-1+1/p} L^{1/p} = \lambda \kappa^{-b}$. Then for each $1 \leq q \leq r \leq \infty$ we have*

$$\mathbf{C}_4^{-1} \left(1 + \lambda \kappa^{a/r - b/q}\right) \leq \mathbf{c}(p,q,r,\mathfrak{S}_{(p,\lambda,a,b,\kappa)}) \leq \mathbf{C}_4 \left(1 + \lambda \kappa^{a/r - b/q}\right),$$

where $\mathbf{C}_4 = \mathbf{C}_4(p,q,r)$ is the constant from Lemma 4.

4. Composite test spaces

In the following two sections by a *composite test space* we mean any metric measure space $\mathfrak{T}$ that arises as a result of applying Proposition 1 to a certain family of test spaces introduced in Section 3. This is a bit imprecise, but one can think of composite test spaces as intermediate objects between test spaces and the spaces we want to obtain in Theorem 2. More precisely, these latter ones will be composite test spaces constructed with the aid of a sequence of simpler composite test spaces. We now briefly explain the details of such a construction.

Proposition 2. Let $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ be a given sequence of composite test spaces. Then there exists a composite test space $\mathfrak{T}$ such that for each $p \in (1,\infty)$ and $1 \leq q \leq r \leq \infty$ we have

$$\mathbf{C}^{-2} \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{T}_n) \leq \mathbf{c}(p,q,r,\mathfrak{T}) \leq \mathbf{C}^2 \sup_{n\in\mathbb{N}} \mathbf{c}(p,q,r,\mathfrak{T}_n),$$

where $\mathbf{C} = \mathbf{C}(p,q,r)$ is the constant from Proposition 1.

Proof. Each space $\mathfrak{T}_n$ is constructed with the aid of some sequence of test spaces, say $(\mathfrak{S}_{n,m})_{m\in\mathbb{N}}$. We let $\mathfrak{T}$ be the space constructed by using Proposition 1 for the whole family of test spaces $\{\mathfrak{S}_{n,m} \colon n, m \in \mathbb{N}\}$. It follows directly from Proposition 1 that $\mathfrak{T}$ satisfies the desired condition. □

Now we will construct some composite test spaces for which the associated maximal operators have very specific properties.

Lemma 5. *Let $p \in (1, \infty)$, $\gamma \in \mathbb{R}$, $a, b, R \in \mathbb{N}$, and $\epsilon \in (0, \infty)$. Then there exists a composite test space $\mathfrak{T} = \mathfrak{T}_{p,\gamma,a,b,R,\epsilon}$ such that for each $1 \le q \le r \le \infty$ we have*

$$\mathbf{c}(p, q, r, \mathfrak{T}) = \infty \quad \text{if } a/r - b/q = \gamma,$$
$$\mathbf{C}_5^{-1} R^{\epsilon d} \le \mathbf{c}(p, q, r, \mathfrak{T}) \le \mathbf{C}_5 R^{2\epsilon d} \quad \text{if } a/r - b/q \in (\gamma - 2\epsilon d, \gamma - \epsilon d),$$
$$\mathbf{c}(p, q, r, \mathfrak{T}) \le \mathbf{C}_5 \quad \text{if } a/r - b/q \le \gamma - 3\epsilon d,$$

where $d = \sqrt{a^2 + b^2}$ and $\mathbf{C}_5 = \mathbf{C}_5(p, q, r)$ is independent of γ, a, b, R, and ϵ.

Figure 2 describes the behavior of the function $\mathbf{c}(p, q, r, \mathfrak{T})$. The parameter d appears here only for purely aesthetical reasons (for example, the Euclidean distance between the lines $a/r - b/q = \gamma$ and $a/r - b/q = \gamma - \epsilon d$ equals ϵ).

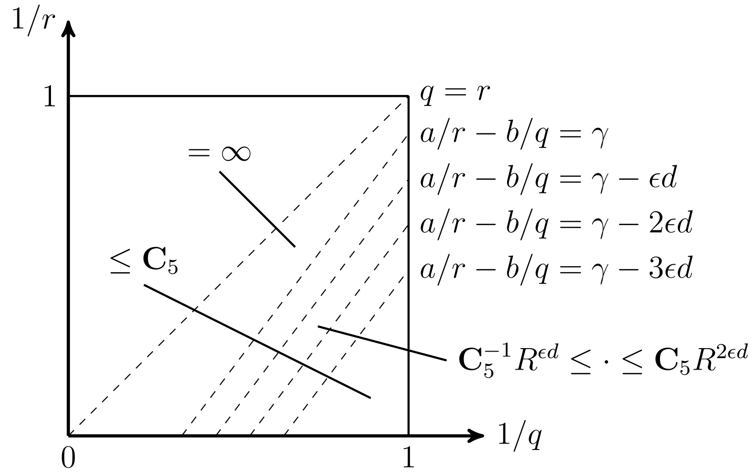

Figure 2. The behavior of the function $\mathbf{c}(p, q, r, \mathfrak{T})$.

Proof. For each $n \in \mathbb{N}$ let $\mathfrak{S}_n$ be the test space $\mathfrak{S}_{(p,\lambda,a,b,\kappa)}$ from Corollary 1 with p, a, and b as above, $\kappa = R^n$, and $\lambda = R^{-n\gamma+(n+2)\epsilon d}$. We let $\mathfrak{T}$ be the space obtained by using Proposition 1 for the sequence $(\mathfrak{S}_n)_{n\in\mathbb{N}}$. The following estimates hold: if $a/r - b/q = \gamma$, then

$$\mathbf{c}(p, q, r, \mathfrak{T}) \ge \frac{\lim_{n\to\infty} R^{-n\gamma+(n+2)\epsilon d} R^{n\gamma}}{\mathbf{C}\mathbf{C}_4} = \infty,$$

if $a/r - b/q \in (\gamma - 2\epsilon d, \gamma - \epsilon d)$, then

$$\mathbf{c}(p, q, r, \mathfrak{T}) \ge \frac{\sup_{n\in\mathbb{N}} R^{-n\gamma+(n+2)\epsilon d} R^{n(\gamma-2\epsilon d)}}{\mathbf{C}\mathbf{C}_4} = \frac{R^{\epsilon d}}{\mathbf{C}\mathbf{C}_4}$$

