Solutions with multiple peaks for nonlinear Kirchhoff equations on $\mathbb{R}^3$

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Abstract. In this paper, we mainly investigate the following nonlinear Kirchhoff equation
\[
\begin{cases}
-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = Q(x)u^{q-1}, & u > 0, \ x \in \mathbb{R}^3, \\
\lim_{|x| \to +\infty} u = 0,
\end{cases}
\]
where $a, b > 0$ are constants, $2 < q < 6$, and $\varepsilon > 0$ is a parameter. Under some suitable assumptions on the function $Q(x)$, we obtain that the equation above has positive multi-peak solutions concentrating at a critical point of $Q(x)$ for $\varepsilon > 0$ sufficiently small, by using the finite dimensional reduction method. Different from the local Schrödinger problem, here the corresponding limit problem is a system. Moreover, the nonlocal term brings some new difficulties which involve some technical and complicated estimates.

1. Introduction and main results

In this paper, we consider the following Kirchhoff equation
\[
\begin{cases}
-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = Q(x)u^{q-1}, & u > 0, \ x \in \mathbb{R}^3, \\
\lim_{|x| \to +\infty} u = 0,
\end{cases}
\]
where $a, b > 0$ are constants, $2 < q < 6$, $\varepsilon > 0$ is a parameter and $Q(x)$ is a smooth bounded function.

Problem (1.1) and its variants have been studied extensively in the literature. To extend the classical D’Alembert’s wave equations for free vibration of elastic strings,
Kirchhoff [22] first proposed the following time-dependent wave equation
\[
\rho \frac{\partial^2 u}{\partial^2 t} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial^2 x} = 0.
\]

Bernstein [4] and Pohozaev [31] studied the above type of Kirchhoff equations quite early. In order to study the problem (1.2) preferably, Lions [25] introduced an abstract functional framework to this problem. After that, many interesting results of Kirchhoff equations can be found in e.g. [2, 18, 13, 17, 12, 33] and the reference therein. From a mathematical point of view, Kirchhoff equation is nonlocal, in the sense that the term \( \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u \) depends not only on \( \Delta u \), but also on the integral of \( |\nabla u|^2 \) over the whole space. This feature brings new mathematical difficulties, which makes the study of Kirchhoff type equations particularly meaningful. We refer to e.g. [19, 20, 21, 30] for mathematical researches on Kirchhoff type equations in bounded domains and in the whole space.

In fact, equation (1.1) is closely related to the Schrödinger equations. When \( a = 1 \) and \( b = 0 \), equation (1.1) is reduced to the perturbed Schrödinger equation. Kwong [23] considered the classical Schrödinger equation
\[
- \Delta u + u = u^p, \quad x \in \mathbb{R}^N,
\]
where \( 1 < p < +\infty \) if \( N = 1, 2 \), and \( 1 < p < \frac{N+2}{N-2} \) if \( N \geq 3 \). Equation (1.3) has a unique radial symmetric and nondegenerate positive solution. Based on this property, Cao, Noussair and Yan [6] proved the existence of multi-peak solutions for equation
\[
- \Delta u + \lambda^2 u = Q(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N,
\]
where \( \lambda \neq 0 \), \( N \geq 3 \) and \( 2 < q < 2N/(N-2) \).

Dancer and Yan [10] studied the following equation
\[
\begin{cases}
-\varepsilon^2 \Delta u + u = Q(y)w^{p-1}, & u > 0, \; y \in \mathbb{R}^N, \\
u \to 0, & \text{as } |y| \to +\infty,
\end{cases}
\]
where \( \varepsilon > 0 \) is a parameter, \( 2 < p < +\infty \) if \( N = 2 \) and \( 2 < p < 2N/(N-2) \) if \( N > 2 \). They not only proved that the Schrödinger equation has positive multi-peak solutions concentrating at a designated saddle point or a strictly local minimum point of \( Q(y) \) in \( \mathbb{R}^N \), but also showed that there is no multi-peak solution concentrating at a strictly local maximum point of \( Q(y) \) in \( \mathbb{R}^N \). Besides, many interesting results of multi-peak solutions can be found in e.g. [32, 14, 5, 16, 11, 15, 26, 28, 29] and the reference therein.

Based on the uniqueness and nondegeneracy property of equation (1.3), the authors in [24] proved the uniqueness and nondegeneracy of positive solutions for equation
\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = u^p, \quad u > 0 \text{ in } \mathbb{R}^3,
\]
where \( 1 < p < 5 \). Then, by using Lyapunov–Schmidt reduction method, they constructed the existence and the uniqueness of single peak solutions to equation
\[
- \left( \varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = u^p, \quad u > 0 \text{ in } \mathbb{R}^3,
\]
where \( 1 < p < 5 \) and \( \varepsilon > 0 \) is sufficiently small.
Luo, Peng and Wang [27] proved equation (1.7) has positive multi-peak solutions concentrating at different points if $\varepsilon > 0$ is sufficiently small. It should be pointed out that, they constructed the multi-peak solutions of equation (1.7) based on the following system

$$
(1.8) \quad (a + b \sum_{j=1}^{k} \int_{\mathbb{R}^3} |\nabla u_j|^2) \Delta u_i + V(a_i)u_i = (u_i)^p, \quad u_i > 0, \ i = 1, \ldots, k, \ x \in \mathbb{R}^3.
$$

They also showed that $a_i$ ($1 \leq i \leq k$) are critical points of $V$, and there exist the multi-peak solutions of the form $u_\varepsilon = \sum_{i=1}^{k} u_i(\frac{x-y_{i,\varepsilon}}{\varepsilon}) + \varphi_\varepsilon$.

Note that in [27], they did not consider the existence of multi-peak solutions concentrating at the same point ($a_1 = \cdots = a_k$) to equation (1.7). When $a_1 = a_2 = \cdots = a_k$, $|y_{i,\varepsilon} - y_{j,\varepsilon}| \to 0$ ($i \neq j$) as $\varepsilon \to 0$, but we cannot get the result of $|y_{i,\varepsilon} - y_{j,\varepsilon}|/\varepsilon \to +\infty$ ($i \neq j$) as $\varepsilon \to 0$. Therefore, we must impose additional conditions on $y_{i,\varepsilon}$.

Recently, Cui et al. [7] proved the existence and local uniqueness of normalized multi-peak solutions to the following Kirchhoff equation

$$
(1.9) \quad \begin{cases} 
- (a + b_\varepsilon \int_{\mathbb{R}^3} |\nabla u_\lambda|^2) \Delta u_\lambda + (\lambda + V(x))u_\lambda = \beta_\lambda u_\lambda^p, & u_\lambda > 0, \ x \in \mathbb{R}^3, \\
\lambda \in H^1(\mathbb{R}^3),
\end{cases}
$$

where $a > 0$, $1 < p < 5$, $\lambda$, $b_\varepsilon$, $\beta_\lambda > 0$ are parameters, $\int_{\mathbb{R}^3} u_\lambda^2 = 1$, and $V(x): \mathbb{R}^3 \to \mathbb{R}$ is a bounded continuous function.

Inspired by the literatures [10, 24, 27, 7], we apply the conditions of $Q(x)$ in [10] to $\mathbb{R}^3$. Then, we consider the existence of positive multi-peak solutions to the equation (1.1) concentrating at a critical point of $Q(x)$.

Now we give some definitions and assumptions.

**Definition 1.1.** Let $m \in N_+$, $a_0 \in \mathbb{R}^3$, we say that $u_\varepsilon$ is a m-peak solution of (1.1) if $u_\varepsilon$ satisfies

(i) $u_\varepsilon$ has $m$ local maximum points $y_{\varepsilon,i} \in \mathbb{R}^3$, $i = 1, \ldots, m$, satisfying

$$
y_{\varepsilon,i} \to a_0
$$

as $\varepsilon \to 0$ for each $i$;

(ii) For any given $\tau > 0$, there exists $R > 1$, such that

$$
|u_\varepsilon(x)| \leq \tau, \quad x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^{m} B_R(y_{\varepsilon,i});
$$

(iii) There exists $C > 0$ such that

$$
\int_{\mathbb{R}^3} (\varepsilon^2 a |\nabla u_\varepsilon|^2 + u_\varepsilon^2) \leq C \varepsilon^3.
$$

We assume that $Q(x)$ satisfies the following conditions:

(Q1) $Q(x)$ is a smooth bounded function in $\mathbb{R}^3$.

(Q2) $x_0 = 0$ is a critical point of $Q(x)$.

(Q3) $Q(x)$ has the following expansion (after suitably rotating the coordinate system)

$$
(1.10) \quad Q(x) = Q(0) + P_1(x') - P_2(x'') + R(x), \quad x \in B_\delta(0),
$$

where $Q(0) > 0$, $\delta > 0$, $x = (x', x'')$, $x' = (x_1, \ldots, x_l)$, $x'' = (x_{l+1}, \ldots, x_3)$, $t \in \{1, 2, 3\}$, $P_1$ and $P_2$ satisfy

$$
(1.11) \quad P_1(x') = \lambda |x'|^{h_1}, \quad |x'| \leq \delta,
$$
Then we want to construct k-peak solutions to equation (1.1) concentrating at the critical point \( x \).

The energy functional corresponding to equation (1.1) is

\[
I_\varepsilon(u) = \frac{1}{2} \|u\|_\varepsilon^2 + \frac{\varepsilon b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \frac{1}{q} \int_{\mathbb{R}^3} Q(x) u^q, \quad u \in H_\varepsilon,
\]

where \( u_+ = \max(u, 0) \). It is standard to verify that \( I_\varepsilon \in C^2(H_\varepsilon) \). So we just need to find a critical point of \( I_\varepsilon \in C^2(H_\varepsilon) \).

For \( k \in N_+ \), let \( w \) be the unique positive radial solution (see Lemma 2.2) to equation

\[
\Delta u + u = Q(0) u^{q-1}.
\]

Then we want to construct k-peak solutions to equation (1.1) concentrating at the critical point \( x_0 = 0 \) of \( Q(x) \) by using the uniqueness and nondegeneracy property of \( w \).

Our main result is as follows.

**Theorem 1.3.** Assume that \( Q(x) \) satisfies \((Q_1)\)–\((Q_3)\). Then, for any \( k \in N_+ \), there exists \( \varepsilon_0 = \varepsilon(k) \) such that for \( \varepsilon \in (0, \varepsilon_0] \), equation (1.1) has at least one k-peak solution of the form

\[
u_\varepsilon = \sum_{i=1}^{k} w_{\varepsilon,y_{\varepsilon,i}} + \varphi_\varepsilon,
\]

where \( y_{\varepsilon,i} \in \mathbb{R}^3, \ i = 1, \ldots, k, \ \varphi_\varepsilon \in H^1(\mathbb{R}^3), \) and as \( \varepsilon \to 0 \),

\[
y_{\varepsilon,i} \to 0, \quad i = 1, \ldots, k,
\]

\[
\frac{|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon} \to +\infty, \quad i \neq j, \ i, j = 1, \ldots, k,
\]

\[
\|\varphi_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{q}{2}}).
\]

**Remark 1.4.** Theorem 1.3 extend the result got by Dancer and Yan [10] about the existence of solutions for the nonlinear Schrödinger equation to the nonlinear Kirchhoff equation (1.1).

In this paper, we prove Theorem 1.3 by using the finite dimensional reduction method. Although the method is standard, we have to make some modifications. Since there is a nonlocal term, we encounter some new difficulties which involve
some complicated and technical estimates. To our knowledge, the result we obtain is new.

Our notations are standard. We use $B_r(x)$ (and $\overline{B_r(x)}$) to denote open (and close) balls in $\mathbb{R}^3$ centred at $x$ with radius $r$, and $B_r^c(x)$ to denote the complementary set of $B_r(x)$ in $\mathbb{R}^3$. Unless otherwise stated, we write $\int u$ to denote Lebesgue integrals over $\mathbb{R}^3$, and $\|u\|_{L^p}$, $\|u\|_{H^1}$ to mean $L^p$-norm, $H^1$-norm respectively. We will use $C, C_j (j \in \mathbb{N})$ to denote various positive constants, and $O(t), o(t), o(t)/t \to 0, o(t)/t \to 0$ as $t \to 0$ and $o_R(1) \to 0$ as $R \to +\infty$ respectively.

This paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, we give the finite dimensional reduction process and in Section 4, we prove Theorem 1.3.

2. Some preliminaries

In this section, we introduce some preliminaries.

