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Solutions with multiple peaks for nonlinear Kirchhoff equations on $\mathbb{R}^3$

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Abstract. In this paper, we mainly investigate the following nonlinear Kirchhoff equation

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + u = Q(x) u^{q-1}, \quad u > 0,\ x \in \mathbb{R}^3, \\ u \to 0, \quad \text{as } |x| \to +\infty, \end{cases}$$

where $a, b > 0$ are constants, $2 < q < 6$, and $\varepsilon > 0$ is a parameter. Under some suitable assumptions on the function $Q(x)$, we obtain that the equation above has positive multi-peak solutions concentrating at a critical point of $Q(x)$ for $\varepsilon > 0$ sufficiently small, by using the finite dimensional reduction method. Different from the local Schrödinger problem, here the corresponding limit problem is a system. Moreover, the nonlocal term brings some new difficulties which involve some technical and complicated estimates.

Epälineaaristen Kirchhoffin yhtälöiden monihuippuisia ratkaisuita avaruudessa $\mathbb{R}^3$

Tiivistelmä. Tässä työssä tutkimme pääasiassa seuraavaa epälineaarista Kirchhoffin yhtälöä

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + u = Q(x) u^{q-1}, \quad u > 0,\ x \in \mathbb{R}^3, \\ u \to 0, \quad \text{kun } |x| \to +\infty, \end{cases}$$

missä $a, b > 0$ ovat vakioita, $2 < q < 6$, ja $\varepsilon > 0$ on parametri. Tehtyämme sopivia oletuksia funktiosta $Q(x)$, voimme osoittaa äärellisulotteisella pelkistysmenetelmällä, että em. yhtälöllä on positiivisia monihuippuisia ratkaisuita, jotka kasaantuvat funktion $Q(x)$ kriittiseen pisteeseen, kun $\varepsilon > 0$ on riittävän pieni. Toisin kuin paikallisessa Schrödingerin ongelmassa, vastaava rajaongelma on nyt yhtälösysteemi. Lisäksi ei-paikallinen termi aiheuttaa tiettyjä uusia hankaluuksia, joihin liittyy teknisiä ja monimutkaisia arvioita.

1. Introduction and main results

In this paper, we consider the following Kirchhoff equation

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + u = Q(x) u^{q-1}, \quad u > 0,\ x \in \mathbb{R}^3, \\ u \to 0, \quad \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where $a, b > 0$ are constants, $2 < q < 6$, $\varepsilon > 0$ is a parameter and $Q(x)\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth bounded function.

Problem (1.1) and its variants have been studied extensively in the literature. To extend the classical D'Alembert's wave equations for free vibration of elastic strings,

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Kirchhoff [22] first proposed the following time-dependent wave equation

$$\rho\frac{\partial^2 u}{\partial^2 t}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial^2 x}=0. \tag{1.2}$$

Bernstein [4] and Pohozaev [31] studied the above type of Kirchhoff equations quite early. In order to study the problem (1.2) preferably, Lions [25] introduced an abstract functional framework to this problem. After that, many interesting results of Kirchhoff equations can be found in e.g. [2, 18, 13, 17, 12, 33] and the reference therein. From a mathematical point of view, Kirchhoff equation is nonlocal, in the sense that the term $\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u$ depends not only on Δu, but also on the integral of $|\nabla u|^2$ over the whole space. This feature brings new mathematical difficulties, which makes the study of Kirchhoff type equations particularly meaningful. We refer to e.g. [19, 20, 21, 30] for mathematical researches on Kirchhoff type equations in bounded domains and in the whole space.

In fact, equation (1.1) is closely related to the Schrödinger equations. When $a = 1$ and $b = 0$, equation (1.1) is reduced to the perturbed Schrödinger equation. Kwong [23] considered the classical Schrödinger equation

$$-\Delta u+u=u^p,\quad x\in\mathbb{R}^N, \tag{1.3}$$

where $1 < p < +\infty$ if $N = 1, 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. Equation (1.3) has a unique radial symmetric and nondegenerate positive solution. Based on this property, Cao, Noussair and Yan [6] proved the existence of multi-peak solutions for equation

$$-\Delta u+\lambda^2 u=Q(x)|u|^{q-2}u,\quad x\in\mathbb{R}^N, \tag{1.4}$$

where $\lambda \neq 0$, $N \geq 3$ and $2 < q < 2N/(N-2)$.

Dancer and Yan [10] studied the following equation

$$\begin{cases}-\varepsilon^2\Delta u+u=Q(y)u^{p-1}, & u>0,\ y\in\mathbb{R}^N,\\ u\to 0, & \text{as } |y|\to+\infty,\end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a parameter, $2 < p < +\infty$ if $N = 2$ and $2 < p < 2N/(N-2)$ if $N > 2$. They not only proved that the Schrödinger equation has positive multi-peak solutions concentrating at a designated saddle point or a strictly local minimum point of $Q(y)$ in $\mathbb{R}^N$, but also showed that there is no multi-peak solution concentrating at a strictly local maximum point of $Q(y)$ in $\mathbb{R}^N$. Besides, many interesting results of multi-peak solutions can be found in e.g. [32, 14, 5, 16, 11, 15, 26, 28, 29] and the reference therein.

Based on the uniqueness and nondegeneracy property of equation (1.3), the authors in [24] proved the uniqueness and nondegeneracy of positive solutions for equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=u^p,\quad u>0\text{ in }\mathbb{R}^3, \tag{1.6}$$

where $1 < p < 5$. Then, by using Lyapunov–Schmidt reduction method, they constructed the existence and the uniqueness of single peak solutions to equation

$$-\left(\varepsilon^2 a+\varepsilon b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=u^p,\quad u>0\ \text{ in }\mathbb{R}^3, \tag{1.7}$$

where $1 < p < 5$ and $\varepsilon > 0$ is sufficiently small.

Luo, Peng and Wang [27] proved equation (1.7) has positive multi-peak solutions concentrating at different points if $\varepsilon > 0$ is sufficiently small. It should be pointed out that, they constructed the multi-peak solutions of equation (1.7) based on the following system

$$-\left(a+b\sum_{j=1}^{k}\int_{\mathbb{R}^3}|\nabla u_j|^2\right)\Delta u_i+V(a_i)u_i=(u_i)^p,\ \ u_i>0,\ i=1,\dots,k,\ x\in\mathbb{R}^3. \tag{1.8}$$

They also showed that a_i $(1 \le i \le k)$ are critical points of V, and there exist the multi-peak solutions of the form $u_\varepsilon = \sum_{i=1}^{k} u_i(\frac{x-y_{\varepsilon,i}}{\varepsilon}) + \varphi_\varepsilon$.

Note that in [27], they did not consider the existence of multi-peak solutions concentrating at the same point $(a_1 = \cdots = a_k)$ to equation (1.7). When $a_1 = a_2 = \cdots = a_k$, $|y_{\varepsilon,i} - y_{\varepsilon,j}| \to 0$ $(i \ne j)$ as $\varepsilon \to 0$, but we cannot get the result of $|y_{\varepsilon,i} - y_{\varepsilon,j}|/\varepsilon \to +\infty$ $(i \ne j)$ as $\varepsilon \to 0$. Therefore, we must impose additional conditions on $y_{\varepsilon,i}$.

Recently, Cui et al. [7] proved the existence and local uniqueness of normalized multi-peak solutions to the following Kirchhoff equation

$$\begin{cases} -\left(a+b_\lambda\int_{\mathbb{R}^3}|\nabla u_\lambda|^2\right)\Delta u_\lambda+(\lambda+V(x))\,u_\lambda=\beta_\lambda u_\lambda^p, \quad u_\lambda>0,\ x\in\mathbb{R}^3,\\ u_\lambda\in H^1(\mathbb{R}^3), \end{cases} \tag{1.9}$$

where $a > 0$, $1 < p < 5$, λ, b_λ, $\beta_\lambda > 0$ are parameters, $\int_{\mathbb{R}^3} u_\lambda^2 = 1$, and $V(x)\colon \mathbb{R}^3 \to \mathbb{R}$ is a bounded continuous function.

Inspired by the literatures [10, 24, 27, 7], we apply the conditions of $Q(x)$ in [10] to $\mathbb{R}^3$. Then, we consider the existence of positive multi-peak solutions to the equation (1.1) concentrating at a critical point of $Q(x)$.

Now we give some definitions and assumptions.