(here and in the remaining two cases the supremum is attained at $n = 1$) and

$$\mathbf{c}(p, q, r, \mathfrak{T}) \le \mathbf{C}\mathbf{C}_4 \sup_{n\in\mathbb{N}} \left(1 + R^{-n\gamma+(n+2)\epsilon d} R^{n(\gamma-\epsilon d)}\right) \le 2\mathbf{C}\mathbf{C}_4 R^{2\epsilon d},$$

and, if $a/r - b/q \le \gamma - 3\epsilon d$, then

$$\mathbf{c}(p, q, r, \mathfrak{T}) \le \mathbf{C}\mathbf{C}_4 \sup_{n\in\mathbb{N}} \left(1 + R^{-n\gamma+(n+2)\epsilon d} R^{n(\gamma-3\epsilon d)}\right) = 2\mathbf{C}\mathbf{C}_4.$$

Therefore, $\mathfrak{T}$ satisfies the desired properties. □

At the end of this section we present another result for composite test spaces, which is particularly helpful if the domain of F in Theorem 2 is of the form $(\delta, 1]$, or if the domain is of the form $[\delta, 1]$, but either $\delta = 1$ or F is not continuous at δ.

Lemma 6. *Let $p \in (1,\infty)$, $\delta \in [0,1]$, and $\omega \in [0,\delta]$. Then there exists*

- *a composite test space $\mathfrak{T}^{\leq} = \mathfrak{T}^{\leq}_{p,\delta,\omega}$ such that $\mathbf{c}(p,q,r,\mathfrak{T}^{\leq}) < \infty$ if and only if $1/q > \delta$, $1/r \leq 1/q$ or $1/q = \delta$, $1/r \leq \omega$,*
- *a composite test space $\mathfrak{T}^{<} = \mathfrak{T}^{<}_{p,\delta,\omega}$ such that $\mathbf{c}(p,q,r,\mathfrak{T}^{<}) < \infty$ if and only if $1/q > \delta$, $1/r \leq 1/q$ or $1/q = \delta$, $1/r < \omega$.*

Proof. Fix $p \in (1,\infty)$, $\delta \in [0,1]$, and $\omega \in [0,\delta]$. First we construct $\mathfrak{T}^{\leq}$. For each n take $a_n = n$, $b_n = n^2$, and γ_n satisfying $a_n\omega - b_n\delta = \gamma_n - 3d_n\epsilon_n$, where $d_n = \sqrt{a_n^2 + b_n^2}$ and $\epsilon_n = 1/3n^2$. Let $\mathfrak{T}_n$ be the composite test space $\mathfrak{T}$ from Lemma 5 with p as above, $\gamma = \gamma_n$, $a = a_n$, $b = b_n$, $R = n$, and $\epsilon = \epsilon_n$. Since $\lim_{n\to\infty} b_n/a_n = \infty$ and $a_n(\omega + 1/n) - b_n\delta > \gamma_n - 2d_n\epsilon_n$, it is routine to check that $\mathfrak{T}^{\leq}$ may be chosen to be the space obtained by using Proposition 2 for the sequence of composite test spaces $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ (to obtain $\mathbf{c}(p,q,r,\widetilde{\mathfrak{T}}) = \infty$ for $1/q > \delta$, $1/r > 1/q$ we use Remark 2, see Section 6). Finally, to construct $\mathfrak{T}^{<}$ we take $a_n = n$, $b_n = n^2$, and γ_n satisfying $a_n(\omega - 1/n) - b_n\delta = \gamma_n - 3d_n\epsilon_n$ with d_n and ϵ_n are as before, and use the fact that $a_n\omega - b_n\delta > \gamma_n - 2d_n\epsilon_n$. ∎

We note that Lemma 6 may also be used to construct $\mathfrak{X}$ such that $\Omega^p_{\mathrm{HL}}(\mathfrak{X}) = \emptyset$. Indeed, it suffices to take $\mathfrak{T}^{<}$ with p as above, $\delta = 1$, and $\omega = 0$.

5. Proof of Theorem 2

Case 1: $F\colon [\delta,1] \to [0,1]$, F is continuous at δ. Fix $p \in (1,\infty)$ and $\delta \in [0,1]$, and take $F\colon [\delta,1] \to [0,1]$ concave, nondecreasing, continuous at δ, and such that $F(u) \leq u$ for each $u \in [\delta,1]$.

First we construct $\mathfrak{Y}$. We can assume that $\delta < 1$, since the case $\delta = 1$ is covered by Lemma 6. Consider the countable set

$$\Big\{ (\tfrac{1}{q}, \tfrac{1}{r}) \in \big([0,1] \cap \mathbb{Q}\big)^2 \colon \big(\tfrac{1}{q} \geq \delta \text{ and } \tfrac{1}{r} > F(\tfrac{1}{q})\big) \text{ or } \big(\tfrac{1}{q} < \delta\big) \Big\}$$

and enumerate it to get a sequence $(P_n)_{n\in\mathbb{N}}$. Fix $n \in \mathbb{N}$ and let $P_n = (\frac{1}{q_n}, \frac{1}{r_n})$. Since F is concave and nondecreasing, we can choose $\gamma_n \in \mathbb{R}$, $a_n, b_n \in \mathbb{N}$, and $\epsilon_n \in (0,\infty)$ such that

- $a_n/r_n - b_n/q_n = \gamma_n$,
- if $a_n/r - b_n/q > \gamma_n - 3\epsilon_n d_n$, then $\frac{1}{q} \geq \delta$, $\frac{1}{r} > F(\frac{1}{q})$ or $\frac{1}{q} < \delta$, where $d_n = \sqrt{a_n^2 + b_n^2}$.

Let $\mathfrak{T}_n$ be the composite test space $\mathfrak{T}$ from Lemma 5 with p as above, $\gamma = \gamma_n$, $a = a_n$, $b = b_n$, $R = 1$, and $\epsilon = \epsilon_n$. It is routine to check that $\mathfrak{Y}$ may be chosen to be the space obtained by using Proposition 2 for the sequence of composite test spaces $(\mathfrak{T}_n)_{n\in\mathbb{N}}$.