**Lemma 2.1.** [27, Lemma 2.1] For any $2 \leq q \leq 6$, there exists a constant $C > 0$ independent of $\varepsilon$, such that

$$
\|u\|_{L^q} \leq C \varepsilon^{\frac{3}{q} - \frac{3}{2}} \|u\|_{\varepsilon}, \quad \forall u \in H_{\varepsilon}.
$$

Before stating the lemma that follows, we first give a truth that $U$ is a unique positive radial solution to equation

$$
\begin{cases}
-\Delta u + u = u^{q-1}, & x \in \mathbb{R}^3, \\
u \in H^1(\mathbb{R}^3), \\
u(0) = \max_{x \in \mathbb{R}^3} u(x),
\end{cases}
$$

which satisfies

$$
\begin{cases}
\lim_{|x| \to \infty} \frac{|x| e^{|x|} U(|x|)}{|x|^2} = C > 0, \\
\lim_{|x| \to \infty} \frac{U'(|x|)}{U(|x|)} = -1.
\end{cases}
$$

The following result is very crucial for applying the finite dimensional reduction.

**Lemma 2.2.** [24, Theorem 1.2] For any $k \in \mathbb{N}_+$, $a, b > 0$, $2 < q < 6$, and $Q(x)$ satisfying $(Q_1)-(Q_3)$, there exists a unique positive radial solution $w \in H^1(\mathbb{R}^3)$ satisfying

$$
- \left( a + bk \int_{\mathbb{R}^3} |\nabla w|^2 \right) \Delta w + u = Q(0) u^{q-1}.
$$

Moreover, $w$ is nondegenerate in $H^1(\mathbb{R}^3)$ in the sense that there holds

$$
\text{Ker } L = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3} \right\},
$$

where $L: H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is the linear operator defined as

$$
L \varphi = - \left( a + bk \int_{\mathbb{R}^3} |\nabla w|^2 \right) \Delta \varphi + 2bk \left( \int \nabla w \nabla \varphi \right) \Delta w + \varphi - (q-1)Q(0) w^{q-2} \varphi, \quad \forall \varphi \in H^1(\mathbb{R}^3).
$$
The proof of Theorem 1.2 in [24] implies that \( w(x) = \lambda U(\eta x) \) where \( \eta = \frac{1}{\sqrt{|a|}} \) and \( c_k = a + bk \int |\nabla u|^2 \). And since \( U \) decays exponentially at infinity, we infer that
\[
\nabla w(x), \quad w(x) = O( |x|^{-1} e^{-\eta |x|} ) \quad \text{as } |x| \to \infty.
\]

**Lemma 2.3.** [1, Lemma 3.7] Assume \( u, u': \mathbb{R}^n \to \mathbb{R} \) are positive radial continuous functions satisfying
\[
u u(x) \sim |x|^a e^{-b|x|}, \quad u'(x) \sim |x|^a e^{-b'|x|} \quad (|x| \to \infty),
\]
where \( a, a' \in \mathbb{R}, \; b, b' > 0 \). Let \( \xi \in \mathbb{R}^n \) tend to infinity and \( u_\xi(x) = u(x - \xi) \). Then the following asymptotic estimates hold:

(i) If \( b < b' \), then
\[
\int_{\mathbb{R}^n} u_\xi u' \sim e^{-b|x|} |\xi|^a.
\]

If \( b > b' \), then replace \( a \) and \( b \) with \( a' \) and \( b' \).

(ii) If \( b = b' \), suppose, for simplicity, that \( a \geq a' \), then
\[
\int_{\mathbb{R}^n} u_\xi u' \sim \begin{cases}
 e^{-b|x|} |\xi|^{a+a'+n+1}, & a' > -\frac{n+1}{2}; \\
 e^{-b|x|} |\xi|^a \log |\xi|, & a' = -\frac{n+1}{2}; \\
 e^{-b|x|} |\xi|^a, & a' < -\frac{n+1}{2}.
\end{cases}
\]

Combining (2.5) and Lemma 2.3 yields, for any \( r, s > 0 \) with \( r \neq s \), we have, as \( \varepsilon \to 0 \),
\[
\int_{\mathbb{R}^n} w_r^{\varepsilon,y_i} w_s^{\varepsilon,y_j} = O \left( \varepsilon^3 e^{-\min\{r,s\}} |\eta_{1-y_{1-y_j}}^{y_{1-y_j}}| |y_i - y_j|^{-\min\{r,s\}} \right),
\]
\[
\int_{\mathbb{R}^3} |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| = O \left( \varepsilon e^{-\frac{|y_{1-y_j}|}{\varepsilon}} \right).
\]

Particularly, there exist some constants \( C > 0 \), such that
\[
\int_{\mathbb{R}^3} w_r^{\varepsilon,y_i} w_s^{\varepsilon,y_j} \leq C \varepsilon^3 e^{-\min\{r,s\}} |\eta_{1-y_{1-y_j}}^{y_{1-y_j}}|,
\]
\[
\int_{\mathbb{R}^3} |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| \leq C \varepsilon e^{-\frac{|y_{1-y_j}|}{\varepsilon}}.
\]

In the following sections, we will use inequalities (2.1), (2.8) and (2.9) repeatedly. When \( r = s \), the situation is complicated. But when \( r = s \geq 1 \), inequality (2.8) still holds.

**Definition 2.4.** [8, Definition B.1] Let \( Y \) and \( A \) be closed subsets of a topological space \( X \). Then \( Cat_X(A, Y) \) is the least integer \( k \) such that \( A = \bigcup_{j=0}^{k} A_j \), where, for \( 0 \leq j \leq k \), \( A_j \) is closed, and there exists \( h_j \in C([0,1] \times A_j, X) \) such that

(i) \( h_j(0,x) = x \) for \( x \in A_j \), \( 0 \leq j \leq k \);

(ii) \( h_0(1,x) \in Y \) for \( x \in A_0 \) and \( h_0(t,x) = x \) for \( x \in A_0 \cap Y \) and \( t \in [0,1] \);

(iii) \( h_j(1,x) = x_j \) for \( x \in A_j \) and some \( x_j \in X \), \( 1 \leq j \leq k \).

Particularly, if \( Y \) is empty, then write \( Cat_X(A) = Cat_X(A, \emptyset) \).

From the Definition 2.4, we see \( Cat_X(A, Y) \geq 1 \) if \( A \) can not be deformed into a subset of \( Y \) within \( X \).
Lemma 2.5. [8, Proposition 2.2] Suppose that $F(x)$ is a $C^2$ function defined in a bounded domain $\Omega \subset \mathbb{R}^k$. If $F$ satisfies either $F(x) > c$ or $\frac{\partial F(x)}{\partial n} > 0$ at each $x \in \partial \Omega$, where $n$ is the outward unit normal of $\partial \Omega$ at $x$, then

\[
(2.10) \quad \# \{x : DF(x) = 0, \; x \in F^c\} \geq \text{Cat}_{F^c}(F^c),
\]

where $F^c = \{x : x \in \Omega, \; F(x) \leq c\}$. In particular, $F(x)$ has at least one critical point in $F^c$.

Lemma 2.6. [8, Proposition 2.3] Suppose that $F(x)$ is a $C^2$ function defined in a bounded domain $\Omega \subset \mathbb{R}^k$. Let $c_1, c_2$ be two constants such that neither $c_2$ nor $c_1$ is a critical value of $F(x)$. If $F$ satisfies either $F(x) < c_1$ or $\frac{\partial F(x)}{\partial n} > 0$ for each $x \in \partial \Omega$, then

\[
(2.11) \quad \# \{x : DF(x) = 0, \; x \in F^{c_2}\} \geq \text{Cat}_{F^{c_2}}(F^{c_2}, F^{c_1}).
\]

In particular, if $F^{c_2}$ cannot be deformed into $F^{c_1}$, $F$ has at least one critical point in $F^{c_2}\backslash F^{c_1}$.

Lemma 2.7. [9, Proposition B.1] Suppose that $X$ and $\Gamma$ are two compact sets in $\mathbb{R}^3$ satisfying $\Gamma \subset X$. Let

\[
(2.12) \quad K = \underbrace{X \times \cdots \times X}_k,
\]

\[
(2.13) \quad L_1 = \underbrace{\Gamma \times X \times \cdots \times X}_k \cup \underbrace{X \times \cdots \times X}_k \cup \underbrace{X \times \cdots \times X}_k \times \Gamma,
\]

\[
(2.14) \quad L_2 = L_1 \cup D, \quad D = \{Y = (y_1, \ldots, y_k) : y_i = y_j \text{ for some } i \neq j\}.
\]

If $H^m(X, \Gamma) \neq 0$ for some $m \geq 1$, then $H_s(K, L_2) \neq 0$. In particular, $K$ cannot be deformed into $L_2$.

3. Finite dimensional reduction

In this section we complete the finite dimensional reduction process. Denote

\[
Y = (y_1, \ldots, y_k) \in \mathbb{R}^3 \times \cdots \times \mathbb{R}^3, \quad W_{\varepsilon,Y} = \sum_{i=1}^k w_{\varepsilon,y_i},
\]

\[
D_{\varepsilon,\delta}^k = \left\{Y : y_i \in B_{\delta}(0), \; \frac{\eta |y_i - y_j|}{\varepsilon} \geq R_1, \; i, j = 1, \ldots, k, \; i \neq j \right\},
\]

where $R_1 > 0$ is a fixed large constant. Let

\[
E_{\varepsilon,Y}^k = \left\{\varphi \in H_\varepsilon : \left\langle \varphi, \frac{\partial w_{\varepsilon,y_i}}{\partial y_{ij}} \right\rangle_{\varepsilon} = 0, \; i = 1, \ldots, k, \; j = 1, 2, 3 \right\}
\]

and define

\[
J_\varepsilon(Y, \varphi) = I_\varepsilon(W_{\varepsilon,Y} + \varphi), \quad \forall (Y, \varphi) \in D_{\varepsilon,\delta}^k \times E_{\varepsilon,Y}^k.
\]

Expand $J_\varepsilon(Y, \varphi)$ near $\varphi = 0$ for each fixed $Y$:

\[
(3.1) \quad J_\varepsilon(Y, \varphi) = J_\varepsilon(Y, 0) + l_{\varepsilon,Y}(\varphi) + \frac{1}{2}Q_{\varepsilon,Y}(\varphi) + R_{\varepsilon,Y}(\varphi),
\]

where

\[
l_{\varepsilon,Y}(\varphi) = \langle L_\varepsilon'(W_{\varepsilon,Y}), \varphi \rangle
\]

\[
= \langle W_{\varepsilon,Y}, \varphi \rangle_\varepsilon + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int \nabla W_{\varepsilon,Y} \nabla \varphi - \int Q(x)W_{\varepsilon,Y}^{-1}(x) \varphi,
\]
\[ Q_{\varepsilon,Y}(\varphi) = \langle I_{\varepsilon''} (W_{\varepsilon,Y}) [\varphi], \varphi \rangle \]

\[ = \langle \varphi, \varphi \rangle_{\varepsilon} + 2\varepsilon b \left( \int |\nabla W_{\varepsilon,Y}|^2 \right)^2 + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 |\nabla \varphi|^2 - (q - 1) \int Q(x) W_{\varepsilon,Y}^{q-2} \varphi^2, \]

and

\[ R_{\varepsilon,Y}(\varphi) = \frac{\varepsilon b}{4} \left( \int |\nabla \varphi|^2 \right)^2 + \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi \int |\nabla \varphi|^2 - \frac{1}{q} \int Q(x) (W_{\varepsilon,Y} + \varphi)^q + \frac{1}{q} \int Q(x) W_{\varepsilon,Y}^{q-1} \varphi + \frac{1}{2} (q - 1) \int Q(x) W_{\varepsilon,Y}^{q-2} \varphi^2. \]

In terms of \( Q_{\varepsilon,Y}(\varphi) \), \( \mathcal{L}_{\varepsilon,Y} : E_{\varepsilon,Y}^k \to E_{\varepsilon,Y}^k \) is a bounded linear mapping defined by:

\[ \langle \mathcal{L}_{\varepsilon,Y} \varphi_1, \varphi_2 \rangle_{\varepsilon} = \langle \varphi_1, \varphi_2 \rangle_{\varepsilon} + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int \nabla \varphi_1 \nabla \varphi_2 \]

\[ + 2\varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi_1 \int \nabla W_{\varepsilon,Y} \nabla \varphi_2 \]

\[ - (q - 1) \int Q(x) W_{\varepsilon,Y}^{q-2} \varphi_1 \varphi_2, \quad \forall \varphi_1, \varphi_2 \in E_{\varepsilon,Y}^k. \]

The following result shows that \( \mathcal{L}_{\varepsilon,Y} \) is invertible when restricted on \( E_{\varepsilon,Y}^k \).

**Lemma 3.1.** There exist \( \rho > 0, \varepsilon_0 > 0 \) and \( \delta_0 > 0 \), such that for every \( \varepsilon \in (0, \varepsilon_0] \) and \( \delta \in (0, \delta_0] \), there holds

\[ \| \mathcal{L}_{\varepsilon,Y} \varphi \|_{\varepsilon} \geq \rho \| \varphi \|_{\varepsilon}, \quad \forall \varphi \in E_{\varepsilon,Y}^k, \]

uniformly with respect to \( Y \in D_{\varepsilon,\delta}^k \).