Definition 1.1. Let $m \in N_+$, $a_0 \in \mathbb{R}^3$, we say that u_ε is a m-peak solution of (1.1) if u_ε satisfies

(i) u_ε has m local maximum points $y_{\varepsilon,i} \in \mathbb{R}^3$, $i = 1, \dots, m$, satisfying

$$y_{\varepsilon,i} \to a_0$$

as $\varepsilon \to 0$ for each i;

(ii) For any given $\tau > 0$, there exists $R \gg 1$, such that

$$|u_\varepsilon(x)| \le \tau, \quad x \in \mathbb{R}^3 \setminus \textstyle\bigcup_{i=1}^{m} B_{R\varepsilon}(y_{\varepsilon,i});$$

(iii) There exists $C > 0$ such that

$$\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla u_\varepsilon|^2+u_\varepsilon^2\right)\le C\varepsilon^3.$$

We assume that $Q(x)$ satisfies the following conditions:

(Q_1) $Q(x)$ is a smooth bounded function in $\mathbb{R}^3$.

(Q_2) $x_0 = 0$ is a critical point of $Q(x)$.

(Q_3) $Q(x)$ has the following expansion (after suitably rotating the coordinate system)

$$Q(x) = Q(0) + P_1(x') - P_2(x'') + R(x), \quad x \in B_\delta(0), \tag{1.10}$$

where $Q(0) > 0$, $\delta > 0$, $x = (x', x'')$, $x' = (x_1, ..., x_t)$, $x'' = (x_{t+1}, \dots, x_3)$, $t \in \{1, 2, 3\}$, P_1 and P_2 satisfy

$$P_1(x') = \lambda|x'|^{h_1}, \quad |x'| \le \delta, \tag{1.11}$$

$$\langle DP_2(x''), x''\rangle \geq \lambda |x''|^h, \quad |x''| \leq \delta, \tag{1.12}$$

$$|D^m P_2(x'')| = O(|x''|^{h-m}), \quad m = 0, \dots, [h],\ |x''| \leq \delta, \tag{1.13}$$

for some $h_1 \geq h \geq 2$ and some positive constant $\lambda > 0$, and

$$R(x) = O(|x|^{h_1+\sigma}),\ \text{for some } \sigma > 0 \text{ as } |x| \to 0. \tag{1.14}$$

Remark 1.2. $Q(x) = \sin |x|^2 + 1,\ x \in \mathbb{R}^3$, satisfies the conditions (Q_1)–(Q_3).

For $u, v \in H^1(\mathbb{R}^3)$, denote

$$u_{\varepsilon,y}(x) = u\left(\frac{x-y}{\varepsilon}\right), \quad y \in \mathbb{R}^3,$$

$$\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^3} \left(\varepsilon^2 a \nabla u \nabla v + uv\right), \quad \|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2},$$

$$H_\varepsilon = \{u \in H^1(\mathbb{R}^3) \colon \|u\|_\varepsilon < +\infty\}.$$

The energy functional corresponding to equation (1.1) is

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{\varepsilon b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \frac{1}{q}\int_{\mathbb{R}^3} Q(x) u_+^q, \quad u \in H_\varepsilon, \tag{1.15}$$

where $u_+ = \max(u, 0)$. It is standard to verify that $I_\varepsilon \in C^2(H_\varepsilon)$. So we just need to find a critical point of $I_\varepsilon \in C^2(H_\varepsilon)$.

For $k \in N_+$, let w be the unique positive radial solution (see Lemma 2.2) to equation

$$-\left(a + bk\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + u = Q\,(0)\, u^{q-1}. \tag{1.16}$$

Then we want to construct k-peak solutions to equation (1.1) concentrating at the critical point $x_0 = 0$ of $Q(x)$ by using the uniqueness and nondegeneracy property of w.

Our main result is as follows.

Theorem 1.3. *Assume that $Q(x)$ satisfies (Q_1)–(Q_3). Then, for any $k \in N_+$, there exists $\varepsilon_0 = \varepsilon(k)$ such that for $\varepsilon \in (0, \varepsilon_0]$, equation (1.1) has at least one k-peak solution of the form*

$$u_\varepsilon = \sum_{i=1}^k w_{\varepsilon, y_{\varepsilon,i}} + \varphi_\varepsilon,$$

where $y_{\varepsilon,i} \in \mathbb{R}^3,\ i = 1, \dots, k,\ \varphi_\varepsilon \in H^1(\mathbb{R}^3)$, and as $\varepsilon \to 0$,

$$y_{\varepsilon,i} \to 0, \quad i = 1, \dots, k,$$

$$\frac{|y_{\varepsilon,i} - y_{\varepsilon,j}|}{\varepsilon} \to +\infty, \quad i \neq j,\ i, j = 1, \dots, k,$$

$$\|\varphi_\varepsilon\|_\varepsilon = o(\varepsilon^{\frac{3}{2}}).$$

Remark 1.4. Theorem 1.3 extend the result got by Dancer and Yan [10] about the existence of solutions for the nonlinear Schrödinger equation to the nonlinear Kirchhoff equation (1.1).

In this paper, we prove Theorem 1.3 by using the finite dimensional reduction method. Although the method is standard, we have to make some modifications. Since there is a nonlocal term, we encounter some new difficulties which involve some complicated and technical estimates. To our knowledge, the result we obtain is new.

Our notations are standard. We use $B_r(x)$ (and $\overline{B_r(x)}$) to denote open (and close) balls in $\mathbb{R}^3$ centred at x with radius r, and $B_r^c(x)$ to denote the complementary set of $B_r(x)$ in $\mathbb{R}^3$. Unless otherwise stated, we write $\int u$ to denote Lebesgue integrals over $\mathbb{R}^3$, and $\|u\|_{L^p}$, $\|u\|_{H^1}$ to mean L^p-norm, H^1-norm respectively. We will use $C, C_j (j \in N)$ to denote various positive constants, and $O(t)$, $o(t)$, $o_t(1)$, $o_R(1)$ to mean $|O(t)| \le C|t|$, $o(t)/t \to 0$, $o_t(1) \to 0$ as $t \to 0$ and $o_R(1) \to 0$ as $R \to +\infty$ respectively.

This paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, we give the finite dimensional reduction process and in Section 4, we prove Theorem 1.3.

2. Some preliminaries

In this section, we introduce some preliminaries.

Lemma 2.1. [27, Lemma 2.1] *For any* $2 \le q \le 6$, *there exists a constant* $C > 0$ *independent of* ε, *such that*

$$\|u\|_{L^q} \le C\varepsilon^{\frac{3}{q}-\frac{3}{2}}\|u\|_{\varepsilon}, \quad \forall\, u \in H_\varepsilon. \tag{2.1}$$

Before stating the lemma that follows, we first give a truth that U is a unique positive radial solution to equation

$$\begin{cases} -\Delta u + u = u^{q-1}, \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \\ u(0) = \max\limits_{x\in\mathbb{R}^3} u(x), \end{cases} \tag{2.2}$$

which satisfies

$$\begin{cases} \lim\limits_{|x|\to\infty} |x|e^{|x|}U(|x|) = C > 0, \\ \lim\limits_{|x|\to\infty} \frac{U'(|x|)}{U(|x|)} = -1. \end{cases}$$

The following result is very crucial for applying the finite dimensional reduction.

Lemma 2.2. [24, Theorem 1.2] *For any* $k \in N_+$, $a, b > 0$, $2 < q < 6$, *and* $Q(x)$ *satisfying* (Q_1)–(Q_3), *there exists a unique positive radial solution* $w \in H^1(\mathbb{R}^3)$ *satisfying*

$$-\left(a + bk\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + u = Q(0)u^{q-1}. \tag{2.3}$$

Moreover, w *is nondegenerate in* $H^1(\mathbb{R}^3)$ *in the sense that there holds*

$$\operatorname{Ker}\mathcal{L} = \operatorname{span}\left\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3}\right\},$$

where $\mathcal{L}\colon H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ *is the linear operator defined as*

$$\begin{aligned} \mathcal{L}\varphi = &-\left(a + bk\int |\nabla w|^2\right)\Delta\varphi - 2bk\Big(\int \nabla w\nabla\varphi\Big)\Delta w \\ &+ \varphi - (q-1)Q(0)w^{q-2}\varphi, \quad \forall \varphi \in H^1(\mathbb{R}^3). \end{aligned} \tag{2.4}$$

The proof of Theorem 1.2 in [24] implies that $w(x)=\lambda_1 U(\eta x)$ where $\eta=\frac{1}{\sqrt{c_k}}$ and $c_k=a+bk\int|\nabla u|^2$. And since U decays exponentially at infinity, we infer that

$$\nabla w(x),\ w(x)=O(\ |x|^{-1}e^{-\eta|x|})\quad \text{as } |x|\to\infty. \tag{2.5}$$

Lemma 2.3. [1, Lemma 3.7] *Assume $u,u'\colon\mathbb{R}^n\to\mathbb{R}$ are positive radial continuous functions satisfying*

$$u(x)\sim|x|^a e^{-b|x|},\ u'(x)\sim|x|^{a'}e^{-b'|x|}\ (|x|\to\infty),$$

where $a,a'\in\mathbb{R},\ b,b'>0$. Let $\xi\in\mathbb{R}^n$ tend to infinity and $u_\xi(x)=u(x-\xi)$. Then the following asymptotic estimates hold:

(i) *If $b<b'$, then*

$$\int_{\mathbb{R}^n}u_\xi u'\sim e^{-b|\xi|}|\xi|^a.$$

If $b>b'$, then replace a and b with a' and b'.