Now we construct $\mathfrak{Z}$. Again we assume that $\delta < 1$, since the case $\delta = 1$ is covered by Lemma 6. For each $n \in \mathbb{N}$ and $u \in [\delta,1]$ we choose $\gamma_{n,u} \in \mathbb{R}$ and $a_{n,u}, b_{n,u} \in \mathbb{N}$ such that

- $\gamma_{n,u} - 2d_{n,u}/n < a_{n,u}u - b_{n,u}F(u) < \gamma_{n,u} - d_{n,u}/n$, where $d_{n,u} = \sqrt{a_{n,u}^2 + b_{n,u}^2}$,
- if $a_{n,u}/r - b_{n,u}/q \geq \gamma_{n,u} - d_{n,u}/n$, then $\frac{1}{q} \geq \delta$, $\frac{1}{r} > F\big(\frac{1}{q}\big)$ or $\frac{1}{q} < \delta$.

Let $\mathfrak{T}_{n,u}$ be the composite test space $\mathfrak{T}$ from Lemma 5 with p as above, $\gamma = \gamma_{n,u}$, $a = a_{n,u}$, $b = b_{n,u}$, $R = n^n$, and $\epsilon = 1/n$. Fix $n \in \mathbb{N}$ and observe that for each $u \in [\delta,1]$ the set

$$E_{n,u} = \big\{ v \in [\delta,1] \colon \gamma_{n,u} - 2d_n/n < av - bF(v) < \gamma_{n,u} - d_n/n \big\}$$

is open in $[\delta,1]$ with its natural topology. Thus $\{E_{n,u} \colon u \in [\delta,1]\}$ is an open cover of $[\delta,1]$ and we can find a finite subset $U_n \subset [\delta,1]$ such that $\bigcup_{u\in U_n} E_{n,u} = [\delta,1]$. Finally, we let $\mathfrak{Z}$ be the space obtained by using Proposition 2 for the family $\{\mathfrak{T}_{n,u}\colon n \in \mathbb{N}, u \in U_n\}$. We will show that $\mathfrak{Z}$ satisfies the desired properties. Fix $u_0 \in [\delta, 1]$ and observe that for each $n \in \mathbb{N}$ there exists $u_n \in U_n$ such that $u_0 \in E_{n,u_n}$. Therefore, in view of Lemma 5,

$$\mathbf{c}(p, 1/u_0, 1/F(u_0), \mathfrak{Z}) \geq \mathbf{C}^{-2}\, \mathbf{c}(p, 1/u_0, 1/F(u_0), \mathfrak{T}_{n,u_n}) \geq \mathbf{C}^{-2} n^{d_{n,u}}.$$

Since n is arbitrary and $d_{n,u} \geq 1$, we conclude that $\mathbf{c}(p, 1/u_0, 1/F(u_0), \mathfrak{Z}) = \infty$. As a result, we obtain $\mathbf{c}(p, q, r, \mathfrak{Z}) = \infty$ for each pair (q, r) satisfying $\frac{1}{q} \geq \delta$, $\frac{1}{r} \geq F\big(\frac{1}{q}\big)$ or $\frac{1}{q} < \delta$. Now let us consider a pair (q, r) satisfying $\frac{1}{q} \geq \delta$, $\frac{1}{r} < F\big(\frac{1}{q}\big)$. Then we have

$$d(q, r, F) \coloneqq \min\big\{d_{\mathrm{E}}\big((\tfrac{1}{q}, \tfrac{1}{r}), (u, F(u))\big)\colon u \in [\delta, 1]\big\} > 0,$$

where d_{E} is the standard Euclidean metric on the plane. Observe that for each $n \in \mathbb{N}$ and $u \in U_n$ we have the following implication

$$a_{n,u}/r - b_{n,u}/q > \gamma_{n,u} - 3d_{n,u}/n \implies d(q, r, F) \leq 2/n.$$

Hence if $n > 2/d(q, r, F)$, then for each $u \in U_n$ we have $a_{n,u}/r - b_{n,u}/q \leq \gamma_{n,u} - 3d_{n,u}/n$, which implies $\mathbf{c}(p, q, r, \mathfrak{T}_{n,u}) \leq \mathbf{C}_5$. Finally, since for each of the finitely many pairs (n, u) satisfying $n \leq 2/d(q, r, F)$ and $u \in U_n$ there is $\mathbf{c}(p, q, r, \mathfrak{T}_{n,t}) < \infty$, we conclude that $\mathbf{c}(p, q, r, \mathfrak{Z}) < \infty$.

Case 2: $F\colon [\delta, 1] \to [0, 1]$, F is not continuous at δ. Fix $p \in (1, \infty)$ and $\delta \in (0, 1)$, and take $F\colon [\delta, 1] \to [0, 1]$ concave, nondecreasing, satisfying $F(\delta) = \omega < \lim_{u\to\delta} F(u)$ for some $\omega \in [0, \delta)$, and such that $F(u) \leq u$ for each $u \in [\delta, 1]$. We modify F to get $\widetilde{F}\colon [\delta, 1] \to [0, 1]$ continuous at δ, by setting $\widetilde{F}(u) = F(u)$ for $u \in (\delta, 1)$ and $\widetilde{F}(\delta) = \lim_{u\to\delta} F(u)$. Then $\widetilde{F}$ satisfies the conditions specified in Case 1. Let $\widetilde{\mathfrak{Y}}$ (resp. $\widetilde{\mathfrak{Z}}$) be the space obtained in Case 1 for $\widetilde{F}$. We also let $\widetilde{\mathfrak{T}}$ be the composite test space $\mathfrak{T}^{\leq}$ (resp. $\mathfrak{T}^{<}$) from Lemma 6 with p, δ, and ω as above. It is routine to check that $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) may be chosen to be the space obtained by using Proposition 2 for $\widetilde{\mathfrak{Y}}$ (resp. $\widetilde{\mathfrak{Z}}$) and countably many copies of $\widetilde{\mathfrak{T}}$.