**Proof.** We use a contradiction argument. Assume that there exist \( \varepsilon_n \to 0, \delta_n \to 0, Y_n \in D_{\varepsilon_n,\delta_n}^k \) and \( \varphi_n \in E_n \equiv E_{\varepsilon_n,Y_n}^k \) such that

\[ \langle \mathcal{L}_{\varepsilon_n,Y_n} \varphi_n, h_n \rangle_{\varepsilon_n} = o_n(1) \| \varphi_n \|_{\varepsilon_n} \| h_n \|_{\varepsilon_n}, \quad \forall \ h_n \in E_n. \]

With no loss of generality, we assume that \( \| \varphi_n \|_{\varepsilon_n}^2 = \varepsilon_n^3 \), and denote

\[ \varphi_{n,i_0}(x) = \varphi_n(\varepsilon_n x + y_{n,i_0}), \quad i_0 = 1, 2, \ldots, k, \]

\[ \mathcal{E}_n = \{ h_{n,i_0}(x) : h_n(x) \in E_n, 1 \leq i_0 \leq k \}. \]

Substituting (3.5) into (3.6), we obtain

\[ \int \left( a \nabla \varphi_{n,i_0} \nabla h_{n,i_0} + \varphi_{n,i_0} h_{n,i_0} \right) \]

\[ + b \int \left( k |\nabla w|^2 + \sum_{i \neq j} \nabla w \left( x + \frac{y_{n,i_0} - y_{n,i}}{\varepsilon_n} \right) \nabla w \left( x + \frac{y_{n,i_0} - y_{n,j}}{\varepsilon_n} \right) \right) \int \nabla \varphi_{n,i_0} \nabla h_{n,i_0} \]

\[ + 2b \int \sum_{i=1}^k \nabla w \left( x + \frac{y_{n,i_0} - y_{n,i}}{\varepsilon_n} \right) \nabla \varphi_{n,i_0} \int \sum_{i=1}^k \nabla w \left( x + \frac{y_{n,i_0} - y_{n,j}}{\varepsilon_n} \right) \nabla h_{n,i_0} \]

\[ - (q - 1) \int Q(\varepsilon_n x + y_{n,i_0}) \left( \sum_{i=1}^k w \left( x + \frac{y_{n,i_0} - y_{n,i}}{\varepsilon_n} \right) \right)^{q-2} \varphi_{n,i_0} h_{n,i_0} \]
Solutions with multiple peaks for nonlinear Kirchhoff equations on $\mathbb{R}^3$

\[(3.7)\]
\[= a_n(1) \left( \int (a|\nabla h_{n,i0}|^2 + (h_{n,i0})^2) \right)^{\frac{1}{2}}.\]

Since
\[\|\varphi_n\|_n^2 = \varepsilon_n^2 \Rightarrow \int (a|\nabla \varphi_{n,i0}|^2 + (\varphi_{n,i0})^2) = 1,\]
we infer that \(\{\varphi_{n,i0}\}\) is a bounded sequence in \(H^1(\mathbb{R}^3)\) for any \(1 \leq i_0 \leq k\). Hence, up to a subsequence, there exists \(\varphi \in H^1(\mathbb{R}^3)\) such that
\[\varphi_{n,i0} \rightarrow \varphi \quad \text{in} \quad H^1(\mathbb{R}^3),\]
\[\varphi_{n,i0} \rightarrow \varphi \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^3), \quad 1 \leq p < 6,\]
\[\varphi_{n,i0} \rightarrow \varphi \quad \text{a.e. in} \quad \mathbb{R}^3.\]

Next, we will prove that \(\varphi \equiv 0\). For any \(i = 1, \ldots, k, \quad j = 1, 2, 3,\)
\[\int \left( \varepsilon_n^2 a \nabla \varphi_n \nabla \left( \frac{\partial w_{e,y_j}}{\partial y_j} \right) + \varphi_n \frac{\partial w_{e,y_j}}{\partial y_j} \right) = 0\]
is equivalent to
\[(3.8)\]
\[\int \left( a \nabla \varphi_{n,i0} \nabla \left( \frac{\partial w}{\partial x_j} \right) + \varphi_{n,i0} \frac{\partial w}{\partial x_j} \right) = 0.\]

Thus, we can define an equivalent norm \(\|u\|_1^2 = \int (a|\nabla u|^2 + u^2)\) in \(H^1(\mathbb{R}^3)\), then
\[\varphi_n \in E_n\]
is equivalent to
\[(3.9)\]
\[\varphi_{n,i0} \in (\ker \mathcal{L})^\perp.\]

Since \(w\) is radially symmetric, we obtain
\[\left\langle \frac{\partial w}{\partial x_i}, \frac{\partial w}{\partial x_j} \right\rangle_1 = 0, \quad \forall \, i \neq j.\]

For every \(h \in C^\infty_0(\mathbb{R}^3)\), define
\[(3.10)\]
\[h_{n,i0} = h - \sum_{j=1}^3 a_{n,j} \frac{\partial w}{\partial x_j}\]
where \(a_{n,j} = \frac{\langle h, \partial_x w \rangle}{\langle \partial_x w, \partial_x w \rangle}\), then \(h_n \in E_n\). Substituting (3.10) into (3.7) and letting \(n \rightarrow \infty\), we obtain
\[\langle \mathcal{L} \varphi, h \rangle - \langle \mathcal{L} \varphi, \sum_{j=1}^3 a_{n,j} \partial_x w \rangle = 0.\]

Since \(\partial_x w \in \text{Ker} \mathcal{L}\),
\[\langle \mathcal{L} \varphi, h \rangle = 0, \quad \forall \, h \in C^\infty_0(\mathbb{R}^3),\]
which implies that
\[(3.11)\]
\[\varphi \in \text{Ker} \mathcal{L}.\]

Since \(\varphi_n \in E_n\), letting \(n \rightarrow \infty\) in (3.8), we obtain
\[\int \left( a \nabla \varphi \nabla \left( \frac{\partial w}{\partial x_j} \right) + \varphi \frac{\partial w}{\partial x_j} \right) = 0, \quad j = 1, 2, 3.\]
Then
\[(3.12) \quad \varphi \in (\text{Ker } L)^\perp.\]

Combining (3.11) and (3.12), we claim that \(\varphi = 0\).

Now we deduce contradiction. Note that \(\varphi_{n,i_0} \to 0\) in \(L^p_{\text{loc}}(\mathbb{R}^3)\) (1 \(\leq p < 6\)), so there exists \(R > 0\) sufficiently large such that
\[
\int_{\mathbb{R}^3} w^{q-2}(x)(\varphi_{n,i_0})^2 = \int_{B_R(0)} w^{q-2}(x)(\varphi_{n,i_0})^2 + \int_{B_R^c(0)} w^{q-2}(x)(\varphi_{n,i_0})^2 = o_n(1) + o_R(1).
\]

Then
\[
\left| (q - 1) \int_{\mathbb{R}^3} Q(x)W_{\varepsilon_n,Y_n}^{q-2}(\varphi_n)^2 \right|
\leq C\varepsilon_n^3 \int_{\mathbb{R}^3} \left( \sum_{i=1}^{k} w \left( x + \frac{y_{n,i_0} - y_{n,i}}{\varepsilon_n} \right) \right)^{q-2}(\varphi_{n,i_0})^2
\leq C\varepsilon_n^3 \int_{\mathbb{R}^3} w^{q-2}(x)(\varphi_{n,i_0})^2 + C\varepsilon_n^3 \sum_{i \neq i_0} \int_{\mathbb{R}^3} w^{q-2} \left( x + \frac{y_{n,i_0} - y_{n,i}}{\varepsilon_n} \right)(\varphi_{n,i_0})^2
\leq \frac{1}{2}\varepsilon_n^3.
\]

However,
\[
o_n(1)\|\varphi_n\|_{\varepsilon_n}^2 = \langle L_{\varepsilon_n,Y_n}\varphi_n, \varphi_n \rangle
\geq \|\varphi_n\|_{\varepsilon_n}^2 - (q - 1) \int_{\mathbb{R}^3} Q(y)W_{\varepsilon_n,Y_n}^{q-2}(\varphi_n(y))^2
\geq \frac{1}{2}\|\varphi_n\|_{\varepsilon_n}^2.
\]

We reach a contradiction. The proof is complete. \(\square\)

To apply contraction mapping principle to find a critical point of \(J_{\epsilon,Y}(\varphi)\), we first need to estimate \(l_{\epsilon,Y}(\varphi)\) and \(R_{\epsilon,Y}(\varphi)\) for \(i = 0, 1, 2.\)

**Lemma 3.2.** There exists a constant \(C > 0\), independent of \(\varepsilon, \delta\), such that for any \(Y \in D_{\epsilon,\delta}^k\) and \(\varphi \in H_\varepsilon\), there holds
\[
|l_{\epsilon,Y}(\varphi)| \leq C\varepsilon^2 \left( \sum_{i=1}^{k} |Q(y_i) - Q(0)| + \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^m |D^mQ(y_i)| + \varepsilon^{[h]+1} \right)
+ \sum_{i \neq j} e^{-\frac{\theta|y_i - y_j|}{\varepsilon}}\|\varphi\|_{\varepsilon},
\]
where \(\bar{\theta} = \min\{\frac{n-1}{2}, 1\} \).

**Proof.** Since \(w\) is the solution of (2.3), we obtain that \(w_{\epsilon,y_i} (1 \leq i \leq k)\) satisfies
\[
-\left( \varepsilon^2 a + \varepsilon bk \int |\nabla w_{\epsilon,y_i}|^2 \right) \Delta w_{\epsilon,y_i} + w_{\epsilon,y_i} = Q(0)w_{\epsilon,y_i}^{q-1}, \quad j = 1, \ldots, k.
\]
We sum from \(i = 1\) to \(k\) and get
\[
-\left( \varepsilon^2 a + \varepsilon bk \int |\nabla w_{\epsilon,y_j}|^2 \right) \Delta W_{\epsilon,Y} + W_{\epsilon,Y} = Q(0) \sum_{i=1}^{k} w_{\epsilon,y_i}^{q-1}.
\]
Multiplying $\varphi$ on both sides of the above equation and integrating, we obtain
\[
\left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \right) \int \nabla W_{\varepsilon,Y} \nabla \varphi + \int W_{\varepsilon,Y} \varphi = \int Q(0) \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1} \varphi.
\]

Then
\[
(W_{\varepsilon,Y}, \varphi)_{\varepsilon} = \int \left( \varepsilon^2 a \nabla W_{\varepsilon,Y} \nabla \varphi + W_{\varepsilon,Y} \varphi \right)
\]
\[= -\varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \int \nabla W_{\varepsilon,Y} \nabla \varphi + \int Q(0) \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1} \varphi.
\]

Substituting (3.14) into (3.2), we obtain
\[
\begin{align*}
l_{\varepsilon,Y} (\varphi) &= \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi \left( \int |\nabla W_{\varepsilon,Y}|^2 - \int \sum_{j=1}^{k} |\nabla w_{\varepsilon,y_j}|^2 \right) \\
&\quad - \left( \int Q(x) W_{\varepsilon,Y}^{q-1} \varphi - \int Q(0) \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1} \varphi \right) \\
&=: l_1 - l_2.
\end{align*}
\]

To estimate $l_1$, combining Hölder inequality and (2.9) yields
\[
\begin{align*}
|l_1| &= \left| \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi \left( \int |\nabla W_{\varepsilon,Y}|^2 - \int \sum_{j=1}^{k} |\nabla w_{\varepsilon,y_j}|^2 \right) \right| \\
&\leq \varepsilon b \left( \sum_{i=1}^{k} \int |\nabla w_{\varepsilon,y_i} \nabla \varphi| \left( \sum_{i\neq j} \int |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| \right) \right) \\
&\leq \varepsilon^2 b \sum_{i\neq j} e^{-\frac{|y_i - y_j|}{\varepsilon}} \left( \sum_{i=1}^{k} \|\nabla w_{\varepsilon,y_i}\|_{L^2} \|\nabla \varphi\|_{L^2} \right) \\
&\leq \varepsilon^2 b \sum_{i\neq j} e^{-\frac{|y_i - y_j|}{\varepsilon}} \left( k \varepsilon^{\frac{1}{2}} \|\nabla w\|_{L^2} \frac{1}{\sqrt{a\varepsilon}} \|\varphi\|_{\varepsilon} \right) \\
&\leq C \varepsilon^{\frac{3}{2}} \sum_{i\neq j} e^{-\frac{|y_i - y_j|}{\varepsilon}} \|\varphi\|_{\varepsilon}.
\end{align*}
\]

Next, we split $l_2$ into two parts:
\[
l_2 = \int Q(x) \left( \sum_{i=1}^{k} w_{\varepsilon,y_i} \right)^{q-1} \varphi - \sum_{i=1}^{k} Q(0) w_{\varepsilon,y_i}^{q-1} \varphi
\]
\[= \int Q(x) \left( \left( \sum_{i=1}^{k} w_{\varepsilon,y_i} \right)^{q-1} - \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1} \right) \varphi + \int (Q(x) - Q(0)) \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1} \varphi
\]
\[=: l_{21} + l_{22}.
\]