(ii) *If $b=b'$, suppose, for simplicity, that $a\ge a'$, then*

$$\int_{\mathbb{R}^n}u_\xi u'\sim\begin{cases}e^{-b|\xi|}|\xi|^{a+a'+\frac{n+1}{2}}, & a'>-\frac{n+1}{2};\\ e^{-b|\xi|}|\xi|^a\log|\xi|, & a'=-\frac{n+1}{2};\\ e^{-b|\xi|}|\xi|^a, & a'<-\frac{n+1}{2}.\end{cases}$$

Combining (2.5) and Lemma 2.3 yields, for any $r,s>0$ with $r\neq s$, we have, as $\varepsilon\to0$,

$$\int_{\mathbb{R}^3}w_{\varepsilon,y_i}^r w_{\varepsilon,y_j}^s=O\left(\varepsilon^3 e^{-\min\{r,s\}\frac{\eta|y_i-y_j|}{\varepsilon}}\left|\frac{y_i-y_j}{\varepsilon}\right|^{-\min\{r,s\}}\right), \tag{2.6}$$

$$\int_{\mathbb{R}^3}|\nabla w_{\varepsilon,y_i}\nabla w_{\varepsilon,y_j}|=O\left(\varepsilon e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\right). \tag{2.7}$$

Particularly, there exist some constants $C>0$, such that

$$\int_{\mathbb{R}^3}w_{\varepsilon,y_i}^r w_{\varepsilon,y_j}^s\le C\varepsilon^3 e^{-\min\{r,s\}\frac{\eta|y_i-y_j|}{\varepsilon}}, \tag{2.8}$$

$$\int_{\mathbb{R}^3}|\nabla w_{\varepsilon,y_i}\nabla w_{\varepsilon,y_j}|\le C\varepsilon e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}. \tag{2.9}$$

In the following sections, we will use inequalities (2.1), (2.8) and (2.9) repeatedly. When $r=s$, the situation is complicated. But when $r=s\ge1$, inequality (2.8) still holds.

Definition 2.4. [8, Definition B.1] Let Y and A be closed subsets of a topological space X. Then $Cat_X(A,Y)$ is the least integer k such that $A=\cup_{j=0}^k A_j$, where, for $0\le j\le k$, A_j is closed, and there exists $h_j\in C\big([0,1]\times A_j,X\big)$ such that

(i) $h_j(0,x)=x$ for $x\in A_j,\ 0\le j\le k$;
(ii) $h_0(1,x)\in Y$ for $x\in A_0$ and $h_0(t,x)=x$ for $x\in A_0\cap Y$ and $t\in[0,1]$;
(iii) $h_j(1,x)=x_j$ for $x\in A_j$ and some $x_j\in X,\ 1\le j\le k$.

Particularly, if Y is empty, then write $Cat_X(A)=Cat_X(A,\varnothing)$.

From the Definition 2.4, we see $Cat_X(A,Y)\ge1$ if A can not be deformed into a subset of Y within X.

Lemma 2.5. [8, Proposition 2.2] *Suppose that $F(x)$ is a C^2 function defined in a bounded domain $\Omega \subset \mathbb{R}^{3k}$. If F satisfies either $F(x) > c$ or $\frac{\partial F(x)}{\partial n} > 0$ at each $x \in \partial\Omega$, where n is the outward unit normal of $\partial\Omega$ at x, then*

$$\#\big\{x \colon DF(x) = 0,\ x \in F^c\big\} \geq Cat_{F^c}(F^c), \tag{2.10}$$

where $F^c = \big\{x \colon x \in \Omega,\ F(x) \leq c\big\}$. In particular, $F(x)$ has at least one critical point in F^c.

Lemma 2.6. [8, Proposition 2.3] *Suppose that $F(x)$ is a C^2 function defined in a bounded domain $\Omega \subset \mathbb{R}^{3k}$. Let $c_1,\ c_2$ be two constants such that neither c_2 nor c_1 is a critical value of $F(x)$. If F satisfies either $F(x) < c_1$ or $\frac{\partial F(x)}{\partial n} > 0$ for each $x \in \partial\Omega$, then*

$$\#\big\{x \colon DF(x) = 0,\ x \in F^{c_2} \backslash F^{c_1}\big\} \geq Cat_{F^{c_2}}(F^{c_2}, F^{c_1}). \tag{2.11}$$

In particular, if F^{c_2} cannot be deformed into F^{c_1}, F has at least one critical point in $F^{c_2} \backslash F^{c_1}$.

Lemma 2.7. [9, Proposition B.1] *Suppose that X and Γ are two compact sets in $\mathbb{R}^3$ satisfying $\Gamma \subset X$. Let*

$$K = \underbrace{X \times \cdots \times X}_{k}, \tag{2.12}$$

$$L_1 = \underbrace{\Gamma \times X \times \cdots \times X}_{k} \cup \underbrace{X \times \Gamma \times \cdots \times X}_{k} \cup \underbrace{X \times \cdots X \times \Gamma}_{k}, \tag{2.13}$$

$$L_2 = L_1 \cup D, \quad D = \big\{Y = (y_1, \ldots, y_k) \colon y_i = y_j \text{ for some } i \neq j\big\}. \tag{2.14}$$

If $H^m(X, \Gamma) \neq 0$ for some $m \geq 1$, then $H_*(K, L_2) \neq 0$. In particular, K cannot be deformed into L_2.

3. Finite dimensional reduction

In this section we complete the finite dimensional reduction process. Denote

$$Y = (y_1, \ldots, y_k) \in \mathbb{R}^3 \times \cdots \times \mathbb{R}^3, \quad W_{\varepsilon,Y} = \sum_{i=1}^{k} w_{\varepsilon,y_i},$$

$$D^k_{\varepsilon,\delta} = \left\{Y : y_i \in \overline{B_\delta(0)},\ \frac{\eta|y_i - y_j|}{\varepsilon} \geq R_1,\ i,j = 1, \ldots, k,\ i \neq j\right\},$$

where $R_1 > 0$ is a fixed large constant. Let

$$E^k_{\varepsilon,Y} = \left\{\varphi \in H_\varepsilon \colon \left\langle \varphi, \frac{\partial w_{\varepsilon,y_i}}{\partial y_{ij}} \right\rangle_\varepsilon = 0,\ i = 1, \ldots, k,\ j = 1,2,3\right\}$$

and define

$$J_\varepsilon\left(Y, \varphi\right) = I_\varepsilon\left(W_{\varepsilon,Y} + \varphi\right), \quad \forall \left(Y, \varphi\right) \in D^k_{\varepsilon,\delta} \times E^k_{\varepsilon,Y}.$$

Expand $J_\varepsilon(Y, \varphi)$ near $\varphi = 0$ for each fixed Y:

$$J_\varepsilon\left(Y, \varphi\right) = J_\varepsilon\left(Y, 0\right) + l_{\varepsilon,Y}\left(\varphi\right) + \frac{1}{2} Q_{\varepsilon,Y}(\varphi) + R_{\varepsilon,Y}\left(\varphi\right), \tag{3.1}$$

where

$$\begin{aligned} l_{\varepsilon,Y}\left(\varphi\right) &= \left\langle I_\varepsilon'\left(W_{\varepsilon,Y}\right), \varphi\right\rangle \\ &= \left\langle W_{\varepsilon,Y}, \varphi\right\rangle_\varepsilon + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int \nabla W_{\varepsilon,Y} \nabla \varphi - \int Q\left(x\right) W_{\varepsilon,Y}^{q-1} \varphi, \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
Q_{\varepsilon,Y}(\varphi) &= \langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi], \varphi\rangle \\
&= \langle \varphi,\varphi\rangle_\varepsilon + 2\varepsilon b\Big(\int \nabla W_{\varepsilon,Y}\nabla\varphi\Big)^2 \\
&\quad + \varepsilon b\int |\nabla W_{\varepsilon,Y}|^2\int|\nabla\varphi|^2 - (q-1)\int Q(x)W_{\varepsilon,Y}^{q-2}\varphi^2,
\end{aligned} \tag{3.3}
$$

and

$$
\begin{aligned}
R_{\varepsilon,Y}(\varphi) = \frac{\varepsilon b}{4}\Big(\int|\nabla\varphi|^2\Big)^2 + \varepsilon b\int\nabla W_{\varepsilon,Y}\nabla\varphi\int|\nabla\varphi|^2 - \frac{1}{q}\int Q(x)(W_{\varepsilon,Y}+\varphi)_+^q \\
+\frac{1}{q}\int Q(x)W_{\varepsilon,Y}^q + \int Q(x)W_{\varepsilon,Y}^{q-1}\varphi + \frac{1}{2}(q-1)\int Q(x)W_{\varepsilon,Y}^{q-2}\varphi^2.
\end{aligned} \tag{3.4}
$$