Case 3: $F\colon (\delta, 1] \to [0, 1]$. Fix $p \in (1, \infty)$ and $\delta \in [0, 1)$, and take $F\colon (\delta, 1] \to [0, 1]$ concave, nondecreasing and such that $F(u) \leq u$ for each $u \in (\delta, 1]$. We extend F to get $\widetilde{F}\colon [\delta, 1] \to [0, 1]$ continuous at δ, by setting $\widetilde{F}(\delta) = \lim_{u\to\delta} F(u)$. Then $\widetilde{F}$ satisfies the conditions specified in Case 1. Let $\widetilde{\mathfrak{Y}}$ and $\widetilde{\mathfrak{Z}}$ be the spaces obtained in Case 1 for $\widetilde{F}$. We also let $\widetilde{\mathfrak{T}}$ be the composite test space $\mathfrak{T}^{<}$ from Lemma 6 with p and δ as above, and $\omega = 0$. It is routine to check that $\mathfrak{Y}$ (resp. $\mathfrak{Z}$) may be chosen to be the space obtained by using Proposition 2 for $\widetilde{\mathfrak{Y}}$ (resp. $\widetilde{\mathfrak{Z}}$) and countably many copies of $\widetilde{\mathfrak{T}}$.

6. Necessary conditions

In the last section we briefly discuss why there are no alternatives for the shape of $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ other than those mentioned in Theorem 1. We begin with the following simple observation.

Remark 1. Fix $p \in (1, \infty)$ and let $\mathfrak{X}$ be an arbitrary metric measure space. If $(u, w) \in \Omega^p_{\mathrm{HL}}(\mathfrak{X})$, then $[0, u] \times [w, 1] \subset \Omega^p_{\mathrm{HL}}(\mathfrak{X})$.

Indeed, this follows by the fact that the Lorentz spaces $L^{p,q}(\mathfrak{X})$ increase as the parameter q increases.

By Remark 1 we know that either $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ is empty or it consists of points lying under the graph of some nondecreasing function, say F, and the domain of F is of the form $[\delta, 1]$ or $(\delta, 1]$ for some $\delta \in [0,1]$ or $\delta \in [0,1)$, respectively. More precisely, for each u from the domain of F we have $(u,w) \in \Omega^p_{\rm HL}(\mathfrak{X})$ for $w < F(u)$ and $(u,w) \notin \Omega^p_{\rm HL}(\mathfrak{X})$ for $w > F(u)$ (here we do not focus on whether $(u, F(u))$ belongs to $\Omega^p_{\rm HL}(\mathfrak{X})$ or not, except for the case $F(u) = 0$, which forces that the first option actually takes place).

Remark 2 below, in turn, explains why the assumption $F(u) \le u$ is needed.

Remark 2. Let $\mathfrak{X}$ be a metric measure space such that $\mu(X \setminus \mathrm{supp}(\mu)) = 0$. Assume that there exists an infinite family $\mathfrak{B}$ of disjoint balls B satisfying $0 < \mu(B) < \infty$. Then for each $p \in (1,\infty)$ we have $\Omega^p_{\rm HL}(\mathfrak{X}) \subset \{(u,w) \in [0,1]^2 \colon w \le u\}$.

Indeed, fix $p \in (1,\infty)$ and $1 \le r < q < \infty$ (the case $q = \infty$ is similar). Let $n_0 \in \mathbb{N}$. We can find a sequence $(E_n)_{n \in [n_0]}$ of disjoint sets with the following properties:

- each E_n is a union of finitely many elements from $\mathfrak{B}$,
- $\mu(E_n) \ge 2\mu(E_{n-1})$ for each $n \in [n_0] \setminus [1]$.

Precisely, if $\mathfrak{B}$ contains balls with arbitrarily small measures, then we can start from $n = n_0$ and choose each E_n to be a single ball. If not, then we can start from $n = 1$ and choose suitable finite unions of balls. Consider $g_{n_0} \in L^{p,q}(\mathfrak{X})$ defined by

$$g_{n_0} \coloneqq \sum_{n \in [n_0]} n^{-\frac{2}{q+r}} \mu(E_n)^{-1/p} \mathbb{1}_{E_n}.$$

By Lemma 1 the following estimates hold

$$\|g_{n_0}\|_{p,q} \le \mathbf{C}_1(p,q)(p/q)^{1/q} \Bigg(\sum_{n \in [n_0]} n^{-\frac{2q}{q+r}} \Bigg)^{1/q}, \ \|g_{n_0}\|_{p,r} \ge \frac{(p/r)^{1/r}}{\mathbf{C}_1(p,r)} \Bigg(\sum_{n \in [n_0]} n^{-\frac{2r}{q+r}} \Bigg)^{1/r}.$$

Observe that for each $x \in \mathrm{supp}(\mu)$ we have $\mathcal{M}_{\mathfrak{X}} g_{n_0}(x) \ge g_{n_0}(x)$. Therefore, $\|\mathcal{M}_{\mathfrak{X}} g_{n_0}\|_{p,r} \ge \|g_{n_0}\|_{p,r}$ follows in view of $\mu(X \setminus \mathrm{supp}(\mu)) = 0$. Since $2r/(q+r) < 1 < 2q/(q+r)$, we obtain $\lim_{n_0 \to \infty} \frac{\|g_{n_0}\|_{p,r}}{\|g_{n_0}\|_{p,q}} = \infty$ and, consequently, $(\frac{1}{q}, \frac{1}{r}) \notin \Omega^p_{\rm HL}(\mathfrak{X})$.

One additional comment should be made here. Namely, if $\mathfrak{B}$ from Remark 2 does not exist, then there are only finitely many points $x \in \mathrm{supp}(\mu)$ such that $\mu(B(x, s(x))) < \infty$ for some $s(x) \in (0,\infty)$. In this case $\Omega^p_{\rm HL}(\mathfrak{X}) = [0,1]^2$ holds trivially for each $p \in (1,\infty)$, provided that $\mu(X \setminus \mathrm{supp}(\mu)) = 0$ is satisfied. Indeed, let E be the set of all points with this property. If E is infinite, then we can find $x \in E$ and $s_x \in (0, s(x)]$ such that $E \setminus B(x, s_x)$ is infinite. Next, we can find $x' \in E \setminus B(x, s_x)$ and $s_{x'} \in (0, s(x')]$ such that $B(x', s_{x'}) \cap B(x, s_x) = \emptyset$ and $E \setminus (B(x, s_x) \cup B(x, s_{x'}))$ is still infinite. By repeating this procedure, we construct $\mathfrak{B}$. On the other hand, if E is finite, then $\mathcal{M}_{\mathfrak{X}} f$ is negligible on $X \setminus \mathrm{supp}(\mu)$, bounded by $\max_{x \in E : \mu(\{x\}) > 0} |f(x)|$ on E, and equal to 0 on $\mathrm{supp}(\mu) \setminus E$, since each open ball containing at least one point from $\mathrm{supp}(\mu) \setminus E$ has infinite measure.