To estimate $l_{21}$, for $2 < q \leq 3$, by the following inequality
\[
|a + b|^{q-1} - |a|^{q-1} - |b|^{q-1} \leq \begin{cases} C|a||b|^{q-2}, & \text{if } |a| \leq |b|, \\ C|b||a|^{q-2}, & \text{if } |b| \leq |a|, \\ C|a|^{q-1} |b|^{\frac{q-1}{2}}, & \end{cases}
\]
we obtain

\[ |l_{21}| = \left| \int Q(x) \left( \left( \sum_{i=1}^{k} w_{e,y_i} \right)^{q-1} - \sum_{i=1}^{k} w_{e,y_i}^{q-1} \right) \varphi \right| \]

\[ \leq C \int \sum_{i \neq j} w_{e,y_i}^{q-2} w_{e,y_j}^{q-1} |\varphi| \leq C \int \left( \int w_{e,y_i}^{q-1} w_{e,y_j}^{q-1} \right)^{\frac{1}{2}} \|\varphi\|_{L^2} \]

\[ \leq C \varepsilon^2 \sum_{i \neq j} e^{\frac{q-1}{2} \frac{|y_j - y_i|}{\varepsilon}} \|\varphi\|_{e}. \]

For \( q > 3 \), we have

\[ |l_{21}| = \left| \int Q(x) \left( \left( \sum_{i=1}^{k} w_{e,y_i} \right)^{q-1} - \sum_{i=1}^{k} w_{e,y_i}^{q-1} \right) \varphi \right| \]

\[ \leq C \int \sum_{i \neq j} w_{e,y_i}^{q-2} w_{e,y_j} |\varphi| \leq C \varepsilon^2 \sum_{i \neq j} e^{\frac{q-1}{2} \frac{|y_j - y_i|}{\varepsilon}} \|\varphi\|_{e}. \]

Combining (3.17) and (3.18) yields

\[ |l_{21}| \leq C \varepsilon^2 \sum_{i \neq j} e^{-\min\left(\frac{q-1}{2}, 1\right) \frac{|y_j - y_i|}{\varepsilon}} \|\varphi\|_{e}. \]

To estimate \( l_{22} \), we split \( l_{22} \) into two parts:

\[ |l_{22}| = \int \sum_{i=1}^{k} \left( Q(x) - Q(0) \right) w_{e,y_i}^{q-1} \varphi \]

\[ = \int \sum_{i=1}^{k} \left( Q(x) - Q(y_i) \right) w_{e,y_i}^{q-1} \varphi + \int \sum_{i=1}^{k} \left( Q(y_i) - Q(0) \right) w_{e,y_i}^{q-1} \varphi \]

\[ =: l_{221} + l_{222}. \]

Estimating \( l_{221} \), we have

\[ |l_{221}| \leq \sum_{i=1}^{k} \int |Q(x) - Q(y_i)| w_{e,y_i}^{q-1} \|\varphi\|_{e} \]

\[ \leq \sum_{i=1}^{k} \left( \int |Q(x) - Q(y_i)|^2 w_{e,y_i}^{2(q-1)} \right)^{\frac{1}{2}} \|\varphi\|_{L^2} \]

\[ = \sum_{i=1}^{k} \left( \int_{B_{\delta}(y_i)} |Q(x) - Q(y_i)|^2 w_{e,y_i}^{2(q-1)} + \int_{B_{\delta}(y_i)} |Q(x) - Q(y_i)|^2 w_{e,y_i}^{2(q-1)} \right)^{\frac{1}{2}} \|\varphi\|_{L^2} \]
Finally, to estimate $l_{222}$, combining Hölder inequality and Lemma 2.1, we obtain

$$\begin{align*}
|l_{222}| &= \left| \int \sum_{i=1}^{k} (Q(y_i) - Q(0)) w_{\varepsilon,y_i}^{q-1} \varphi \right| \\
&\leq \sum_{i=1}^{k} |Q(y_i) - Q(0)| \int w_{\varepsilon,y_i}^{q-1} |\varphi| \\
&\leq \sum_{i=1}^{k} |Q(y_i) - Q(0)| \left( \int w_{\varepsilon,y_i}^{q} \right)^{\frac{q-1}{q}} \left( \int |\varphi|^{q} \right)^{\frac{1}{q}} \\
&\leq C \varepsilon^{\frac{3}{2}} \sum_{i=1}^{k} |Q(y_i) - Q(0)| \|\varphi\|_{\varepsilon}. 
\end{align*}$$

Combining (3.15) and (3.19)–(3.20) yields (3.13). \qed

**Lemma 3.3.** There exists a constant $C > 0$, independent of $\varepsilon, \delta$, such that for any $\varphi \in H_{\varepsilon}$, there holds

$$\|R_{\varepsilon,Y}^{(i)}(\varphi)\| \leq C b \varepsilon^{-\frac{3}{2}} \left( 1 + \varepsilon^{-\frac{3}{2}} \|\varphi\|_{\varepsilon} \right) \|\varphi\|_{\varepsilon}^{3-i} + C \varepsilon^{-\frac{2(q-2)}{q}} \|\varphi\|_{\varepsilon}^{q-i}, \; i = 0, 1, 2.$$  

**Proof.** By (3.4), we have

$$R_{\varepsilon,Y}(\varphi) = A_1(\varphi) - A_2(\varphi),$$

where

$$A_1(\varphi) = \frac{\varepsilon b}{4} \left( \int |\nabla \varphi|^2 \right)^2 + \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi \int |\nabla \varphi|^2$$

and

$$A_2(\varphi) = \frac{1}{q} \int Q(x) \left( (W_{\varepsilon,Y} + \varphi)^q_+ - W_{\varepsilon,Y}^q - q W_{\varepsilon,Y}^{q-1} \varphi - \frac{q(q-1)}{2} W_{\varepsilon,Y}^{q-2} \varphi^2 \right).$$

For any $\psi, \xi \in H_{\varepsilon}$, we obtain

$$\langle A_1^{(i)}(\varphi), \psi \rangle = \varepsilon b \left( \int |\nabla \varphi|^2 \int \nabla \varphi \nabla \psi + \int |\nabla \psi|^2 \int \nabla W_{\varepsilon,Y} \nabla \psi \right)$$

$$+ 2 \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi \int \nabla \varphi \nabla \psi,$$
Next, we estimate $A_1^{(i)}(\varphi)$, $i = 0, 1, 2$. Noting that
\begin{equation}
\|\nabla \varphi\|_{L^2} \leq \frac{1}{\sqrt{a\varepsilon}} \|\varphi\|_e
\end{equation}
and
\begin{equation}
\int |\nabla W_{e,Y}|^2 \leq k \sum_{i=1}^{k} \int |\nabla w_{e,y_i}|^2 = k^2 \varepsilon \int |\nabla w|^2,
\end{equation}
we have
\begin{equation}
\|\nabla W_{e,Y}\|_{L^2} \leq C_1 \varepsilon^{\frac{3}{4}}
\end{equation}
where $C_1 = k\|\nabla w\|_{L^2}$. Combining (3.28) and (3.29), we obtain that for any $\psi, \xi, \nu \in H_{e}$, there hold
\begin{equation}
\int |\nabla \varphi\nabla \psi| \int |\nabla W_{e,Y} \nabla \xi| \leq C_1 \varepsilon^{\frac{1}{4}} \|\varphi\|_e \|\psi\|_e \|\xi\|_e,
\end{equation}
\begin{equation}
\int |\nabla \varphi\nabla \psi| \int |\nabla \nu \nabla \xi| \leq C_1 \varepsilon^{\frac{1}{4}} \|\varphi\|_e \|\psi\|_e \|\nu\|_e \|\xi\|_e.
\end{equation}
Combining (3.22), (3.24), (3.26), (3.30) and (3.31) yields
\begin{equation}
\|A_1^{(1)}(\varphi)\| \leq C\varepsilon^{-\frac{3}{4}} \|\varphi\|_{3+i}^{3-i} + C\varepsilon^{-\frac{3}{4}} \|\varphi\|_{3+i}^{4-i}
\leq C\varepsilon^{-\frac{3}{4}} \|\varphi\|_{3+i}^{3+i} \varepsilon^{-\frac{3}{4}} \|\varphi\|_e (\varepsilon^{-\frac{3}{4}} + 1).
\end{equation}
Then, we estimate $A_1^{(i)}(\varphi)$, $i = 0, 1, 2$. For $2 < q \leq 3$, we apply the following elementary inequalities: for $e, f \in \mathbb{R}$, there exist constants $C_1(q)$, $C_2(q)$, $C_3(q) > 0$ such that
\begin{equation}
|(e + f)^q - e^q - qe^{q-1}f - \frac{q(q-1)}{2}e^{q-2}f^2| \leq C_1(q)|f|^q,
\end{equation}
\begin{equation}
|(e + f)^q - e^{q-1} - (q - 1)e^{q-2}f| \leq C_2(q)|f|^{q-1},
\end{equation}
and
\begin{equation}
|(e + f)^q - e^{q-2}| \leq C_3(q)|f|^{q-2}.
\end{equation}
Combining the above inequalities and Lemma 2.1 yields
\begin{equation}
\|A_2(\varphi)\| \leq C\varepsilon^{-\frac{3(q-2)}{4}} \|\varphi\|_e^q,
\end{equation}
\begin{equation}
\|A_1^{(1)}(\varphi)\| \leq C\varepsilon^{-\frac{3(q-2)}{4}} \|\varphi\|_{q-1}^{q-1},
\end{equation}
\begin{equation}
\|A_2^{(2)}(\varphi)\| \leq C\varepsilon^{-\frac{3(q-2)}{4}} \|\varphi\|_{q-2}^{q-2}.
\end{equation}
Similarly, for \(3 < q < 6\) and \(e, f \in \mathbb{R}\), there exist constants \(C_4'(q), C_5'(q), C_6'(q) > 0\) such that
\[
|(e + f)^q - e^q - qe^{q-1}f - \frac{q(q-1)}{2}e^{q-2}f^2| \leq C_4'(q)(|e|^{q-3} + |f|^{q-3})|f|^3,
\]
\[
|(e + f)^{q-1} - e^{q-1} - (q-1)e^{q-2}f| \leq C_5'(q)(|e|^{q-3} + |f|^{q-3})|f|^2,
\]
and
\[
|(e + f)^{q-2} - e^{q-2}| \leq C_6'(q)(|e|^{q-3} + |f|^{q-3})|f|.
\]
Combining the above inequalities and Lemma 2.1 yields
\[
|A_2(\varphi)| \leq C_4'(q) \int \left( |W_{e,Y}|^{q-3} + |\varphi|^{q-3} \right) |\varphi|^3
\]
(3.36)
\[
\leq C \left( \int |W_{e,Y}|^{2(q-3)} \right)^{\frac{1}{2}} \varepsilon^{-3} \|\varphi\|^3 + C \varepsilon^{-\frac{3(q-2)}{2}} \|\varphi\|^2,
\]
\[
\leq C \left( \varepsilon^{-\frac{3}{2}} \|\varphi\|^3 + \varepsilon^{-\frac{3(q-2)}{2}} \|\varphi\|^3 \right).
\]
By the same token, we obtain
(3.37)
\[
\|A_2^{(1)}(\varphi)\| \leq C \left( \varepsilon^{-\frac{3}{2}} \|\varphi\|^2 + \varepsilon^{-\frac{3(q-2)}{2}} \|\varphi\|^{q-1} \right),
\]
(3.38)
\[
\|A_2^{(2)}(\varphi)\| \leq C \left( \varepsilon^{-3} \|\varphi\|^3 + \varepsilon^{-\frac{3(q-2)}{2}} \|\varphi\|^2 \right).
\]
Combining (3.32)–(3.38) yields (3.21). \(\square\)

To state the lemma that follows, we define
\[
N_{\varepsilon} = \left\{ \varphi \in E_{k,e}^{y,k} : \|\varphi\|_e \leq \varepsilon^{\frac{3}{2}} \left( \sum_{i=1}^{k} |Q(y_i) - Q(0)|^{1-\tau} \right) \right. \leq \varepsilon^{\frac{3}{2}} \left( \sum_{i=1}^{h} |Q(y_i) - Q(0)|^{1-\tau} + \sum_{m=1}^{i} \varepsilon^{m-\tau}|D^{m}Q(y_i)|^{1-\tau} + \varepsilon^{[h]+1-\tau} + \sum_{i \neq j} e^{-(\theta - \tau) \frac{|y_i - y_j|}{\varepsilon}} \right) \right\}
\]
where \(0 < \tau < \min\{1, \bar{\theta}\}\).