In terms of $Q_{\varepsilon,Y}(\varphi)$, $\mathcal{L}_{\varepsilon,Y}\colon E^k_{\varepsilon,Y}\to E^k_{\varepsilon,Y}$ is a bounded linear mapping defined by:

$$
\begin{aligned}
\langle\mathcal{L}_{\varepsilon,Y}\varphi_1,\varphi_2\rangle_\varepsilon &= \langle\varphi_1,\varphi_2\rangle_\varepsilon + \varepsilon b\int|\nabla W_{\varepsilon,Y}|^2\int\nabla\varphi_1\nabla\varphi_2 \\
&\quad + 2\varepsilon b\int\nabla W_{\varepsilon,Y}\nabla\varphi_1\int\nabla W_{\varepsilon,Y}\nabla\varphi_2 \\
&\quad - (q-1)\int Q(x)W_{\varepsilon,Y}^{q-2}\varphi_1\varphi_2, \quad \forall\ \varphi_1,\varphi_2\in E^k_{\varepsilon,Y}.
\end{aligned} \tag{3.5}
$$

The following result shows that $\mathcal{L}_{\varepsilon,Y}$ is invertible when restricted on $E^k_{\varepsilon,Y}$.

Lemma 3.1. *There exist $\rho>0$, $\varepsilon_0>0$ and $\delta_0>0$, such that for every $\varepsilon\in(0,\varepsilon_0]$ and $\delta\in(0,\delta_0]$, there holds*

$$
\|\mathcal{L}_{\varepsilon,Y}\varphi\|_\varepsilon \ge \rho\|\varphi\|_\varepsilon, \quad \forall\ \varphi\in E^k_{\varepsilon,Y},
$$

uniformly with respect to $Y\in D^k_{\varepsilon,\delta}$.

Proof. We use a contradiction argument. Assume that there exist $\varepsilon_n\to 0$, $\delta_n\to 0$, $Y_n\in D^k_{\varepsilon_n,\delta_n}$ and $\varphi_n\in E_n\equiv E^k_{\varepsilon_n,Y_n}$ such that

$$
\langle\mathcal{L}_{\varepsilon_n,Y_n}\varphi_n,h_n\rangle_{\varepsilon_n} = o_n(1)\|\varphi_n\|_{\varepsilon_n}\|h_n\|_{\varepsilon_n},\ \forall\ h_n\in E_n. \tag{3.6}
$$

With no loss of generality, we assume that $\|\varphi_n\|^2_{\varepsilon_n}=\varepsilon_n^3$, and denote

$$
\begin{aligned}
\varphi_{n,i_0}(x) &= \varphi_n(\varepsilon_n x + y_{n,i_0}), \quad i_0 = 1,2,\ldots,k, \\
\tilde{E}_n &= \{h_{n,i_0}(x)\colon h_n(x)\in E_n, 1\le i_0\le k\}.
\end{aligned}
$$

Substituting (3.5) into (3.6), we obtain

$$
\begin{aligned}
&\int(a\nabla\varphi_{n,i_0}\nabla h_{n,i_0} + \varphi_{n,i_0}h_{n,i_0}) \\
&+ b\int\Big(k|\nabla w|^2 + \sum_{i\ne j}\nabla w\Big(x+\frac{y_{n,i_0}-y_{n,i}}{\varepsilon_n}\Big)\nabla w\Big(x+\frac{y_{n,i_0}-y_{n,j}}{\varepsilon_n}\Big)\Big)\int\nabla\varphi_{n,i_0}\nabla h_{n,i_0} \\
&+ 2b\int\sum_{i=1}^k\nabla w\Big(x+\frac{y_{n,i_0}-y_{n,i}}{\varepsilon_n}\Big)\nabla\varphi_{n,i_0}\int\sum_{i=1}^k\nabla w\Big(x+\frac{y_{n,i_0}-y_{n,j}}{\varepsilon_n}\Big)\nabla h_{n,i_0} \\
&- (q-1)\int Q(\varepsilon_n x + y_{n,i_0})\Big(\sum_{i=1}^k w\Big(x+\frac{y_{n,i_0}-y_{n,i}}{\varepsilon_n}\Big)\Big)^{q-2}\varphi_{n,i_0}h_{n,i_0}
\end{aligned}
$$

$$= o_n(1)\bigg(\int \big(a|\nabla h_{n,i_0}|^2 + (h_{n,i_0})^2\big)\bigg)^{\frac{1}{2}}. \tag{3.7}$$

Since

$$\|\varphi_n\|_{\varepsilon_n}^2 = \varepsilon_n^3 \Rightarrow \int \big(a|\nabla \varphi_{n,i_0}|^2 + (\varphi_{n,i_0})^2\big) = 1,$$

we infer that $\{\varphi_{n,i_0}\}$ is a bounded sequence in $H^1(\mathbb{R}^3)$ for any $1 \le i_0 \le k$. Hence, up to a subsequence, there exists $\varphi \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &\varphi_{n,i_0} \rightharpoonup \varphi \quad \text{in } H^1(\mathbb{R}^3),\\ &\varphi_{n,i_0} \to \varphi \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^3),\ 1 \le p < 6,\\ &\varphi_{n,i_0} \to \varphi \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Next, we will prove that $\varphi \equiv 0$. For any $i = 1, \dots, k$, $j = 1, 2, 3$,

$$\int \bigg(\varepsilon_n^2 a \nabla \varphi_n \nabla\bigg(\frac{\partial w_{\varepsilon,y_i}}{\partial y_{ij}}\bigg) + \varphi_n \frac{\partial w_{\varepsilon,y_i}}{\partial y_{ij}}\bigg) = 0$$

is equivalent to

$$\int \bigg(a \nabla \varphi_{n,i_0} \nabla\bigg(\frac{\partial w}{\partial x_j}\bigg) + \varphi_{n,i_0} \frac{\partial w}{\partial x_j}\bigg) = 0. \tag{3.8}$$

Thus, we can define an equivalent norm $\|u\|_1^2 = \int (a|\nabla u|^2 + u^2)$ in $H^1(\mathbb{R}^3)$, then

$$\varphi_n \in E_n$$

is equivalent to

$$\varphi_{n,i_0} \in (\ker \mathcal{L})^{\perp}. \tag{3.9}$$

Since w is radially symmetric, we obtain

$$\bigg\langle \frac{\partial w}{\partial x_i}, \frac{\partial w}{\partial x_j}\bigg\rangle_1 = 0, \quad \forall\ i \ne j.$$

For every $h \in C_0^\infty(\mathbb{R}^3)$, define

$$h_{n,i_0} = h - \sum_{j=1}^{3} a_{n,j} \frac{\partial w}{\partial x_j} \tag{3.10}$$

where $a_{n,j} = \frac{\langle h, \partial_{x_j} w\rangle_1}{\langle \partial_{x_j} w, \partial_{x_j} w\rangle_1}$, then $h_n \in E_n$. Substituting (3.10) into (3.7) and letting $n \to \infty$, we obtain

$$\langle \mathcal{L}\varphi, h\rangle - \bigg\langle \mathcal{L}\varphi, \sum_{j=1}^{3} a_{n,j} \partial_{x_j} w \bigg\rangle = 0.$$

Since $\partial_{x_j} w \in \operatorname{Ker} \mathcal{L}$,

$$\langle \mathcal{L}\varphi, h\rangle = 0, \quad \forall\ h \in C_0^\infty(\mathbb{R}^3),$$

which implies that

$$\varphi \in \operatorname{Ker} \mathcal{L}. \tag{3.11}$$

Since $\varphi_n \in E_n$, letting $n \to \infty$ in (3.8), we obtain

$$\int \bigg(a \nabla \varphi \nabla\bigg(\frac{\partial w}{\partial x_j}\bigg) + \varphi \frac{\partial w}{\partial x_j}\bigg) = 0, \quad j = 1, 2, 3.$$

Then

$$\varphi \in (\operatorname{Ker} \mathcal{L})^{\perp}. \tag{3.12}$$

Combining (3.11) and (3.12), we claim that $\varphi \equiv 0$.