Finally, the fact that $\Omega^p_{\rm HL}(\mathfrak{X})$ is convex, and hence F must be concave, is justified by the following interpolation argument.

Theorem 3. *Fix $p \in [1,\infty)$, $1 \le q_0 \le q_1 \le \infty$, and $1 \le r_0, r_1 \le \infty$ such that $q_i \le r_i$ for $i \in \{0,1\}$. Let $\mathfrak{X}$ be an arbitrary metric measure space and assume that the associated maximal operator $\mathcal{M}_{\mathfrak{X}}$ is bounded from $L^{p,q_i}(\mathfrak{X})$ to $L^{p,r_i}(\mathfrak{X})$ for $i \in \{0,1\}$. Then for each $\theta \in (0,1)$ the operator $\mathcal{M}_{\mathfrak{X}}$ is bounded from $L^{p,q_\theta}(\mathfrak{X})$ to $L^{p,r_\theta}(\mathfrak{X})$, where*

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

We explain briefly how Theorem 3 can be inferred from the general theory of interpolation. We begin with the comment that Lorentz spaces in this context were considered for the first time by Hunt in [5]. However, the theorem formulated there does not cover Theorem 3. Hence, we are forced to refer to the literature where some more advanced interpolation methods are developed. The appropriate variant of Theorem 3 for linear operators can be directly deduced from [3, Theorem 5.3.1] (see also [13], where the K-functional for the couple (L^{p,q_0}, L^{p,q_1}) is computed). Then, a suitable linearization argument (see [6], for example) allows us to extend this result to the class of sublinear operators and thus the maximal operator $\mathcal{M}_{\mathfrak{X}}$ is also included.

Although there are several ways to deduce Theorem 3 from the theorems that appear in the literature, each of them, to the author's best knowledge, requires a deep understanding of the interpolation theory. As the author found an elegant, elementary proof of Theorem 3, he decided to present it in Appendix.

The last issue we want to mention is the boundary problem. Denote be $\bar{\partial}\Omega^p_{\mathrm{HL}}(\mathfrak{X})$ the upper part of the boundary of $\Omega^p_{\mathrm{HL}}(\mathfrak{X})$, that is, the set $\{(u, F(u)) \colon u \in \mathrm{Dom(F)}\}$, where Dom(F) is the domain of F. According to this, for each space constructed in Theorem 2 one of the following two possibilities holds

$$\bar{\partial}\Omega^p_{\mathrm{HL}}(\mathfrak{X}) \subset \Omega^p_{\mathrm{HL}}(\mathfrak{X}) \quad \text{or} \quad \bar{\partial}\Omega^p_{\mathrm{HL}}(\mathfrak{X}) \cap \Omega^p_{\mathrm{HL}}(\mathfrak{X}) = \emptyset.$$

In fact, Proposition 2 combined with Lemma 5 and Lemma 6 can provide a wide range of other cases. For example, if F is strictly concave, then for a given set $E \subset \mathrm{Dom(F)}$ such that $\overline{E}$ is countable we can find $\mathfrak{X}$ such that $\mathcal{M}_{\mathfrak{X}}$ is bounded from $L^{p,1/u}(\mathfrak{X})$ to $L^{p,1/F(u)}(\mathfrak{X})$ if and only if $u \notin E$. Nevertheless, it is probably very difficult to describe precisely all forms that the intersections $\bar{\partial}\Omega^p_{\mathrm{HL}}(\mathfrak{X}) \cap \Omega^p_{\mathrm{HL}}(\mathfrak{X})$ can take.

Appendix. Proof of Theorem 3

Here we give an elementary proof of Theorem 3. In what follows we replace $\mathcal{M}_{\mathfrak{X}}$ by an arbitrary operator $\mathcal{H}$ such that, if f is of the form $f_0 + f_1$ for $f_i \in L^{p,q_i}(\mathfrak{X})$, $i \in \{0,1\}$, then $\mathcal{H}f$ is well defined, measurable, and satisfies $|\mathcal{H}f| \le 2^{n_{\mathcal{H}}}(|\mathcal{H}f_0| + |\mathcal{H}f_1|)$ for some numerical $n_{\mathcal{H}} \in \mathbb{N}$. These are the only properties needed in the proof.

First we observe that it suffices to consider the case $q_0 < q_1$ and $r_0 < r_1$. Indeed, in each of the remaining cases our claim is an easy consequence of Fact 3. Fix $\theta \in (0,1)$ and let $\mathbf{C}_{\rightarrow}$ be such that

$$\|\mathcal{H}g\|_{p,r_i} \le \mathbf{C}_{\rightarrow}\|g\|_{p,q_i}, \quad g \in L^{p,q_i}(\mathfrak{X}),\ i \in \{0,1\}.$$

Our aim is to show

$$\|\mathcal{H}g\|_{p,r_\theta} \le C\|g\|_{p,q_\theta} \tag{5}$$

for all $g \in L^{p,q_\theta}(\mathfrak{X})$ with C independent of g. For any measurable function $g \colon X \to \mathbb{C}$ we introduce $\mathcal{S}g, \mathcal{T}g \colon \mathbb{Z} \to [0,\infty]$ given by

$$\mathcal{S}g(n) \coloneqq 2^n d_g(2^n)^{1/p}, \quad n \in \mathbb{Z},$$

and

$$\mathcal{T}g(n) \coloneqq \mathcal{S}\mathcal{H}g(n) = 2^n d_{\mathcal{H}g}(2^n)^{1/p}, \quad n \in \mathbb{Z}.$$

We observe that for each $q \in [1,\infty]$ there is a numerical constant $\mathbf{C}_{\square}(p,q)$ such that

$$\mathbf{C}_{\square}(p,q)^{-1}\,\|\mathcal{S}g\|_q \le \|g\|_{p,q} \le \mathbf{C}_{\square}(p,q)\,\|\mathcal{S}g\|_q, \quad g \in L^{p,q}(\mathfrak{X}),$$

where $\|\cdot\|_q$ denotes the standard norm on $\ell^q(\mathbb{Z})$. Let

$$\mathbf{C}_{\square} \coloneqq \max\{\mathbf{C}_{\square}(p,q_0), \mathbf{C}_{\square}(p,q_\theta), \mathbf{C}_{\square}(p,q_1), \mathbf{C}_{\square}(p,r_0), \mathbf{C}_{\square}(p,r_\theta), \mathbf{C}_{\square}(p,r_1)\}.$$

Thus for each $i \in \{0,1\}$ we have

$$\|\mathcal{T} g\|_{r_i} \leq \mathbf{C}_{\square}^2 \, \mathbf{C}_{\rightarrow} \, \|\mathcal{S} g\|_{q_i} \tag{6}$$

and our goal is to prove

$$\|\mathcal{T} g\|_{r_\theta} \leq \widetilde{C} \|\mathcal{S} g\|_{q_\theta}, \tag{7}$$

which would imply (5) with $C = \widetilde{C}\mathbf{C}_{\square}^2$.