**Lemma 3.4.** There exist \(\varepsilon_0, \delta_0\) sufficiently small such that for every \(\varepsilon \in (0, \varepsilon_0]\) and \(\delta \in (0, \delta_0]\), there exists a \(C^1\) map \(\varphi_\varepsilon : D_{\varepsilon,\delta}^{k} \rightarrow N_{\varepsilon}; \ Y \mapsto \varphi_{\varepsilon,Y} \) satisfying
\[
\left\langle \frac{\partial J_\varepsilon(Y, \varphi_{\varepsilon,Y})}{\partial \varphi}, \psi \right\rangle = 0, \quad \forall \ \psi \in H_\varepsilon, \ \forall \ Y \in D_{\varepsilon,\delta}^{k}
\]
(3.40)
Moreover, we can choose \(0 < \tau < \min\{1, \bar{\theta}\}\) sufficiently small, such that
\[
\|\varphi_{\varepsilon,Y}\|_e \leq \varepsilon^{\frac{3}{2}} \left( \sum_{i=1}^{k} |Q(y_i) - Q(0)|^{1-\tau} + \sum_{m=1}^{k} \sum_{i=1}^{[h]} \varepsilon^{m-\tau}|D^{m}Q(y_i)|^{1-\tau} + \varepsilon^{[h]+1-\tau} + \sum_{i \neq j} e^{-(\theta - \tau) \frac{|y_i - y_j|}{\varepsilon}} \right),
\]
(3.41)
Proof. Recall that
\[
J_\varepsilon(Y, \varphi) = J_\varepsilon(Y, 0) + \left( I'_\varepsilon(W_{\varepsilon,Y}, \varphi) \right) + \frac{1}{2} \left( I''_\varepsilon(W_{\varepsilon,Y})(\varphi, \varphi) \right) + R_{\varepsilon,Y}(\varphi),
\]
so we have
\[
\left\langle \frac{\partial J_\varepsilon}{\partial \varphi}, \psi \right\rangle = \left( I'_\varepsilon(W_{\varepsilon,Y}, \psi) \right) + \left( I''_\varepsilon(W_{\varepsilon,Y})(\varphi, \psi) \right) + \left( R_{\varepsilon,Y}(\varphi, \psi) \right), \ \forall \ \psi \in H_\varepsilon,
\]
i.e.

\[
\frac{\partial J_\varepsilon}{\partial \varphi} = I'_\varepsilon(W_{\varepsilon,Y}) + I''_\varepsilon(W_{\varepsilon,Y})[\varphi] + R'_{\varepsilon,Y}(\varphi)
\]

\[
= l_{\varepsilon,Y} + I''_\varepsilon(W_{\varepsilon,Y})[\varphi] + R'_{\varepsilon,Y}(\varphi).
\]

Then \( \frac{\partial J_\varepsilon}{\partial \varphi} \) is a bounded linear functional in \( N_\varepsilon \). Denote

\[
\mathfrak{W} = \{ f : f \text{ is a bounded linear functional defined on } H_\varepsilon \}.
\]

For any \( f \in \mathfrak{W} \), by Riesz representation theorem, there exists a unique \( \hat{f} \in H_\varepsilon \) such that

\[
f(\psi) = \langle \hat{f}, \psi \rangle_\varepsilon, \quad \forall \psi \in H_\varepsilon.
\]

So we can define a map \( \sigma : \mathfrak{W} \to H_\varepsilon \); \( f \mapsto \hat{f} \).

Let \( \mathfrak{W}^* = \sigma(\mathfrak{W}) \). Next, we prove \( \sigma \) is a linear isomorphic map from \( \mathfrak{W} \) to \( \mathfrak{W}^* \).

In fact, if \( \sigma(f_1) = \sigma(f_2) \), in the sense that \( \hat{f}_1 = \hat{f}_2 \), we obtain

\[
f_1(\psi) = \langle \hat{f}_1, \psi \rangle_\varepsilon = \langle \hat{f}_2, \psi \rangle_\varepsilon = f_2(\psi), \quad \forall \psi \in H_\varepsilon.
\]

Then \( f_1 = f_2 \) and \( \sigma \) is injective. Besides, for any \( f_1, f_2 \in \mathfrak{W} \),

\[
\langle \hat{f}_1 + \hat{f}_2, \psi \rangle_\varepsilon = (f_1 + f_2)(\psi) = f_1(\psi) + f_2(\psi) = \langle \hat{f}_1, \psi \rangle_\varepsilon + \langle \hat{f}_2, \psi \rangle_\varepsilon = \langle \hat{f}_1 + \hat{f}_2, \psi \rangle_\varepsilon,
\]

which implies \( \hat{f}_1 + \hat{f}_2 = \hat{f}_1 + \hat{f}_2 \), in the sense that \( \sigma(f_1 + f_2) = \sigma(f_1) + \sigma(f_2) \).

And for any \( k \in \mathbb{R} \) and \( f \in \mathfrak{W} \), we obtain

\[
\langle k\hat{f}, \psi \rangle_\varepsilon = (k f)(\psi) = k(\hat{f}, \psi)_\varepsilon = \langle k\hat{f}, \psi \rangle_\varepsilon.
\]

Thus, \( \hat{kf} = k\hat{f} \) and \( \sigma(kf) = k\sigma(f) \).

Therefore, (3.42) is equivalent to

\[
(3.43) \quad \frac{\partial J_\varepsilon}{\partial \varphi} = \hat{l}_{\varepsilon,Y} + \mathcal{L}_{\varepsilon,Y}(\varphi) + \hat{R}'_{\varepsilon,Y}(\varphi).
\]

Since \( \mathcal{L}_{\varepsilon,Y} \) is invertible in \( E_{\varepsilon,Y}^k \), by Lemma 3.1, it is sufficient to find \( \varphi \in N_\varepsilon \) that satisfies

\[
(3.44) \quad \varphi = -\mathcal{L}_{\varepsilon,Y}^{-1}(\hat{l}_{\varepsilon,Y}) - \mathcal{L}_{\varepsilon,Y}^{-1}(\hat{R}'_{\varepsilon,Y}(\varphi)) =: \mathcal{A}_{\varepsilon,Y}(\varphi).
\]

Next, we prove that \( \mathcal{A}_{\varepsilon,Y} \) is a contraction map on \( N_\varepsilon \). First, for any \( \varphi \in N_\varepsilon \), we have

\[
(3.45) \quad \| \mathcal{A}_{\varepsilon,Y}(\varphi) \|_\varepsilon \leq \frac{1}{\rho} \| \hat{l}_{\varepsilon,Y} \|_\varepsilon + \frac{1}{\rho} \| \hat{R}'_{\varepsilon,Y}(\varphi) \|_\varepsilon
\]

\[
= \frac{1}{\rho} \| l_{\varepsilon,Y} \| + \frac{1}{\rho} \| R'_{\varepsilon,Y}(\varphi) \|.
\]

By Lemma 3.2, we obtain

\[
\| l_{\varepsilon,Y} \| \leq C\varepsilon^{\frac{2}{\tau}} \left( \varepsilon^{[h]+1} + \sum_{i=1}^{k} |Q(y_i) - Q(0)| + \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^m |D^m Q(y_i)| + \sum_{i \neq j} e^{-\rho \frac{\|y_i - y_j\|}{\varepsilon}} \right).
\]

Choose \( \varepsilon, \delta \) sufficiently small such that

\[
\begin{cases}
C\varepsilon^\tau < \frac{\rho}{2}, \\
C|Q(y_i) - Q(0)|^\tau < \frac{\rho}{2}, \quad i = 1, \ldots, k, \\
C\varepsilon^\tau |D^m Q(y_i)|^\tau < \frac{\rho}{2}, \quad i = 1, \ldots, k, m = 1, \ldots, [h], \\
C e^{-\tau \frac{\|y_i - y_j\|}{\varepsilon}} < \frac{\rho}{2}, \quad i \neq j,
\end{cases}
\]
then

\[
\|l_{\varepsilon,Y}\| \leq \frac{\rho}{2} \varepsilon^\frac{3}{4} \left( \sum_{i=1}^{k} |Q(y_i) - Q(0)|^{1-\tau} + \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^{m-\tau} |D^m Q(y_i)|^{1-\tau} \right) \\
+ \frac{\rho}{2} \varepsilon^\frac{3}{4} \left( \varepsilon^{[h]+1-\tau} + \sum_{i \neq j} e^{-\frac{1}{2} \varepsilon^{[h]+1-\tau}} \right).
\]

(3.46)

As \( \varphi \in N_\varepsilon \),

\[ \varepsilon^{-\frac{3}{2}} \|\varphi\| \leq o_\varepsilon(1) + o_\delta(1). \]

So for \( \varepsilon, \delta \) sufficiently small, by Lemma 3.3, we have

\[ \|R_{\varepsilon,Y}'(\varphi)\| = (o_\varepsilon(1) + o_\delta(1)) \|\varphi\| \leq \frac{\rho}{2} \|\varphi\| \varepsilon. \]

(3.47)

Combining (3.45)–(3.47) yields

\[
\|A_{\varepsilon,Y}(\varphi)\| \leq \varepsilon^\frac{3}{4} \left( \sum_{i=1}^{k} |Q(y_i) - Q(0)|^{1-\tau} + \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^{m-\tau} |D^m Q(y_i)|^{1-\tau} \right) \\
+ \varepsilon^\frac{3}{4} \left( \varepsilon^{[h]+1-\tau} + \sum_{i \neq j} e^{-\frac{1}{2} \varepsilon^{[h]+1-\tau}} \right).
\]

Hence, \( A_{\varepsilon,Y}(N_\varepsilon) \subset N_\varepsilon \). On the other hand, for every \( \varphi, \psi \in N_\varepsilon \),

\[
\|A_{\varepsilon,Y}(\varphi) - A_{\varepsilon,Y}(\psi)\| = \|L_{\varepsilon,Y}^{-1}(\tilde{R}_{\varepsilon,Y}'(\varphi)) - L_{\varepsilon,Y}^{-1}(\tilde{R}_{\varepsilon,Y}'(\psi))\|_\varepsilon \\
\leq \frac{1}{\rho} \|R_{\varepsilon,Y}'(\varphi) - R_{\varepsilon,Y}'(\psi)\| \\
= \frac{1}{\rho} \|R_{\varepsilon,Y}''(\xi \varphi + (1 - \xi)\psi)\| \|\varphi - \psi\|_\varepsilon, \quad 0 < \xi < 1.
\]

By Lemma 3.3, we obtain

\[
\|R_{\varepsilon,Y}''(\xi \varphi + (1 - \xi)\psi)\| \leq C \varepsilon^{-\frac{3(\alpha-2)}{2}} \|\xi \varphi + (1 - \xi)\psi\|_\varepsilon^{\frac{\alpha-2}{2}} \\
+ C_3 \varepsilon^{-\frac{3}{2}} (1 + \varepsilon^{-\frac{3}{2}} \|\xi \varphi + (1 - \xi)\psi\|_\varepsilon) \|\xi \varphi + (1 - \xi)\psi\|_\varepsilon \\
= o_\varepsilon(1).
\]

Thus, for \( \varepsilon \) sufficiently small, we have

\[
\|A_{\varepsilon,Y}(\varphi) - A_{\varepsilon,Y}(\psi)\| \leq \frac{1}{2} \|\varphi - \psi\|_\varepsilon.
\]

So \( A_{\varepsilon,Y} \) is a contraction map on \( N_\varepsilon \). By contraction mapping principle, we infer that (3.44) has a unique solution. Finally, by similar arguments as that of Cao, Noussair and Yan [6], we can deduce that \( \varphi_\varepsilon \) belongs to \( C^1 \).

\[ \□ \]

4. Proof of Theorem 1.3

In this section, without loss of generality, we assume \( Q(0) = 1 \). By Lemma 3.4, we can define a \( C^1 \) function on \( D_{\varepsilon,\delta}^k \), in the sense that

\[ K(Y) =: J_\varepsilon(Y, \varphi_\varepsilon, Y), \quad Y \in D_{\varepsilon,\delta}^k. \]

Define

\[ c_{\varepsilon,1} = \varepsilon^3 (kA - k^2B - T \varepsilon^{\alpha h_1}), \quad c_{\varepsilon,2} = \varepsilon^3 (kA - k^2B + \mu), \]
where \( A = \frac{a - 2}{2q} \| w \|_{L^q}^q, \ B = \frac{1}{2} \| \nabla w \|_{L^2}^2, \mu, T \) are positive constants, \( \varepsilon^\alpha \leq \frac{\delta}{2} \) and \( \alpha \in (0, 1) \) is a fixed constant close to 1.