Now we deduce contradiction. Note that $\varphi_{n,i_0} \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ $(1 \le p < 6)$, so there exists $R > 0$ sufficiently large such that

$$\int_{\mathbb{R}^3} w^{q-2}(x)(\varphi_{n,i_0})^2 = \int_{B_R(0)} w^{q-2}(x)(\varphi_{n,i_0})^2 + \int_{B_R^c(0)} w^{q-2}(x)(\varphi_{n,i_0})^2 = o_n(1) + o_R(1).$$

Then

$$\begin{aligned}
&\left|(q-1)\int_{\mathbb{R}^3} Q(x)W^{q-2}_{\varepsilon_n,Y_n}\varphi_n^2\right| \\
&\le C\varepsilon_n^3 \int_{\mathbb{R}^3}\left(\sum_{i=1}^k w\left(x+\frac{y_{n,i_0}-y_{n,i}}{\varepsilon_n}\right)\right)^{q-2}(\varphi_{n,i_0})^2 \\
&\le C\varepsilon_n^3 \int_{\mathbb{R}^3} w^{q-2}(x)(\varphi_{n,i_0})^2 + C\varepsilon_n^3\sum_{i\neq i_0}\int_{\mathbb{R}^3} w^{q-2}\left(x+\frac{y_{n,i_0}-y_{n,i}}{\varepsilon_n}\right)(\varphi_{n,i_0})^2 \\
&\le \frac{1}{2}\varepsilon_n^3.
\end{aligned}$$

However,

$$\begin{aligned}
o_n(1)\|\varphi_n\|^2_{\varepsilon_n} &= \langle \mathcal{L}_{\varepsilon_n,Y_n}\varphi_n, \varphi_n\rangle \\
&\ge \|\varphi_n\|^2_{\varepsilon_n} - (q-1)\int_{\mathbb{R}^3} Q(y)W^{q-2}_{\varepsilon_n,Y_n}(y)\varphi_n^2(y) \\
&\ge \frac{1}{2}\|\varphi_n\|^2_{\varepsilon_n}.
\end{aligned}$$

We reach a contradiction. The proof is complete. □

To apply contraction mapping principle to find a critical point of $J_\varepsilon(Y,\varphi)$, we first need to estimate $l_{\varepsilon,Y}(\varphi)$ and $R^{(i)}_{\varepsilon,Y}(\varphi)$ for $i = 0, 1, 2$.

Lemma 3.2. *There exists a constant $C > 0$, independent of ε, δ, such that for any $Y \in D^k_{\varepsilon,\delta}$ and $\varphi \in H_\varepsilon$, there holds*

$$\begin{aligned}
|l_{\varepsilon,Y}(\varphi)| \le C\varepsilon^{\frac{3}{2}}\Bigg(&\sum_{i=1}^k |Q(y_i)-Q(0)| + \sum_{i=1}^k\sum_{m=1}^{[h]}\varepsilon^m|D^mQ(y_i)| + \varepsilon^{[h]+1} \\
&+ \sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Bigg)\|\varphi\|_\varepsilon,
\end{aligned} \tag{3.13}$$

where $\bar{\theta} = \min\{\frac{q-1}{2}, 1\}$.

Proof. Since w is the solution of (2.3), we obtain that w_{ε,y_i} $(1 \le i \le k)$ satisfies

$$-\left(\varepsilon^2 a + \varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\right)\Delta w_{\varepsilon,y_i} + w_{\varepsilon,y_i} = Q(0)w^{q-1}_{\varepsilon,y_i}, \quad j = 1,\dots,k.$$

We sum from $i = 1$ to k and get

$$-\left(\varepsilon^2 a + \varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\right)\Delta W_{\varepsilon,Y} + W_{\varepsilon,Y} = Q(0)\sum_{i=1}^k w^{q-1}_{\varepsilon,y_i}.$$

Multiplying φ on both sides of above equation and integrating, we obtain

$$\Big(\varepsilon^2 a+\varepsilon bk\int|\nabla w_{\varepsilon,y_j}|^2\Big)\int\nabla W_{\varepsilon,Y}\nabla\varphi+\int W_{\varepsilon,Y}\varphi=\int Q(0)\sum_{i=1}^{k}w_{\varepsilon,y_i}^{q-1}\varphi.$$

Then

$$\begin{aligned}\langle W_{\varepsilon,Y},\varphi\rangle_\varepsilon&=\int\big(\varepsilon^2 a\nabla W_{\varepsilon,Y}\nabla\varphi+W_{\varepsilon,Y}\varphi\big)\\&=-\varepsilon bk\int|\nabla w_{\varepsilon,y_j}|^2\int\nabla W_{\varepsilon,Y}\nabla\varphi+\int Q(0)\sum_{i=1}^{k}w_{\varepsilon,y_i}^{q-1}\varphi.\end{aligned}\tag{3.14}$$

Substituting (3.14) into (3.2), we obtain

$$\begin{aligned}l_{\varepsilon,Y}(\varphi)=\varepsilon b&\int\nabla W_{\varepsilon,Y}\nabla\varphi\Big(\int|\nabla W_{\varepsilon,Y}|^2-\int\sum_{j=1}^{k}|\nabla w_{\varepsilon,y_j}|^2\Big)\\&-\Big(\int Q(x)W_{\varepsilon,Y}^{q-1}\varphi-\int Q(0)\sum_{i=1}^{k}w_{\varepsilon,y_i}^{q-1}\varphi\Big)\\=:&\ l_1-l_2.\end{aligned}$$

To estimate l_1, combining Hölder inequality and (2.9) yields

$$\begin{aligned}|l_1|&=\Big|\varepsilon b\int\nabla W_{\varepsilon,Y}\nabla\varphi\Big(\int|\nabla W_{\varepsilon,Y}|^2-\int\sum_{j=1}^{k}|\nabla w_{\varepsilon,y_j}|^2\Big)\Big|\\&\le\varepsilon b\Big(\sum_{i=1}^{k}\int|\nabla w_{\varepsilon,y_i}\nabla\varphi|\Big)\Big(\sum_{i\neq j}\int|\nabla w_{\varepsilon,y_i}\nabla w_{\varepsilon,y_j}|\Big)\\&\le\varepsilon^2 b\sum_{i\neq j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big(\sum_{i=1}^{k}\|\nabla w_{\varepsilon,y_i}\|_{L^2}\|\nabla\varphi\|_{L^2}\Big)\\&\le\varepsilon^2 b\sum_{i\neq j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big(k\varepsilon^{\frac12}\|\nabla w\|_{L^2}\frac{1}{\sqrt{a}\varepsilon}\|\varphi\|_\varepsilon\Big)\\&\le C\varepsilon^{\frac32}\sum_{i\neq j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\|\varphi\|_\varepsilon.\end{aligned}\tag{3.15}$$

Next, we split l_2 into two parts:

$$\begin{aligned}l_2&=\int Q(x)\Big(\sum_{i=1}^{k}w_{\varepsilon,y_i}\Big)^{q-1}\varphi-\sum_{i=1}^{k}Q(0)w_{\varepsilon,y_i}^{q-1}\varphi\\&=\int Q(x)\Big(\Big(\sum_{i=1}^{k}w_{\varepsilon,y_i}\Big)^{q-1}-\sum_{i=1}^{k}w_{\varepsilon,y_i}^{q-1}\Big)\varphi+\int(Q(x)-Q(0))\sum_{i=1}^{k}w_{\varepsilon,y_i}^{q-1}\varphi\\&=:l_{21}+l_{22}.\end{aligned}\tag{3.16}$$

To estimate l_{21}, for $2<q\le3$, by the following inequality

$$\begin{aligned}\big||a+b|^{q-1}-|a|^{q-1}-|b|^{q-1}\big|&\le\begin{cases}C|a||b|^{q-2}, & \text{if } |a|\le|b|,\\ C|b||a|^{q-2}, & \text{if } |b|\le|a|,\end{cases}\\&\le C|a|^{\frac{q-1}{2}}|b|^{\frac{q-1}{2}},\end{aligned}$$

we obtain

$$
\begin{aligned}
|l_{21}| &= \Big|\int Q(x)\Big(\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^{q-1} - \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1}\Big)\varphi\Big| \\
&\le C\int \sum_{i\neq j} w_{\varepsilon,y_i}^{\frac{q-1}{2}} w_{\varepsilon,y_j}^{\frac{q-1}{2}} |\varphi| \le C\sum_{i\neq j}\Big(\int w_{\varepsilon,y_i}^{q-1} w_{\varepsilon,y_j}^{q-1}\Big)^{\frac{1}{2}} \|\varphi\|_{L^2} \\
&\le C\varepsilon^{\frac{3}{2}} \sum_{i\neq j} e^{-\frac{q-1}{2}\frac{\eta|y_i-y_j|}{\varepsilon}} \|\varphi\|_{\varepsilon}.
\end{aligned} \tag{3.17}
$$

For $q > 3$, we have

$$
\begin{aligned}
|l_{21}| &= \Big|\int Q(x)\Big(\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^{q-1} - \sum_{i=1}^{k} w_{\varepsilon,y_i}^{q-1}\Big)\varphi\Big| \\
&\le C\int \sum_{i\neq j} w_{\varepsilon,y_i}^{q-2} w_{\varepsilon,y_j} |\varphi| \le C\varepsilon^{\frac{3}{2}} \sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}} \|\varphi\|_{\varepsilon}.
\end{aligned} \tag{3.18}
$$