In order to deduce (7) from (6) we follow the classical proof of the Marcinkiewicz interpolation theorem for operators acting on the Lebesgue spaces (see [19, Theorem 1]). It turns out that this strategy can be successfully applied but we must take into account certain additional difficulties. Namely, our "map" is given by $\mathcal{S} g \mapsto \mathcal{T} g$, hence it cannot be interpreted as a well defined operator because there are usually many different functions with the same distribution function. Thus, we proceed with the details.

Assume that $r_1 < \infty$ and fix $f \in L^{p,q_\theta}(\mathfrak{X})$. For each $\lambda \in (0,\infty)$ we set $N_\lambda \coloneqq \{n \in \mathbb{Z} \colon \mathcal{S} f(n) > \lambda\}$. Also, for $n \in \mathbb{Z}$ let $E_n \coloneqq \{x \in X \colon |f(x)| \geq 2^n\}$. We define

$$f_0^\lambda \coloneqq f \cdot \sum_{n \in N_\lambda} \mathbb{1}_{E_n \setminus E_{n+1}}, \quad f_1^\lambda \coloneqq f \cdot \sum_{n \in \mathbb{Z} \setminus N_\lambda} \mathbb{1}_{E_n \setminus E_{n+1}}.$$

Note that $f = f_0^\lambda + f_1^\lambda$. Moreover, we have $f_i^\lambda \in L^{p,q_1}(\mathfrak{X})$ because

$$\|\mathcal{S} f_i^\lambda\|_{q_i} \leq \big(1 + 2^{-q_i} + 4^{-q_i} + \cdots\big)^{1/q_i} \, \|(\mathcal{S} f)_i^\lambda\|_{q_i}, \tag{8}$$

where $(\mathcal{S} f)_0^\lambda \coloneqq \mathcal{S} f \cdot \mathbb{1}_{N_\lambda}$ and $(\mathcal{S} f)_1^\lambda \coloneqq \mathcal{S} f \cdot \mathbb{1}_{\mathbb{Z} \setminus N_\lambda}$. Indeed, to verify (8) we note that $d_{f_0^\lambda}(2^n) = d_{f_0^\lambda}(2^{n+1})$ when $n \in \mathbb{Z} \setminus N_\lambda$, and analogously for f_1^λ and $n \in N_\lambda$. Then for $n \in N_\lambda$ we have $\mathcal{S} f_0^\lambda(n) \leq \mathcal{S} f(n) = (\mathcal{S} f)_0^\lambda(n)$, while if $n \in \mathbb{Z} \setminus N_\lambda$, then either $\mathcal{S} f_0^\lambda(n) = 0$ or there exists $j \in \mathbb{N}$ such that $n + j \in N_\lambda$ and $d_{f_0^\lambda}(2^n) = d_{f_0^\lambda}(2^{n+j})$, so that $\mathcal{S} f_0^\lambda(n) = 2^{-j} \mathcal{S} f_0^\lambda(n+j) \leq 2^{-j} (\mathcal{S} f)_0^\lambda(n+j)$ follows. Similar arguments works for $\mathcal{S} f_1^\lambda$. Summing all these estimates with respect to the appropriate norms gives (8).

Next we study the distribution functions of $(\mathcal{S} f)_i^\lambda$, $i \in \{0,1\}$, more carefully. Observe that we have $d_{(\mathcal{S} f)_0^\lambda}(y) \leq d_{\mathcal{S} f}(\lambda)$ for $y \in (0,\lambda)$ and $d_{(\mathcal{S} f)_0^\lambda}(y) \leq d_{\mathcal{S} f}(y)$ for $y \in (\lambda,\infty)$. Combining both estimates and the fact that $d_{(\mathcal{S} f)_0^\lambda}$ is nonincreasing with the equality

$$2^{q_0} \int_0^{\lambda/2} y^{q_0 - 1} \, \mathrm{d}y = \int_0^\lambda y^{q_0 - 1} \, \mathrm{d}y,$$

we conclude that

$$\begin{aligned} \int_0^\infty y^{q_0-1} d_{(\mathcal{S} f)_0^\lambda}(y) \, \mathrm{d}y &\leq \frac{2^{q_0}}{2^{q_0} - 1} \int_{\lambda/2}^\infty y^{q_0 - 1} d_{\mathcal{S} f}(y) \, \mathrm{d}y \\ &\leq 2^{q_0} \int_{\lambda/4}^\infty (y - \tfrac{\lambda}{4})^{q_0 - 1} d_{\mathcal{S} f}(y) \, \mathrm{d}y. \end{aligned} \tag{9}$$

Similarly, we note that $d_{(\mathcal{S} f)_1^\lambda}(y) \leq d_{\mathcal{S} f}(y)$ for $y \in (0,\lambda)$ and $d_{(\mathcal{S} f)_1^\lambda}(y) = 0$ for $y \in (\lambda,\infty)$, which gives

$$\int_0^\infty y^{q_1-1} d_{(\mathcal{S} f)_1^\lambda}(y) \, \mathrm{d}y \leq \int_0^\lambda y^{q_1-1} d_{\mathcal{S} f}(y) \, \mathrm{d}y \leq 2^{2q_1} \int_0^{\lambda/4} y^{q_1-1} d_{\mathcal{S} f}(y) \, \mathrm{d}y. \tag{10}$$