Denote
\[
\Omega_\gamma = \left\{ Y = (y_1, \ldots, y_k): y_i \in B_\delta(0) \times B_{\gamma_\epsilon}(0), \ i = 1, \ldots, k, \ \frac{|y_i - y_j|}{\varepsilon} \geq R_1, \ i \neq j \right\},
\]
where \( B_\delta(0) = \{ y \in \mathbb{R}^l: |y| \leq \tau \}, \ R_1 > 0 \) is a large constant, and
\[
K^c = \{ Y: Y \in \Omega_\varepsilon^\alpha, \ K(Y) \leq c \}.
\]

**Lemma 4.1.** For any \( \varphi \in E^k_{\epsilon,Y} \), there holds
\[
(4.1) \quad \langle L_{\epsilon,Y} \varphi, \varphi \rangle_\varepsilon = O (\| \varphi \|^2).
\]

**Proof.** By the definition of \( L_{\epsilon,Y} \), we have
\[
\langle L_{\epsilon,Y} \varphi, \varphi \rangle_\varepsilon = \langle \varphi, \varphi \rangle_\varepsilon + \varepsilon b \int |\nabla W_{\epsilon,Y}|^2 \int |\nabla \varphi|^2 \tag{4.2}
\]
\[
+ 2\varepsilon b \left( \int |\nabla W_{\epsilon,Y} \nabla \varphi|^2 \right) - (q - 1) \int Q(x) W_{\epsilon,Y}^{q-2} \varphi^2.
\]
Calculating directly yields
\[
(4.3) \quad \varepsilon b \int |\nabla W_{\epsilon,Y}|^2 \int |\nabla \varphi|^2 \leq \varepsilon bk \int \sum_{i=1}^k |\nabla w_{\epsilon,y_i}|^2 \int |\nabla \varphi|^2 \leq C \| \varphi \|_\varepsilon^2.
\]
By Hölder inequality, we obtain
\[
(4.4) \quad \varepsilon b \left( \int |\nabla W_{\epsilon,Y} \nabla \varphi|^2 \right) \leq \varepsilon b \int |\nabla W_{\epsilon,Y}|^2 \int |\nabla \varphi|^2 \leq C \| \varphi \|_\varepsilon^2.
\]
Finally, as \( Q(x) \) is bounded, we have
\[
(4.5) \quad \int Q(x) W_{\epsilon,Y}^{q-2} \varphi^2 \leq C \left( \int W_{\epsilon,Y}^{q-\frac{2}{q}} \right) \left( \int |\varphi|^q \right)^\frac{2}{q} \leq C \| \varphi \|_\varepsilon^2.
\]
Combining (4.2)–(4.5) yields (4.1). \( \square \)

**Lemma 4.2.** There exist constants \( \varepsilon_0, \delta_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0] \) and \( \delta \in (0, \delta_0] \), \( (Y, \varphi) \) is a critical point of \( J_\varepsilon \) on \( D^k_{\epsilon,\delta} \times E^k_{\epsilon,Y} \) is equivalent to
\[
\sum_{i=1}^k w_{\epsilon,y_i} + \varphi
\]
is a critical point of \( I_\varepsilon \).

**Proof.** This lemma can be proved by the same arguments as that of \([3, 6]\) with minor modifications. We omit the details. \( \square \)

**Lemma 4.3.** For every \( Y \in \partial \Omega_\varepsilon^\alpha \), we have either \( K(Y) < c_{\varepsilon,1} \) or \( \frac{\partial K(Y)}{\partial m} > 0 \), where \( n \) is the outward unit normal of \( \partial \Omega_\varepsilon^\alpha \) at \( Y \).

**Proof.** We divide the proof of this lemma into two steps.

**Step 1:** Suppose that \( \frac{y_i - y_j}{\varepsilon} = R_1 \) for some \( i \neq j \), or \( y_i \in \partial B_\delta(0) \times B_{\gamma_\epsilon}(0) \) for some \( i \in \{1, \ldots, k\} \). We claim that \( Y \in K_{\varepsilon,1}^{c_{\varepsilon,1}} \).

In fact, since \( \varphi_{\epsilon,Y} \in N_\varepsilon \), by Lemma 3.2, we obtain
\[
(4.6) \quad |l_{\epsilon,Y} (\varphi_{\epsilon,Y})| = O (\| \varphi_{\epsilon,Y} \|_\varepsilon^2).
\]
And by Lemma 3.3, we have
\begin{equation}
|R_{\varepsilon,Y}(\varphi_{\varepsilon,Y})| = o_{\varepsilon}(1)\|\varphi_{\varepsilon,Y}\|_{\varepsilon}^2.
\end{equation}

Combining (3.1), (4.1), (4.6) and (4.7) yields
\begin{equation}
J_{\varepsilon}(Y, \varphi_{\varepsilon,Y}) = J_{\varepsilon}(Y, 0) + O(\|\varphi_{\varepsilon,Y}\|_{\varepsilon}^2).
\end{equation}

Then combining Lemma A.1, (4.8) and (3.41) yields
\begin{equation}
K(Y) = \varepsilon^3 (kA - k^2B) - \frac{2}{q-2}A\varepsilon^3 \sum_{i=1}^{k} (Q(y_i) - 1)
\end{equation}

\begin{equation}
\quad + \int \sum_{i=1}^{k-1} w_{\varepsilon,y_i} \left( \sum_{j=i+1}^{k} w_{\varepsilon,y_j} \right) q^{-1} + O(\varepsilon^{4+[\bar{\epsilon}]})
\end{equation}

\begin{equation}
\quad + O\left( \sum_{i=1}^{k} |Q(y_i)| - 1 \right)^{2(1-\tau)} + \sum_{i=1}^{k} \sum_{m=1}^{[\bar{\epsilon}]} \varepsilon^{3+m} |D^m Q(y_i)|^{2(1-\tau)}
\end{equation}

\begin{equation}
\quad + \varepsilon^3 O\left( \varepsilon^{2([\bar{\epsilon}]+1-\tau)} + \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right).
\end{equation}

Choose \( \tau \) sufficiently small such that
\[ 2([\bar{\epsilon}] + 1 - \tau) > [\bar{\epsilon}] + 1, \ 2(m-\tau) > m, \ 2(\bar{\tau} - \tau) > 1. \]

Then by (4.9), we have
\begin{equation}
K(Y) = \varepsilon^3 (kA - k^2B) - \frac{2}{q-2}A\varepsilon^3 \sum_{i=1}^{k} (Q(y_i) - 1)
\end{equation}

\begin{equation}
\quad - \int \sum_{i=1}^{k-1} w_{\varepsilon,y_i} \left( \sum_{j=i+1}^{k} w_{\varepsilon,y_j} \right) q^{-1} + O(\varepsilon^{4+[\bar{\epsilon}]})
\end{equation}

\begin{equation}
\quad + O\left( \sum_{i=1}^{k} \sum_{m=1}^{[\bar{\epsilon}]} \varepsilon^{3+m} |D^m Q(y_i)| + \varepsilon^3 \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right).
\end{equation}

Combining the above equality and the condition (Q3) yields
\begin{equation}
K(Y) \leq \varepsilon^3 \left( kA - k^2B - C \sum_{i=1}^{k} P_1(y_i') - C \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right) + O(\varepsilon^4).
\end{equation}

If \( \frac{\eta|y_i - y_j|}{\varepsilon} = R_1 \) for some \( i \neq j \), taking \( R_1 = \frac{ak_1 \ln \frac{1}{\varepsilon} - \ln T}{2} \), by (4.11) we obtain
\begin{equation}
K(Y) \leq \varepsilon^3 \left( kA - k^2B - Te^{\alpha h} \right) - C\varepsilon^3 \sum_{i=1}^{k} P_1(y_i') + O(\varepsilon^4) < c_{\varepsilon,1}.
\end{equation}
If $y_i \in \partial B_{\delta}(0) \times B_{\varepsilon_{n-t}}(0)$ for some $i \in \{1, \ldots, k\}$, combining (4.11) and (1.11) yields

$$K(Y) \leq \varepsilon^3 \left( kA - k^2B - C\lambda \sum_{i=1}^{k} |y_i|^{h_1} \right) - C\varepsilon^3 \sum_{i<j} e^{-\frac{\varepsilon|y_i - y_j|}{\tau}} + O(\varepsilon^4)$$

$$\leq \varepsilon^3 \left( kA - k^2B - C\varepsilon^{\alpha h_1} \right) - C\varepsilon^3 \sum_{i<j} e^{-\frac{\varepsilon|y_i - y_j|}{\tau}} + O(\varepsilon^4).$$

Let $T$ be sufficiently small such that $T < C\lambda$, then we have $K(Y) < c_{\varepsilon,1}$.

Step 2: Suppose $t \in \{1, 2\}$ and $y_j \in B_{\delta}(0) \times \partial B_{\varepsilon_{n-t}}(0)$ for some $j \in \{1, \ldots, k\}$. Without loss of generality, we assume $j = 1$. We claim that either $K(Y) < c_{\varepsilon,1}$ or $\frac{\partial K(Y)}{\partial y_{1}} > 0$, where $n$ is the outward unit normal of $B_{\delta}(0) \times \partial B_{\varepsilon_{n-t}}(0)$ at $y_1$.

In fact, for any $y_i \in B_{\delta}(0) \times B_{\varepsilon_{n-t}}(0)$ and $m \geq 1$, we have

$$\varepsilon^m |D^m Q(y_i)| = O \left( \varepsilon^m |y_i|^{h_1-m} + \varepsilon^m |y_i|^{h-m} \right)$$

$$= O \left( \varepsilon^m |y_i|^{h_1-m} + \varepsilon^{\alpha h+m(1-\alpha)} \right).$$

By Lemma A.2, we obtain

$$\frac{\partial K}{\partial y_{1}} = -C\varepsilon^3 D_1 Q(y_1) - (q-1) \sum_{i=2}^{k} \int w_{\varepsilon_{y_i}}(y_{i}, y_{i}) \frac{\partial w_{\varepsilon_{y_i}}}{\partial y_{1}}$$

$$+ O \left( \sum_{i=1}^{k} \sum_{m=2}^{[h]} \varepsilon^{2+m-\tau} |D^m Q(y_i)|^{1-\tau} + \varepsilon^{3+[h]-\tau} \right)$$

$$+ O \left( \varepsilon^3 \sum_{i=1}^{k} |Q(y_i) - 1|^{1-\tau} \right) + O \left( \varepsilon^2 \sum_{i \neq j} e^{-\frac{\varepsilon|y_i - y_j|}{\tau}} \right).$$

Denote $\bar{\eta} = \min_{i \neq j} \eta |y_i - y_j|$. We divide it into two cases.

(i) Suppose that $\varepsilon \frac{-\bar{\eta}}{\tau} > L^\alpha h$ or $|y_i| > L^{\alpha h/h_1}$ for some $i \in \{1, \ldots, k\}$, where $L > T$ is a large constant. We claim that $K(Y) < c_{\varepsilon,1}$.

In fact, combining (4.10) and (1.11) yields

$$K(Y) \leq \varepsilon^3 \left( kA - k^2B \right) - C\varepsilon^3 \sum_{i=1}^{k} |y_i|^{h_1} - C_1 \varepsilon^3 e^{-\frac{\bar{\eta}}{\tau}}$$

$$+ O \left( \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^{3+m} |y_i|^{h_1-m} + \varepsilon^{3+\alpha h} \right)$$

$$\leq \varepsilon^3 \left( kA - k^2B \right) - (C_1 - \tau') \varepsilon^3 \sum_{i=1}^{k} |y_i|^{h_1} - C_1 \varepsilon^3 e^{-\frac{\bar{\eta}}{\tau}} + C_\tau \varepsilon^{3+\alpha h},$$

where $\tau' > 0$ is a constant. When $L > T$ is large enough, we have $K(Y) < c_{\varepsilon,1}$.

(ii) Suppose that $\varepsilon \frac{-\bar{\eta}}{\tau} \leq L^{\alpha h}$ and $|y_i| \leq L^{\alpha h/h_1}$, $i = 1, \ldots, k$. We claim that $\frac{\partial K(Y)}{\partial y_{1}} > 0$. First, we can see

$$|1 - Q(y_i)| = O(\varepsilon^{\alpha h}),$$

$$|D^m Q(y_i)|^m = O(\varepsilon^{\alpha h(h_1-m)/h_1} \varepsilon^m + \varepsilon^{\alpha h+m(1-\alpha)}) = O(\varepsilon^{\alpha h+m(1-\alpha)}),$$

$$\varepsilon^2 e^{-\frac{\bar{\eta}}{\tau}} = O(\varepsilon^{2+\bar{\eta}}).$$
Since for any \( i \neq 1 \),

\[
\int w^{q-2}_{\varepsilon,y_1} w_{\varepsilon,y_i} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} = (C + o(1)) \varepsilon^2 w \left( \frac{|y_i - y_1|}{\varepsilon} \right) \frac{y_i - y_1}{|y_i - y_1|},
\]

(4.19)

\[
\left\langle \frac{y_i - y_1}{|y_i - y_1|}, n \right\rangle \leq 0, \quad \forall \, y_i \in B^i_\delta(0) \times B^{3-t}_{c_\varepsilon}(0),
\]

(4.20)

where

\[
K = \{ 0, \frac{\varepsilon^{1.5}}{(\varepsilon^{1.5})^2 + (\varepsilon^{1.5})^2 + \varepsilon^{1.5})^{1/2} \}, \quad t = 1,
\]

\[
0, \frac{\varepsilon^{1.5}}{(\varepsilon^{1.5})^2 + (\varepsilon^{1.5})^2 + \varepsilon^{1.5})^{1/2}, \quad t = 2,
\]

combining (1.12), (4.14) and (4.16)–(4.20) yields

\[
\frac{\partial K(Y)}{\partial n} \geq C \varepsilon^3 \langle -DQ(y_1), n \rangle + O(\varepsilon^{3+\alpha h+\tau''})
\]

\[
\geq C \varepsilon^3 |y_1|^{\alpha} + O(\varepsilon^{3+\alpha h+\tau''})
\]

\[
> 0,
\]

(4.21)

where \( \tau'' > 0 \) is a constant.