Combining (3.17) and (3.18) yields

$$
|l_{21}| \le C\varepsilon^{\frac{3}{2}} \sum_{i\neq j} e^{-\min\{\frac{q-1}{2},1\}\frac{\eta|y_i-y_j|}{\varepsilon}} \|\varphi\|_{\varepsilon}. \tag{3.19}
$$

To estimate l_{22}, we split l_{22} into two parts:

$$
\begin{aligned}
|l_{22}| &= \int \sum_{i=1}^{k} (Q(x) - Q(0))\, w_{\varepsilon,y_i}^{q-1}\varphi \\
&= \int \sum_{i=1}^{k} (Q(x) - Q(y_i))\, w_{\varepsilon,y_i}^{q-1}\varphi + \int \sum_{i=1}^{k} (Q(y_i) - Q(0))\, w_{\varepsilon,y_i}^{q-1}\varphi \\
&=: l_{221} + l_{222}.
\end{aligned}
$$

Estimating l_{221}, we have

$$
\begin{aligned}
|l_{221}| &\le \sum_{i=1}^{k} \int |Q(x) - Q(y_i)| w_{\varepsilon,y_i}^{q-1} |\varphi| \\
&\le \sum_{i=1}^{k} \Big(\int |Q(x) - Q(y_i)|^2 w_{\varepsilon,y_i}^{2(q-1)}\Big)^{\frac{1}{2}} \|\varphi\|_{L^2} \\
&= \sum_{i=1}^{k} \Big(\int_{B_\delta(y_i)} |Q(x) - Q(y_i)|^2 w_{\varepsilon,y_i}^{2(q-1)} + \int_{B_\delta^c(y_i)} |(Q(x) - Q(y_i)|^2 w_{\varepsilon,y_i}^{2(q-1)}\Big)^{\frac{1}{2}} \|\varphi\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{i=1}^{k}\bigg(\varepsilon^3\int_{|y|<\frac{\delta}{\varepsilon}}\Big(\sum_{m=1}^{[h]}\varepsilon^{2m}|D^mQ(y_i)|^2|y|^{2m}+\varepsilon^{2([h]+1)}|y|^{2([h]+1)}\Big)w^{2(q-1)}(y)\bigg)^{\frac{1}{2}}\|\varphi\|_{L^2}\\
&\quad+C\sum_{i=1}^{k}\bigg(\varepsilon^3\int_{|y|\geq\frac{\delta}{\varepsilon}}w^{2(q-1)}(y)\bigg)^{\frac{1}{2}}\|\varphi\|_{L^2}\\
&\leq C\varepsilon^{\frac{3}{2}}\sum_{i=1}^{k}\bigg(C_1\sum_{m=1}^{[h]}\varepsilon^m|D^mQ(y_i)|+C_2\varepsilon^{[h]+1}+C_3e^{-(q-1)\frac{\eta\delta}{2\varepsilon}}\bigg)\|\varphi\|_{L^2}\\
&\leq C\varepsilon^{\frac{3}{2}}\bigg(\sum_{i=1}^{k}\sum_{m=1}^{[h]}\varepsilon^m|D^mQ(y_i)|+\varepsilon^{[h]+1}\bigg)\|\varphi\|_{\varepsilon}.
\end{aligned}
$$

Finally, to estimate l_{222}, combining Hölder inequality and Lemma 2.1, we obtain

$$
\begin{aligned}
|l_{222}|&=\left|\int\sum_{i=1}^{k}\left(Q(y_i)-Q(0)\right)w_{\varepsilon,y_i}^{q-1}\varphi\right|\\
&\leq\sum_{i=1}^{k}|Q(y_i)-Q(0)|\int w_{\varepsilon,y_i}^{q-1}|\varphi|\\
&\leq\sum_{i=1}^{k}|Q(y_i)-Q(0)|\left(\int w_{\varepsilon,y_i}^{q}\right)^{\frac{q-1}{q}}\left(\int|\varphi|^q\right)^{\frac{1}{q}}\\
&\leq C\varepsilon^{\frac{3}{2}}\sum_{i=1}^{k}|Q(y_i)-Q(0)|\,\|\varphi\|_{\varepsilon}.
\end{aligned}
\tag{3.20}
$$

Combining (3.15) and (3.19)–(3.20) yields (3.13). □

Lemma 3.3. *There exists a constant $C>0$, independent of ε, δ, such that for any $\varphi\in H_\varepsilon$, there holds*

$$
\|R_{\varepsilon,Y}^{(i)}(\varphi)\|\leq Cb\varepsilon^{-\frac{3}{2}}\big(1+\varepsilon^{-\frac{3}{2}}\|\varphi\|_{\varepsilon}\big)\|\varphi\|_{\varepsilon}^{3-i}+C\varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_{\varepsilon}^{q-i},\ i=0,1,2.
\tag{3.21}
$$

Proof. By (3.4), we have

$$
R_{\varepsilon,Y}(\varphi)=A_1(\varphi)-A_2(\varphi),
$$

where

$$
A_1(\varphi)=\frac{\varepsilon b}{4}\Big(\int|\nabla\varphi|^2\Big)^2+\varepsilon b\int\nabla W_{\varepsilon,Y}\nabla\varphi\int|\nabla\varphi|^2
\tag{3.22}
$$

and

$$
A_2(\varphi)=\frac{1}{q}\int Q(x)\Big((W_{\varepsilon,Y}+\varphi)_+^q-W_{\varepsilon,Y}^q-qW_{\varepsilon,Y}^{q-1}\varphi-\frac{q(q-1)}{2}W_{\varepsilon,Y}^{q-2}\varphi^2\Big).
\tag{3.23}
$$

For any $\psi,\xi\in H_\varepsilon$, we obtain

$$
\begin{aligned}
\langle A_1^{(1)}(\varphi),\psi\rangle&=\varepsilon b\Big(\int|\nabla\varphi|^2\int\nabla\varphi\nabla\psi+\int|\nabla\varphi|^2\int\nabla W_{\varepsilon,Y}\nabla\psi\Big)\\
&\quad+2\varepsilon b\int\nabla W_{\varepsilon,Y}\nabla\varphi\int\nabla\varphi\nabla\psi,
\end{aligned}
\tag{3.24}
$$

$$\langle A_2^{(1)}(\varphi),\psi\rangle = \int Q(x)\big((W_{\varepsilon,Y}+\varphi)_+^{q-1} - W_{\varepsilon,Y}^{q-1} - (q-1)W_{\varepsilon,Y}^{q-2}\varphi\big)\psi, \tag{3.25}$$

$$\begin{aligned}\langle A_1^{(2)}(\varphi)[\psi],\xi\rangle = {}& \varepsilon b\Big(2\int \nabla\varphi\nabla\xi\int\nabla\varphi\nabla\psi + \int|\nabla\varphi|^2\int\nabla\xi\nabla\psi\Big)\\ &+ 2\varepsilon b\Big(\int\nabla W_{\varepsilon,Y}\nabla\xi\int\nabla\varphi\nabla\psi + \int\nabla W_{\varepsilon,Y}\nabla\varphi\int\nabla\xi\nabla\psi\Big)\\ &+ 2\varepsilon b\int\nabla\varphi\nabla\xi\int\nabla W_{\varepsilon,Y}\nabla\psi,\end{aligned} \tag{3.26}$$

$$\langle A_2^{(2)}(\varphi)[\psi],\xi\rangle = (q-1)\int Q(x)\big((W_{\varepsilon,Y}+\varphi)_+^{q-2} - W_{\varepsilon,Y}^{q-2}\big)\psi\xi. \tag{3.27}$$

Next, we estimate $A_1^{(i)}(\varphi),\ i=0,1,2$. Noting that

$$\|\nabla\varphi\|_{L^2} \le \frac{1}{\sqrt{a\varepsilon}}\|\varphi\|_\varepsilon \tag{3.28}$$

and

$$\int|\nabla W_{\varepsilon,Y}|^2 \le k\sum_{i=1}^{k}\int|\nabla w_{\varepsilon,y_i}|^2 = k^2\varepsilon\int|\nabla w|^2,$$

we have

$$\|\nabla W_{\varepsilon,Y}\|_{L^2} \le C_1\varepsilon^{\frac12} \tag{3.29}$$

where $C_1 = k\|\nabla w\|_{L^2}$. Combining (3.28) and (3.29), we obtain that for any $\psi,\xi,\nu\in H_\varepsilon$, there hold

$$\int|\nabla\varphi\nabla\psi|\int|\nabla W_{\varepsilon,Y}\nabla\xi| \le C\varepsilon^{-\frac52}\|\varphi\|_\varepsilon\|\psi\|_\varepsilon\|\xi\|_\varepsilon, \tag{3.30}$$

$$\int|\nabla\varphi\nabla\psi|\int|\nabla\nu\nabla\xi| \le C\varepsilon^{-4}\|\varphi\|_\varepsilon\|\psi\|_\varepsilon\|\nu\|_\varepsilon\|\xi\|_\varepsilon. \tag{3.31}$$