Now we turn our attention to $\mathcal{T} f$. Fix $y \in (0,\infty)$ and $\lambda = \lambda(y)$ which will be specified later on. We have $\mathcal{T} f(n) \leq 2^{1+n_\mathcal{H}} (\mathcal{T} f_0^\lambda(n - 1 - n_\mathcal{H}) + \mathcal{T} f_1^\lambda(n - 1 - n_\mathcal{H}))$

for each $n \in \mathbb{N}$ thanks to $|\mathcal{H}f| \le 2^{n_{\mathcal{H}}}(|\mathcal{H}f_0^\lambda| + |\mathcal{H}f_1^\lambda|)$ and $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$ for $a, b \in [0, \infty)$. Hence

$$d_{\mathcal{T}f}(y) \le d_{\mathcal{T}f_0^\lambda}(y/2^{2+n_{\mathcal{H}}}) + d_{\mathcal{T}f_1^\lambda}(y/2^{2+n_{\mathcal{H}}}). \tag{11}$$

By the hypothesis we obtain

$$d_{\mathcal{T}f_i^\lambda}(y/2^{2+n_{\mathcal{H}}}) \le \frac{\|\mathcal{T}f_i^\lambda\|_{r_i}^{r_i}}{(y/2^{2+n_{\mathcal{H}}})^{r_i}} \le \left(2^{2+n_{\mathcal{H}}}\mathbf{C}_\square^2 \mathbf{C}_\to\right)^{r_i} \frac{\|\mathcal{S}f_i^\lambda\|_{q_i}^{r_i}}{y^{r_i}}. \tag{12}$$

Therefore, combining (8), (9), (10), (11), and (12) gives

$$\begin{aligned}
\|\mathcal{T}f\|_{r_\theta}^{r_\theta} &= r_\theta \int_0^\infty y^{r_\theta - 1} d_{\mathcal{T}f}(y)\,\mathrm{d}y \\
&\le C' \left(\int_0^\infty y^{r_\theta - r_0 - 1} \left(\int_{\lambda(y)/4}^\infty (t - \lambda(y)/4)^{q_0 - 1} d_{\mathcal{S}f}(t)\,\mathrm{d}t \right)^{r_0/q_0} \mathrm{d}y \right. \\
&\qquad \left. + \int_0^\infty y^{r_\theta - r_1 - 1} \left(\int_0^{\lambda(y)/4} t^{q_1 - 1} d_{\mathcal{S}f}(t)\,\mathrm{d}t \right)^{r_1/q_1} \mathrm{d}y \right),
\end{aligned}$$

with some constant C' which may depend on $p, q_0, q_1, r_0, r_1, \theta, n_{\mathcal{H}}$, and $\mathbf{C}_\to$ but is independent of f and the choice of $\lambda(y)$ (we only need that $y \mapsto \lambda(y)$ is measurable).

It is worth noting here that the inequality above reduces the problem to estimating the expression of the form very similar to that appearing in [19, (3.7)] (here $d_{\mathcal{S}f}$, $\lambda/4$, q_0, q_1, r_0, r_1, and r_θ play the roles of m, z, a_2, a_1, b_2, b_1, and b, respectively). Thus, in order to obtain (7), we may repeat the remaining calculations without any further changes. We briefly sketch the rest of the proof for the sake of completeness.

Denote by P and Q the two double integrals in the last estimate. Then

$$P^{q_0/r_0} = \sup_{\omega_0} \int_0^\infty y^{r_\theta - r_0 - 1} \int_{\lambda(y)/4}^\infty (t - \lambda(y)/4)^{q_0 - 1}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t\, \omega_0(y)\,\mathrm{d}y$$

and

$$Q^{q_1/r_1} = \sup_{\omega_1} \int_0^\infty y^{r_\theta - r_1 - 1} \int_0^{\lambda(y)/4} t^{q_1 - 1}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t\, \omega_1(y)\,\mathrm{d}y,$$

where the functions $\omega_i \ge 0$ satisfy

$$\int_0^\infty y^{r_\theta - r_i - 1} \omega_i^{(r_i/q_i)'}(y)\,\mathrm{d}y \le 1,$$

with $(r_i/q_i)'$, the exponent conjugate to r_i/q_i. We set $\lambda(y) \coloneqq 4\|\mathcal{S}f\|_{q_\theta}^{-\tau\xi} y^\xi$, where τ and ξ will be determined later on. Now, by using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^\infty y^{r_\theta - r_0 - 1} \int_{\|\mathcal{S}f\|_{q_\theta}^{-\tau\xi} y^\xi}^\infty (t - \|\mathcal{S}f\|_{q_\theta}^{-\tau\xi} y^\xi)^{q_0 - 1}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t\, \omega_0(y)\,\mathrm{d}y \\
&\le \int_0^\infty t^{q_0 - 1}\, d_{\mathcal{S}f}(t) \int_0^{\|\mathcal{S}f\|_{q_\theta}^\tau t^{\frac{1}{\xi}}} y^{r_\theta - r_0 - 1} \omega_0(y)\,\mathrm{d}y\,\mathrm{d}t \\
&\le \int_0^\infty t^{q_0 - 1} d_{\mathcal{S}f}(t) \left(\int_0^{\|\mathcal{S}f\|_{q_\theta}^\tau t^{\frac{1}{\xi}}} y^{r_\theta - r_0 - 1} \mathrm{d}y \right)^{\frac{q_0}{r_0}} \left(\int_0^{\|\mathcal{S}f\|_{q_\theta}^\tau t^{\frac{1}{\xi}}} y^{r_\theta - r_0 - 1} \omega_0^{(\frac{r_0}{q_0})'}(y)\mathrm{d}y \right)^{\frac{1}{(r_0/q_0)'}} \mathrm{d}t \\
&\le (r_\theta - r_0)^{-q_0/r_0} \|\mathcal{S}f\|_{q_\theta}^{\frac{(r_\theta - r_0)q_0\tau}{r_0}} \int_0^\infty t^{q_0 - 1 + \frac{(r_\theta - r_0)q_0}{\xi r_0}}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t.
\end{aligned}$$