Combining Steps 1 and 2 we complete the proof of this lemma. \( \square \)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** It follows from Lemma 2.6 and Lemma 4.3 that we just need to prove \( K^{c_{\varepsilon,2}} \) cannot be deformed into \( K^{c_{\varepsilon,1}} \). Then \( K \) has at least one critical point in \( K^{c_{\varepsilon,2}} \setminus K^{c_{\varepsilon,1}} \). Finally, by Lemma 4.2, we obtain that \( u = \sum_{i=1}^k w_{\varepsilon,y_i} + \varphi_{\varepsilon,Y} \) is a critical point of \( I_{\varepsilon} \), in the sense that it is a solution of equation (1.1). Next, we prove \( K^{c_{\varepsilon,2}} \) cannot be deformed into \( K^{c_{\varepsilon,1}} \). It’s easy to know

\[
K^{c_{\varepsilon,2}} = \Omega_{c_{\alpha}}.
\]

Denote

\[
M = B^0_\delta(0) \times B^{3-t}_{c_\varepsilon}(0),
\]

(4.22)

\[
\Gamma_i = \{ (y', y'') \in M, \, |y'| \geq i \},
\]

(4.23)

\[
T_\gamma = \cup \{ \eta \mid |y_i - y_j| \leq \gamma, \quad y_i, y_j \in M \},
\]

(4.24)

\[
\mathcal{L}_{\gamma} = \Gamma_i \times \mathcal{M} \times \cdots \times \mathcal{M} \cup \mathcal{M} \times \Gamma_i \times \cdots \times \mathcal{M} \cup \cdots
\]

\[
\cup (\mathcal{M} \times \cdots \times \mathcal{M} \times \Gamma_i) \cup T_\gamma.
\]

(4.25)

\[
\mathcal{L}_{\delta/2,c\varepsilon^{\alpha h} \varepsilon^{-1}} \setminus T_{\varepsilon R_1} \subset K^{c_{\varepsilon,1}} \setminus L_{c' \varepsilon^{\alpha h}/c_{\varepsilon \varepsilon^{-1}}} \setminus T_{\varepsilon R_1}.
\]

(4.26)

**Step 1:** We claim that there exist constants \( C, \, c' \) with \( C > c' > 0 \), such that

\[
\mathcal{L}_{\delta/2,c\varepsilon^{\alpha h} \varepsilon^{-1}} \setminus T_{\varepsilon R_1} \subset K^{c_{\varepsilon,1}} \setminus L_{c' \varepsilon^{\alpha h}/c_{\varepsilon \varepsilon^{-1}}} \setminus T_{\varepsilon R_1}.
\]

In fact, for any \( Y \in K^{c_{\varepsilon,1}} \), we have \( K(Y) < c_{\varepsilon,1} \). Then by (4.10), we obtain

\[
c_{\varepsilon,1} = \varepsilon^3 (kA - k^2B - T\varepsilon^2h_{1})
\]

\[
> K(Y)
\]

(4.27)

\[
\geq \varepsilon^3 (kA - k^2B) - c' \varepsilon^3 k \sum_{i=1}^k |y_i|^h_{1} - c' \varepsilon^3 \sum_{i \neq j} e^{-n|y_i-y_j|/\varepsilon} + O(\varepsilon^{4+\alpha h}).
\]
Thus, \( |y'_i| \geq c'\varepsilon^{3/h_1} \) or \( \eta|y_i - y_j| \leq C\varepsilon\ln\varepsilon^{-1} \) for some \( i \neq j \). Hence,
\[
K_{c,1} \subset L_{c'\varepsilon^{3/h_1},C\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1}.
\]

On the other hand, choose \( c' > 0 \) sufficiently small. When \( |y'_i| \geq \frac{\delta}{2} \) or \( \eta|y_i - y_j| \leq c'\varepsilon\ln\varepsilon^{-1} \) for some \( i \neq j \), by (4.11), we have \( K(Y) < c_{e,1} \). Then
\[
L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1} \subset K_{c,1}.
\]

So the claim follows.

**Step 2:** Since \( L_{c'\varepsilon^{3/h_1},C\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1} \) can be deformed into \( L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1} \), then \( K_{c,1} \) can be deformed into \( L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1} \). Suppose \( K_{c,2} \) can be deformed into \( K_{c,1} \), then we see that \( \Omega_{\varepsilon R_0} = K_{c,2} \) can be deformed into \( L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \setminus \mathcal{E}_{\varepsilon R_1} \). Hence, \( M \times M \times \cdots \times M \) can be deformed into \( L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \). However,
\[
H^t(M, \Gamma_{\delta/2}) = H^t(B^t(\varepsilon,0), \partial B^t(\varepsilon,0)) \neq 0.
\]

By Lemma 2.7, we obtain
\[
H_k(M \times M \times \cdots \times M, L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}}) \neq 0.
\]

Then \( M \times M \times \cdots \times M \) cannot be deformed into \( L_{\delta/2,c'\varepsilon\ln\varepsilon^{-1}} \). This is a contradiction. 

\[ \square \]

**Appendix A. Energy estimates**

**Lemma A.1.** For \( \varepsilon \) sufficiently small and any \( Y \in D^k_{\varepsilon,\delta} \), we have
\[
J_{\varepsilon}(Y, 0) = \varepsilon^3 \left( kA - k^2B - \frac{2}{q-2} A \sum_{i=1}^{k} (Q(y_i) - 1) \right)
\]

\[ \text{(A.1)} \]

\[
- \int \sum_{i=1}^{k-1} w_{\varepsilon, y_i} \left( \sum_{j=i+1}^{k} w_{\varepsilon, y_j} \right)^{q-1} + O \left( \sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^{3+m}|D^m Q(y_i)| \right) + O \left( \varepsilon^3 \sum_{i \neq j} \frac{\eta|y_i - y_j|}{\varepsilon} + \varepsilon^{4+[h]} \right),
\]

where \( A = \frac{q-2}{2q} \|w\|_{L_q}^2 \) and \( B = \frac{b}{4} \|\nabla w\|_{L_2}^2 \).

**Proof.** By the definition of \( J_{\varepsilon}(Y, \varphi) \), we obtain
\[
J_{\varepsilon}(Y, 0) = L_{\varepsilon}(W_{\varepsilon,Y})
\]

\[
= \frac{1}{2} \|W_{\varepsilon,Y}\|_{e}^2 + \frac{\varepsilon b}{4} \left( \int |\nabla W_{\varepsilon,Y}|^2 \right)^2 - \frac{1}{q} \int Q(x) W_{\varepsilon,Y}^q.
\]

\[ \text{(A.2)} \]

\[
= \frac{1}{2} \sum_{i=1}^{k} \left( \varepsilon^2 a |\nabla w_{\varepsilon,y_i}|^2 + w_{\varepsilon,y_i}^2 \right)
\]

\[
+ \frac{\varepsilon b}{4} \left( \sum_{i=1}^{k} \left( |\nabla w_{\varepsilon,y_i}|^2 \right)^2 \right) + \frac{\varepsilon b}{4} \left( \sum_{i \neq j} \left( |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| \right)^2 \right) - \frac{1}{q} \int Q(x) W_{\varepsilon,Y}^q.
\]
As \( Q(0) = 1 \) and \( w \) is the solution of (2.3), we obtain that \( w_{\varepsilon, y_i} (1 \leq i \leq k) \) satisfies

\[
- \left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon, y_j}|^2 \right) \Delta w_{\varepsilon, y_i} + w_{\varepsilon, y_i} = w_{\varepsilon, y_i}^{q-1}, \quad j = 1, \ldots, k.
\]

Multiplying \( w_{\varepsilon, y_i} \) on both sides of the above equality and integrating, we have

\[
\left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon, y_j}|^2 \right) \int |\nabla w_{\varepsilon, y_i}|^2 + \int w_{\varepsilon, y_i}^2 = \int w_{\varepsilon, y_i}^q.
\]

Summing \( i \) from 1 to \( k \), we obtain

\[
\sum_{i=1}^{k} \left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon, y_j}|^2 \right) \int |\nabla w_{\varepsilon, y_i}|^2 + \sum_{i=1}^{k} \int w_{\varepsilon, y_i}^2 = \sum_{i=1}^{k} \int w_{\varepsilon, y_i}^q.
\]

Then

\[
\sum_{i=1}^{k} \int \left( \varepsilon^2 a |\nabla w_{\varepsilon, y_i}|^2 + w_{\varepsilon, y_i}^2 \right) = -\varepsilon b k \int |\nabla w_{\varepsilon, y_j}|^2 \int \sum_{i=1}^{k} |\nabla w_{\varepsilon, y_i}|^2 + \int \sum_{i=1}^{k} w_{\varepsilon, y_i}^q.
\]

Substituting it into (A.2) yields

\[
J_\varepsilon(Y, 0) = \frac{1}{2} \sum_{i=1}^{k} \int w_{\varepsilon, y_i}^q - \frac{\varepsilon b}{4} \left( \sum_{i=1}^{k} \int |\nabla w_{\varepsilon, y_i}|^2 \right)^2
- \frac{1}{q} \int Q(x) W_{\varepsilon, Y}^q + O \left( \varepsilon^3 \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right)
= \left( \frac{1}{2} - \frac{1}{q} \right) \sum_{i=1}^{k} \int w_{\varepsilon, y_i}^q - \frac{\varepsilon b}{4} \left( \sum_{i=1}^{k} \int |\nabla w_{\varepsilon, y_i}|^2 \right)^2
\]

(A.3)

\[
- \frac{1}{q} \int \left( Q(x) \left( \sum_{i=1}^{k} w_{\varepsilon, y_i} \right)^q - \sum_{i=1}^{k} w_{\varepsilon, y_i}^q \right) + O \left( \varepsilon^3 \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right)
= \left( \frac{1}{2} - \frac{1}{q} \right) \varepsilon^3 k \|w\|_{L^q}^q - \frac{\varepsilon^3 k^2 b}{4} \|\nabla w\|_{L^2}^4 - \frac{1}{q} \int \left( \sum_{i=1}^{k} w_{\varepsilon, y_i} \right)^q - \sum_{i=1}^{k} w_{\varepsilon, y_i}^q
- \frac{1}{q} \int \left( Q(x) - 1 \right) \left( \sum_{i=1}^{k} w_{\varepsilon, y_i} \right)^q + O \left( \varepsilon^3 \sum_{i \neq j} e^{-\eta|y_i - y_j|} \right).
\]
Next, we estimate the third and fourth term of the right side of (A.3) respectively. First, to estimate the third term, we use the following inequalities

\[ |a + b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}| \leq \begin{cases} C|b|^{q-2}|a|^2, & |b| \leq |a|; \\ C|a|^{q-2}|b|^2, & |a| \leq |b|, \\ \end{cases} \]

\[ \leq C|a|^\frac{q}{2}|b|^\frac{q}{2}, \quad (2 < q \leq 3), \]

\[ |a + b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}| \leq C(a^{q-2}b^2 + a^2b^{q-2}), \quad (q > 3). \]

Thus,

\[
\begin{align*}
&\left( \sum_{i=1}^{k} w_{e,y_i} \right)^q - w_{e,y_1}^q - \left( \sum_{i=2}^{k} w_{e,y_i} \right)^q \\
&- qw_{e,y_1}^q \left( \sum_{i=2}^{k} w_{e,y_i} \right) - qw_{e,y_1} \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q-1} \\
&\leq \begin{cases} Cw_{e,y_1}^q \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q}, & 2 < q \leq 3; \\ Cw_{e,y_1}^q \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q-2} + Cw_{e,y_1}^q \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q-2}, & q > 3. \\ \end{cases}
\end{align*}
\]

(A.4)

If \(2 < q \leq 3\), by (2.8), we have

\[
\int w_{e,y_1}^q \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q} \leq C \int \sum_{i=2}^{k} w_{e,y_1}^q w_{e,y_i}^q \leq C \sum_{i=2}^{k} \varepsilon e^{-\frac{q|y_j - y_i|}{\varepsilon}}.
\]

(A.5)