Combining (3.22), (3.24), (3.26), (3.30) and (3.31) yields

$$\begin{aligned}\|A_1^{(i)}(\varphi)\| &\le Cb\varepsilon^{-\frac32}\|\varphi\|_\varepsilon^{3-i} + Cb\varepsilon^{-3}\|\varphi\|_\varepsilon^{4-i}\\ &\le Cb\varepsilon^{-\frac32}\|\varphi\|_\varepsilon^{3-i}\big(\varepsilon^{-\frac32}\|\varphi\|_\varepsilon+1\big).\end{aligned} \tag{3.32}$$

Then, we estimate $A_2^{(i)}(\varphi),\ i=0,1,2$. For $2<q\le 3$, we apply the following elementary inequalities: for $e,f\in\mathbb{R}$, there exist constants $C_1(q),\ C_2(q),\ C_3(q)>0$ such that

$$|(e+f)_+^q - e_+^q - qe_+^{q-1}f - \frac{q(q-1)}{2}e_+^{q-2}f^2| \le C_1(q)|f|^q,$$
$$|(e+f)_+^{q-1} - e_+^{q-1} - (q-1)e_+^{q-2}f| \le C_2(q)|f|^{q-1},$$

and

$$|(e+f)_+^{q-2} - e_+^{q-2}| \le C_3(q)|f|^{q-2}.$$

Combining the above inequalities and Lemma 2.1 yields

$$|A_2(\varphi)| \le C\varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^q, \tag{3.33}$$

$$\|A_2^{(1)}(\varphi)\| \le C\varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^{q-1}, \tag{3.34}$$

$$\|A_2^{(2)}(\varphi)\| \le C\varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^{q-2}. \tag{3.35}$$

Similarly, for $3 < q < 6$ and $e, f \in \mathbb{R}$, there exist constants $C_1'(q),\ C_2'(q),\ C_3'(q) > 0$ such that

$$|(e+f)_+^q - e_+^q - qe_+^{q-1}f - \frac{q(q-1)}{2}e_+^{q-2}f^2| \leq C_1'(q)(|e|^{q-3} + |f|^{q-3})|f|^3,$$
$$|(e+f)_+^{q-1} - e_+^{q-1} - (q-1)e_+^{q-2}f| \leq C_2'(q)(|e|^{q-3} + |f|^{q-3})|f|^2,$$

and

$$|(e+f)_+^{q-2} - e_+^{q-2}| \leq C_3'(q)(|e|^{q-3} + |f|^{q-3})|f|.$$

Combining the above inequalities and Lemma 2.1 yields

$$\begin{aligned} |A_2(\varphi)| &\leq C_1'(q)\int \big(|W_{\varepsilon,Y}|^{q-3} + |\varphi|^{q-3}\big)|\varphi|^3 \\ &\leq C\Big(\int |W_{\varepsilon,Y}|^{2(q-3)}\Big)^{\frac{1}{2}}\varepsilon^{-3}\|\varphi\|_\varepsilon^3 + C\varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^q \\ &\leq C\big(\varepsilon^{-\frac{3}{2}}\|\varphi\|_\varepsilon^3 + \varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^q\big). \end{aligned} \tag{3.36}$$

By the same token, we obtain

$$\|A_2^{(1)}(\varphi)\| \leq C\big(\varepsilon^{-\frac{3}{2}}\|\varphi\|_\varepsilon^2 + \varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^{q-1}\big), \tag{3.37}$$

$$\|A_2^{(2)}(\varphi)\| \leq C\big(\varepsilon^{-\frac{3}{2}}\|\varphi\|_\varepsilon + \varepsilon^{-\frac{3(q-2)}{2}}\|\varphi\|_\varepsilon^{q-2}\big). \tag{3.38}$$

Combining (3.32)–(3.38) yields (3.21). $\square$

To state the lemma that follows, we define

$$\begin{aligned} N_\varepsilon = \bigg\{\varphi \in E_{\varepsilon,Y}^k \colon \|\varphi\|_\varepsilon \leq \varepsilon^{\frac{3}{2}}\bigg(&\sum_{i=1}^k |Q(y_i) - Q(0)|^{1-\tau} \\ &+ \sum_{i=1}^k\sum_{m=1}^{[h]} \varepsilon^{m-\tau}|D^m Q(y_i)|^{1-\tau} + \varepsilon^{[h]+1-\tau} + \sum_{i\neq j} e^{-(\bar{\theta}-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\bigg)\bigg\} \end{aligned} \tag{3.39}$$

where $0 < \tau < \min\{1, \bar{\theta}\}$.

Lemma 3.4. *There exist $\varepsilon_0,\ \delta_0$ sufficiently small such that for every $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$, there exists a C^1 map $\varphi_\varepsilon \colon D_{\varepsilon,\delta}^k \to N_\varepsilon;\ Y \mapsto \varphi_{\varepsilon,Y}$ satisfying*

$$\Big\langle \frac{\partial J_\varepsilon(Y, \varphi_{\varepsilon,Y})}{\partial\varphi}, \psi \Big\rangle = 0, \quad \forall\ \psi \in H_\varepsilon,\ \forall\ Y \in D_{\varepsilon,\delta}^k. \tag{3.40}$$

Moreover, we can choose $0 < \tau < \min\{1, \bar{\theta}\}$ sufficiently small, such that

$$\begin{aligned} \|\varphi_{\varepsilon,Y}\|_\varepsilon \leq \varepsilon^{\frac{3}{2}}\bigg(&\sum_{i=1}^k |Q(y_i) - Q(0)|^{1-\tau} + \sum_{i=1}^k\sum_{m=1}^{[h]} \varepsilon^{m-\tau}|D^m Q(y_i)|^{1-\tau} + \varepsilon^{[h]+1-\tau} \\ &+ \sum_{i\neq j} e^{-(\bar{\theta}-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\bigg). \end{aligned} \tag{3.41}$$

Proof. Recall that

$$J_\varepsilon(Y,\varphi) = J_\varepsilon(Y,0) + \langle I_\varepsilon'(W_{\varepsilon,Y}), \varphi\rangle + \frac{1}{2}\langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi], \varphi\rangle + R_{\varepsilon,Y}(\varphi),$$

so we have

$$\Big\langle \frac{\partial J_\varepsilon}{\partial\varphi}, \psi \Big\rangle = \langle I_\varepsilon'(W_{\varepsilon,Y}), \psi\rangle + \langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi], \psi\rangle + \langle R_{\varepsilon,Y}'(\varphi), \psi\rangle,\ \forall\ \psi \in H_\varepsilon,$$

i.e.

$$
\begin{aligned}
\frac{\partial J_\varepsilon}{\partial \varphi} &= I_\varepsilon'(W_{\varepsilon,Y}) + I_\varepsilon''(W_{\varepsilon,Y})[\varphi] + R_{\varepsilon,Y}'(\varphi) \\
&= l_{\varepsilon,Y} + I_\varepsilon''(W_{\varepsilon,Y})[\varphi] + R_{\varepsilon,Y}'(\varphi).
\end{aligned}
\tag{3.42}
$$

Then $\frac{\partial J_\varepsilon}{\partial \varphi}$ is a bounded linear functional in N_ε. Denote

$$\mathfrak{W} = \{f \colon f \text{ is a bounded linear functional defined on } H_\varepsilon\}.$$

For any $f \in \mathfrak{W}$, by Riesz representation theorem, there exists a unique $\hat{f} \in H_\varepsilon$ such that

$$f(\psi) = \langle \hat{f}, \psi\rangle_\varepsilon, \quad \forall\, \psi \in H_\varepsilon.$$

So we can define a map $\sigma \colon \mathfrak{W} \to H_\varepsilon;\ f \mapsto \hat{f}$.