Similarly, we obtain

$$\int_0^\infty y^{r_\theta - r_1 - 1} \int_0^{\|\mathcal{S}f\|_{q_\theta}^{-\tau\xi} y^\xi} t^{q_1-1}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t\,\omega_1(y)\,\mathrm{d}y$$
$$\le (r_1 - r_\theta)^{-q_1/r_1} \|\mathcal{S}f\|_{q_\theta}^{\frac{(r_\theta - r_1)q_1\tau}{r_1}} \int_0^\infty t^{q_1 - 1 + \frac{(r_\theta - r_1)q_1}{\xi r_1}}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t.$$

Collecting these results we conclude that

$$\|\mathcal{T}f\|_{r_\theta}^{r_\theta} \le C'' \sum_{i\in\{0,1\}} \|\mathcal{S}f\|_{q_\theta}^{(r_\theta - r_i)\tau} \left(\int_0^\infty t^{q_i - 1 + \frac{(r_\theta - r_i)q_i}{r_i \xi}}\, d_{\mathcal{S}f}(t)\,\mathrm{d}t \right)^{r_i/q_i},$$

for some C'' independent of f. Choosing

$$\tau := \frac{q_\theta(r_1/q_1 - r_0/q_0)}{r_1 - r_0}, \quad \xi := \frac{q_\theta^{-1}(r_1^{-1} - r_\theta^{-1})}{r_\theta^{-1}(q_1^{-1} - q_\theta^{-1})}, \tag{13}$$

gives that both terms in the sum above equal $\|\mathcal{S}f\|_{q_\theta}^{r_\theta}$. Thus (7) holds with $\widetilde{C} = (2C'')^{1/r_\theta}$, which completes the proof in the case $r_1 < \infty$.

Finally, let us assume that $r_1 = \infty$. If $q_1 = \infty$, then the formulas in (13) reduce to

$$\tau = 0, \quad \xi = 1.$$

We choose $\lambda(y) := cy$ for some sufficiently small constant $c \in (0,\infty)$. In fact, if $c < \mathbf{C}_{\to}^{-1}\mathbf{C}_{\square}^{-2} 2^{-2-n_{\mathcal{H}}}$, then we have $d_{\mathcal{T}f_1^\lambda}(y/2^{2+n_{\mathcal{H}}}) = 0$, while $d_{\mathcal{T}f_0^\lambda}(y/2^{2+n_{\mathcal{H}}})$ may be estimated as before. On the other hand, if $q_1 < \infty$, then the formulas in (13) reduce to

$$\tau = q_\theta/q_1, \quad \xi = q_1/(q_1 - q_\theta).$$

Again, it can be shown that if $\lambda(y) := c'\|f\|_{q_\theta}^{-q_\theta/(q_1-q_\theta)} y^{q_1/(q_1-q_\theta)}$, where $c' \in (0,\infty)$ is sufficiently small (but independent of f and y), then $d_{\mathcal{T}f_1^\lambda}(y/2^{2+n_{\mathcal{H}}}) = 0$ and $d_{\mathcal{T}f_0^\lambda}(y/2^{2+n_{\mathcal{H}}})$ may be estimated as before. This completes the proof in the case $r_1 = \infty$.

References

[1] Aldaz, J. M.: An example on the maximal function associated to a nondoubling measure. - Publ. Mat. 49, 2005, 453–458.

[2] Bennett, C., and R. Sharpley: Interpolation of operators. - Pure Appl. Math. 129, Academic Press, Inc., Boston, MA, 1988.

[3] Bergh, J., and J. Löfström: Interpolation spaces: An introduction. - Grundlehren Math. Wiss. 223, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[4] Folland, G. B.: Real analysis: Modern techniques and their applications. - Wiley, New York, 1999.

[5] Hunt, R.: An extension of the Marcinkiewicz interpolation. - Bull. Amer. Math. Soc. 70, 1964, 803–807.

[6] Janson, S.: On the interpolation of sublinear operators. - Studia Math. 75, 1982, 51–53.

[7] Kosz, D.: On relations between weak and strong type inequalities for maximal operators on non-doubling metric measure spaces. - Publ. Mat. 62, 2018, 37–54.

[8] Kosz, D.: On relations between weak and restricted weak type inequalities for maximal operators on non-doubling metric measure spaces. - Studia Math. 241, 2018, 57–70.

[9] Kosz, D.: Maximal operators on Lorentz spaces in non-doubling setting. - Math. Z. 298, 2021, 1523–1543.

[10] Li, H.-Q.: La fonction maximale de Hardy–Littlewood sur une classe d'espaces métriques mesurables. - C. R. Math. Acad. Sci. Paris 338, 2004, 31–34.

[11] Li, H.-Q.: La fonction maximale non centrée sur les variétés de type cuspidale. - J. Funct. Anal. 229, 2005, 155–183.

[12] Li, H.-Q.: Les fonctions maximales de Hardy–Littlewood pour des mesures sur les variétés cuspidales. - J. Math. Pures Appl. 88, 2007, 261–275.

[13] Maligranda, L.: The K-functional for symmetric spaces. - Lecture Notes in Math. 1070, 1984, 169–182.

[14] Nazarov, F., S. Treil, and A. Volberg: Weak type estimates and Cotlar inequalities for Calderón–Zygmund operators on nonhomogeneous spaces. - Int. Math. Res. Not. IMRN 9, 1998, 463–487.

[15] Sawano, Y.: Sharp estimates of the modified HardyŨLittlewood maximal operator on the nonhomogeneous space via covering lemmas. - Hokkaido Math. J. 34, 2005, 435–458.

[16] Sjögren, P.: A remark on the maximal function for measures in $\mathbb{R}^n$. Amer. J. Math. **105** (1983), 1231–1233.

[17] Stempak, K.: Modified Hardy–Littlewood maximal operators on nondoubling metric measure spaces. - Ann. Acad. Sci. Fenn. Math. 40, 2015, 443–448.

[18] Stempak, K.: Examples of metric measure spaces related to modified Hardy–Littlewood maximal operators. - Ann. Acad. Sci. Fenn. Math. 41, 2016, 313–314.

[19] Zygmund, A.: On a theorem of Marcinkiewicz concerning interpolation of operations. - J. Math. Pures Appl. 34, 1956, 223–248.

Received 14 August 2021 • Revision received 10 July 2023 • Accepted 12 July 2023
Published online 28 July 2023

Dariusz Kosz
Basque Center for Applied Mathematics
48009 Bilbao, Spain
and Wrocław University of Science and Technology
50-370 Wrocław, Poland
dkosz@bcamath.org