If \(q > 3\), by (2.8), we have

\[
\int \left( w_{e,y_1}^{q-2} \left( \sum_{i=2}^{k} w_{e,y_i} \right)^2 + w_{e,y_1}^2 \left( \sum_{i=2}^{k} w_{e,y_i} \right)^{q-2} \right) \]

\[
\leq \begin{cases} C\varepsilon^3 \sum_{i=2}^{k} e^{-(q-2)\frac{|y_j - y_i|}{\varepsilon}}, & 3 < q \leq 4; \\ C\varepsilon^3 \sum_{i=2}^{k} e^{-2\frac{|y_j - y_i|}{\varepsilon}}, & q > 4. \\ \end{cases}
\]

(A.6)

Denote

\[ 1 + \tilde{\sigma} = \begin{cases} \frac{q}{2}, & 2 < q \leq 3; \\ q - 2, & 3 < q \leq 4; \\ 2, & q > 4. \end{cases} \]

Combining (A.4)–(A.6) yields

\[
\int \left( \sum_{i=1}^{k} w_{e,y_i} \right)^q - \sum_{i=1}^{k} w_{e,y_i}^q \right) = q \int \sum_{i<j} w_{e,y_i}^q w_{e,y_j}^q \\
+ q \int \sum_{i=1}^{k-1} w_{e,y_i} \left( \sum_{j=i+1}^{k} w_{e,y_j} \right)^{q-1} + O \left( \varepsilon^3 \sum_{i<j} e^{-\frac{(1+\sigma)q|y_j - y_i|}{\varepsilon}} \right) \\
= q \int \sum_{i=1}^{k-1} w_{e,y_i} \left( \sum_{j=i+1}^{k} w_{e,y_j} \right)^{q-1} + O \left( \varepsilon^3 \sum_{i<j} e^{-\frac{q|y_j - y_i|}{\varepsilon}} \right).
\]

(A.7)
Secondly, to estimate the fourth term of (A.3), we have
\[ (A.8) \quad \int (Q(x) - 1) \left( \sum_{i=1}^{k} w_{\varepsilon, y_i} \right)^q = \sum_{i=1}^{k} \int (Q(x) - 1) w_{\varepsilon, y_i}^q + O \left( \varepsilon^3 \sum_{i \neq j} e^{-\frac{Q(y_i) - Q(y_j)}{\varepsilon}} \right). \]

Estimating the first term of the right side of (A.8), we obtain
\[ (A.9) \quad \int_{\mathbb{R}^3} (Q(x) - 1) w_{\varepsilon, y_i}^q = \int_{B_{\delta}(y_i)} (Q(x) - Q(y_i)) w_{\varepsilon, y_i}^q + \int_{\mathbb{R}^3} (Q(y_i) - 1) w_{\varepsilon, y_i}^q, \]
\[ (A.10) \quad \left| \int_{B_{\delta}(y_i)} (Q(x) - Q(y_i)) w_{\varepsilon, y_i}^q \right| \leq \int_{B_{\delta}(y_i)} |Q(x) - Q(y_i)| w_{\varepsilon, y_i}^q \leq C \varepsilon^3 \int_{|y| \geq \frac{\varepsilon}{\delta}} w^q(y) \]
where \( \tilde{q} = q - \hat{\theta} \) and \( \hat{\theta} > 0 \) is a small constant, and
\[ (A.11) \quad \int_{B_{\delta}(y_i)} (Q(x) - Q(y_i)) w_{\varepsilon, y_i}^q \leq C \varepsilon^3 \int_{|y| < \frac{\varepsilon}{\delta}} \left( \sum_{m=1}^{[h]} \varepsilon^m |y|^m D^m Q(y_i) + \varepsilon^{[h]+1} |y|^{[h]+1} \right) w^q(y) \]
\[ \leq C \left( \sum_{m=1}^{[h]} \varepsilon^{3+m} |D^m Q(y_i)| + \varepsilon^{4+[h]} \right). \]

Combining (A.9)–(A.11) yields
\[ (A.12) \quad \int (Q(x) - 1) w_{\varepsilon, y_i}^q = (Q(y_i) - 1) \int w_{\varepsilon, y_i}^q + \varepsilon^3 O \left( e^{-\frac{\tilde{q}}{\varepsilon^3}} + \varepsilon^{[h]+1} \right) \]
\[ = (Q(y_i) - 1) \varepsilon^3 \|w\|_{L^q}^q + O(\varepsilon^{4+[h]}) + O \left( \sum_{m=1}^{[h]} \varepsilon^{3+m} |D^m Q(y_i)| \right). \]

Combining (A.3), (A.7), (A.8) and (A.12) yields (A.1).

**Lemma A.2.** For any \( Y \in D_{\varepsilon, \delta}^k \), there holds
\[ \frac{\partial K}{\partial y_{il}} = -C \varepsilon^3 D_l Q(y_i) - (q - 1) \sum_{j=1, j \neq i}^{k} \int w_{\varepsilon, y_i}^{q-2} w_{\varepsilon, y_j} \frac{\partial w_{\varepsilon, y_j}}{\partial y_{il}} \]
\[ + O \left( \sum_{i=1}^{k} \sum_{m=2}^{[h]} \varepsilon^{2+m-\tau} |D^m Q(y_i)|^{1-\tau} + \varepsilon^{3+[h]-\tau} \right) \]
\[ + O \left( \varepsilon^2 \sum_{i=1}^{k} |Q(y_i) - 1|^{1-\tau} \right) + O \left( \varepsilon^2 \sum_{i \neq j} e^{-\frac{2|y_i - y_j|}{\varepsilon}} \right), \]
where \( i = 1, \ldots, k \) and \( l = 1, 2, 3 \).
Proof. By the definition of $K(Y)$, we have

$$\frac{\partial K}{\partial y_{il}} = \frac{\partial J_{\varepsilon}}{\partial y_{il}} + \left\langle \frac{\partial J_{\varepsilon}}{\partial \varphi_{\varepsilon,Y}}, \frac{\partial \varphi_{\varepsilon,Y}}{\partial y_{il}} \right\rangle.$$  

First, estimating the first term of (A.14), we obtain

$$\frac{\partial J_{\varepsilon}}{\partial y_{il}} = \varepsilon^2 a \nabla(W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}$$

$$+ \varepsilon b \int |\nabla(W_{\varepsilon,Y} + \varphi_{\varepsilon,Y})|^2 \int \nabla(W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}$$

(A.15)  

$$- \int Q(x)(W_{\varepsilon,Y} + \varphi_{\varepsilon,Y})^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}$$

$$= \left\langle I'_{\varepsilon}(W_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle I''_{\varepsilon}(W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle R'_{\varepsilon,Y}(\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle$$

$$= l_{\varepsilon,Y} \left( \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) + \left\langle I''_{\varepsilon,Y}(W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle R'_{\varepsilon,Y}(\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle.$$  

To estimate the first term of the right side of (A.15), since $Q(0) = 1$ and $w$ is the solution of (2.3), we obtain $w_{\varepsilon,y_j} (1 \leq j \leq k)$ satisfies

$$- \left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \right) \Delta w_{\varepsilon,y_j} + w_{\varepsilon,y_j} = w_{\varepsilon,y_j}^{q-1}, \quad t = 1, \ldots, k.$$  

Summing $j$ from 1 to $k$, we obtain

$$- \left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \right) \Delta W_{\varepsilon,Y} + W_{\varepsilon,Y} = \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1},$$

Multiplying $\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}$ on both sides of the above equality and integrating, we have

$$\left( \varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \right) \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int W_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} = \int \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}.$$  

Then

$$\left\langle W_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = \int \left( \varepsilon^2 a \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + W_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right)$$

$$= -\varepsilon b k \int |\nabla w_{\varepsilon,y_j}|^2 \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}.$$  

Hence,

$$l_{\varepsilon,Y} \left( \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) = \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \left( \int |\nabla W_{\varepsilon,Y}|^2 - \int \sum_{t=1}^{k} |\nabla w_{\varepsilon,y_j}|^2 \right)$$

$$- \left( \int Q(x) W_{\varepsilon,Y}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} - \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right)$$

(A.16)  

$$= \tilde{l}_1 - \tilde{l}_2.$$
By similar estimates of $l_1$ as that of Lemma 3.2, we have

$$(A.17) \quad |\tilde{l}_1| \leq C\varepsilon^2 \sum_{i\neq j} e^{-\eta|y_i-y_j|}. $$

Next, to estimate $\tilde{l}_2$, we have

$$(A.18) \quad \tilde{l}_2 = \int Q(x) \left( \sum_{j=1}^{k} w_{\varepsilon,y_j} \right)^{q-1} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} - \int \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} $$

$$ = \int \left( \left( \sum_{j=1}^{k} w_{\varepsilon,y_j} \right)^{q-1} - \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1} \right) \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} $$

$$ + \int (Q(x) - 1) \left( \sum_{j=1}^{k} w_{\varepsilon,y_j} \right)^{q-1} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} $$

$$ =: \tilde{l}_{21} + \tilde{l}_{22}. $$

Since

$$(A.19) \quad \tilde{l}_{21} \leq (q-1) \sum_{j=1,j\neq i}^{k} \int w_{\varepsilon,y_i}^{q-2} w_{\varepsilon,y_j} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} + O \left( \varepsilon^2 \sum_{i \neq j} e^{-\eta|y_i-y_j|} \right), $$

and

$$(A.20) \quad \tilde{l}_{22} = \frac{1}{q} \int \frac{\partial Q(x)}{\partial x_i} w_{\varepsilon,y_i}^{q-1} + O \left( \varepsilon^2 \sum_{i \neq j} e^{-\eta|y_i-y_j|} \right) $$

$$ = C\varepsilon^3 D_i Q(y_i) + O \left( \sum_{i=1}^{k} \sum_{m=2}^{[h]} \varepsilon^{2+m} |D^m Q(y_i)| + \varepsilon^{3+[h]} \right) $$

combining (A.17)–(A.20) yields

$$(A.21) \quad l_{\varepsilon,Y} \left( \frac{\partial W_{\varepsilon,Y}}{\partial y_i} \right) = -C\varepsilon^3 D_i Q(y_i) - (q-1) \sum_{j=1,j\neq i}^{k} \int w_{\varepsilon,y_i}^{q-2} w_{\varepsilon,y_j} \frac{\partial w_{\varepsilon,y_i}}{\partial y_i} + O \left( \varepsilon^{3+[h]} \right) $$

$$ + O \left( \sum_{i=1}^{k} \sum_{m=2}^{[h]} \varepsilon^{2+m} |D^m Q(y_i)| \right) + O \left( \varepsilon^2 \sum_{i \neq j} e^{-\eta|y_i-y_j|} \right). $$
Next, we estimate the second term of the right side of (A.15). We have
\[
\left\langle I'''_{\varepsilon} (W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = \left\langle \varphi_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int \nabla \varphi_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \\
+ 2\varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \varphi_{\varepsilon,Y} \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \nabla \varphi_{\varepsilon,Y} \\
- (q-1) \int Q(x) W_{\varepsilon,Y}^{q-2} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \varphi_{\varepsilon,Y}.
\]
(A.22)

Since \( \varphi_{\varepsilon,Y} \in E_{\varepsilon,Y}^k \),
\[
\left\langle \varphi_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = 0.
\]
(A.23)

By Hölder inequality, we obtain
\[
\varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int \nabla \varphi_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \leq \varepsilon b \kappa \sum_{i=1}^k \| \nabla w_{\varepsilon,y_i} \|^2 \| \nabla \varphi_{\varepsilon,Y} \|_{L^2} \| \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \|_{L^2} \\
(A.24) \\
\leq C \varepsilon \frac{1}{2} \| \varphi_{\varepsilon,Y} \|_\varepsilon,
\]
(A.25)

\[
\int Q(x) W_{\varepsilon,Y}^{q-2} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \varphi_{\varepsilon,Y} \leq C \left( \int W_{\varepsilon,Y}^q \right)^{\frac{2a-q}{q}} \left( \int |\nabla W_{\varepsilon,Y}|^q \right)^{\frac{1}{q}} \left( \int |\varphi_{\varepsilon,Y}|^q \right)^{\frac{1}{q}} \\
(A.26) \\
\leq C \varepsilon \frac{1}{2} \| \varphi_{\varepsilon,Y} \|_\varepsilon.
\]

Combining (A.22)–(A.26) yields
\[
\left\langle I'''_{\varepsilon} (W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = O(\varepsilon \frac{1}{2} \| \varphi_{\varepsilon,Y} \|_\varepsilon).
\]
(A.27)

Besides, by Lemma 3.3, we have
\[
\left\langle R'_{\varepsilon,Y} (\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = o_\varepsilon(1) \| \varphi_{\varepsilon,Y} \|_\varepsilon.
\]
(A.28)

By Lemma 3.4, we have
\[
\left\langle \frac{\partial J_{\varepsilon}}{\partial \varphi_{\varepsilon,Y}}, \frac{\partial \varphi_{\varepsilon,Y}}{\partial y_{il}} \right\rangle = 0.
\]
(A.29)

Combining (A.21) and (A.27)–(A.29) yields (A.13).

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