Let $\mathfrak{W}^* = \sigma(\mathfrak{W})$. Next, we prove σ is a linear isomorphic map from $\mathfrak{W}$ to $\mathfrak{W}^*$. In fact, if $\sigma(f_1) = \sigma(f_2)$, in the sense that $\hat{f}_1 = \hat{f}_2$, we obtain

$$f_1(\psi) = \langle \hat{f}_1, \psi\rangle_\varepsilon = \langle \hat{f}_2, \psi\rangle_\varepsilon = f_2(\psi), \quad \forall\, \psi \in H_\varepsilon.$$

Then $f_1 = f_2$ and σ is injective. Besides, for any $f_1, f_2 \in \mathfrak{W}$,

$$\langle \widehat{f_1 + f_2}, \psi\rangle_\varepsilon = (f_1+f_2)(\psi) = f_1(\psi) + f_2(\psi) = \langle \hat{f}_1, \psi\rangle_\varepsilon + \langle \hat{f}_2, \psi\rangle_\varepsilon = \langle \hat{f}_1 + \hat{f}_2, \psi\rangle_\varepsilon,$$

which implies $\widehat{f_1 + f_2} = \hat{f}_1 + \hat{f}_2$, in the sense that $\sigma(f_1+f_2) = \sigma(f_1) + \sigma(f_2)$. And for any $k \in \mathbb{R}$ and $f \in \mathfrak{W}$, we obtain

$$\langle \widehat{kf}, \psi\rangle_\varepsilon = (kf)(\psi) = kf(\psi) = k\langle \hat{f}, \psi\rangle_\varepsilon = \langle k\hat{f}, \psi\rangle_\varepsilon.$$

Thus, $\widehat{kf} = k\hat{f}$ and $\sigma(kf) = k\sigma(f)$.

Therefore, (3.42) is equivalent to

$$\frac{\partial \hat{J}_\varepsilon}{\partial \varphi} = \hat{l}_{\varepsilon,Y} + \mathcal{L}_{\varepsilon,Y}(\varphi) + \hat{R}_{\varepsilon,Y}'(\varphi). \tag{3.43}$$

Since $\mathcal{L}_{\varepsilon,Y}$ is invertible in $E^k_{\varepsilon,Y}$ by Lemma 3.1, it is sufficient to find $\varphi \in N_\varepsilon$ that satisfies

$$\varphi = -\mathcal{L}_{\varepsilon,Y}^{-1}(\hat{l}_{\varepsilon,Y}) - \mathcal{L}_{\varepsilon,Y}^{-1}(\hat{R}_{\varepsilon,Y}'(\varphi)) =: \mathcal{A}_{\varepsilon,Y}(\varphi). \tag{3.44}$$

Next, we prove that $\mathcal{A}_{\varepsilon,Y}$ is a contraction map on N_ε. First, for any $\varphi \in N_\varepsilon$, we have

$$
\begin{aligned}
\|\mathcal{A}_{\varepsilon,Y}(\varphi)\|_\varepsilon &\le \frac{1}{\rho}\|\hat{l}_{\varepsilon,Y}\|_\varepsilon + \frac{1}{\rho}\|\hat{R}_{\varepsilon,Y}'(\varphi)\|_\varepsilon \\
&= \frac{1}{\rho}\|l_{\varepsilon,Y}\| + \frac{1}{\rho}\|R_{\varepsilon,Y}'(\varphi)\|.
\end{aligned}
\tag{3.45}
$$

By Lemma 3.2, we obtain

$$\|l_{\varepsilon,Y}\| \le C\varepsilon^{\frac{3}{2}}\left(\varepsilon^{[h]+1} + \sum_{i=1}^k |Q(y_i) - Q(0)| + \sum_{i=1}^k \sum_{m=1}^{[h]} \varepsilon^m |D^m Q(y_i)| + \sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i - y_j|}{\varepsilon}}\right).$$

Choose ε, δ sufficiently small such that

$$
\begin{cases}
C\varepsilon^\tau < \frac{\rho}{2}, \\
C|Q(y_i) - Q(0)|^\tau < \frac{\rho}{2}, & i = 1,\ldots,k, \\
C\varepsilon^\tau |D^m Q(y_i)|^\tau < \frac{\rho}{2}, & i = 1,\ldots,k,\ m = 1,\ldots,[h], \\
Ce^{-\tau\frac{\eta|y^i - y^j|}{\varepsilon}} < \frac{\rho}{2}, & i \neq j,
\end{cases}
$$

then

$$\|l_{\varepsilon,Y}\| \le \frac{\rho}{2}\varepsilon^{\frac{3}{2}}\bigg(\sum_{i=1}^{k}|Q(y_i)-Q(0)|^{1-\tau}+\sum_{i=1}^{k}\sum_{m=1}^{[h]}\varepsilon^{m-\tau}|D^mQ(y_i)|^{1-\tau}\bigg) + \frac{\rho}{2}\varepsilon^{\frac{3}{2}}\bigg(\varepsilon^{[h]+1-\tau}+\sum_{i\neq j}e^{-(\bar{\theta}-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\bigg). \tag{3.46}$$

As $\varphi\in N_\varepsilon$,

$$\varepsilon^{-\frac{3}{2}}\|\varphi\|_\varepsilon = o_\varepsilon(1)+o_\delta(1).$$

So for ε, δ sufficiently small, by Lemma 3.3, we have

$$\|R^{'}_{\varepsilon,Y}(\varphi)\| = (o_\varepsilon(1)+o_\delta(1))\,\|\varphi\|_\varepsilon \le \frac{\rho}{2}\|\varphi\|_\varepsilon. \tag{3.47}$$

Combining (3.45)–(3.47) yields

$$\|\mathcal{A}_{\varepsilon,Y}(\varphi)\|_\varepsilon \le \varepsilon^{\frac{3}{2}}\bigg(\sum_{i=1}^{k}|Q(y_i)-Q(0)|^{1-\tau}+\sum_{i=1}^{k}\sum_{m=1}^{[h]}\varepsilon^{m-\tau}|D^mQ(y_i)|^{1-\tau}\bigg) + \varepsilon^{\frac{3}{2}}\bigg(\varepsilon^{[h]+1-\tau}+\sum_{i\neq j}e^{-(\bar{\theta}-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\bigg).$$

Hence, $\mathcal{A}_{\varepsilon,Y}(N_\varepsilon)\subset N_\varepsilon$. On the other hand, for every $\varphi,\psi\in N_\epsilon$,

$$\begin{aligned}\|\mathcal{A}_{\varepsilon,Y}(\varphi)-\mathcal{A}_{\varepsilon,Y}(\psi)\|_\varepsilon &= \|\mathcal{L}^{-1}_{\varepsilon,Y}\big(\hat{R}^{'}_{\varepsilon,Y}(\varphi)\big)-\mathcal{L}^{-1}_{\varepsilon,Y}\big(\hat{R}^{'}_{\varepsilon,Y}(\psi)\big)\|_\varepsilon\\ &\le \frac{1}{\rho}\|R^{'}_{\varepsilon,Y}(\varphi)-R^{'}_{\varepsilon,Y}(\psi)\|\\ &= \frac{1}{\rho}\|R^{''}_{\varepsilon,Y}(\xi\varphi+(1-\xi)\psi)\|\|\varphi-\psi\|_\varepsilon, \quad 0<\xi<1.\end{aligned}$$

By Lemma 3.3, we obtain

$$\begin{aligned}\|R^{''}_{\varepsilon,Y}(\xi\varphi+(1-\xi)\psi)\| &\le C\varepsilon^{-\frac{3(q-2)}{2}}\|\xi\varphi+(1-\xi)\psi\|^{q-2}_\varepsilon\\ &\quad + Cb\varepsilon^{-\frac{3}{2}}\big(1+\varepsilon^{-\frac{3}{2}}\|\xi\varphi+(1-\xi)\psi\|_\varepsilon\big)\|\xi\varphi+(1-\xi)\psi\|_\varepsilon\\ &= o_\varepsilon(1).\end{aligned}$$

Thus, for ε sufficiently small, we have

$$\|\mathcal{A}_{\varepsilon,Y}(\varphi)-\mathcal{A}_{\varepsilon,Y}(\psi)\|_\varepsilon \le \frac{1}{2}\|\varphi-\psi\|_\varepsilon.$$

So $\mathcal{A}_{\varepsilon,Y}$ is a contraction map on N_ε. By contraction mapping principle, we infer that (3.44) has a unique solution. Finally, by similar arguments as that of Cao, Noussair and Yan [6], we can deduce that φ_ε belongs to C^1. □

4. Proof of Theorem 1.3

In this section, without loss of generality, we assume $Q(0)=1$. By Lemma 3.4, we can define a C^1 function on $D^k_{\varepsilon.\delta}$, in the sense that

$$K(Y) =: J_\varepsilon(Y,\varphi_{\varepsilon,Y}), \quad Y\in D^k_{\varepsilon,\delta}.$$

Define

$$c_{\varepsilon,1}=\varepsilon^3(kA-k^2B-T\varepsilon^{\alpha h_1}), \quad c_{\varepsilon,2}=\varepsilon^3(kA-k^2B+\mu),$$

where $A = \frac{q-2}{2q}\|w\|_{L^q}^q$, $B = \frac{b}{4}\|\nabla w\|_{L^2}^4$, μ, T are positive constants, $\varepsilon^\alpha \le \frac{\delta}{2}$ and $\alpha \in (0,1)$ is a fixed constant close to 1.

Denote

$$\Omega_\gamma = \Big\{Y = (y_1, \dots, y_k)\colon y_i \in B_\delta^t(0) \times B_\gamma^{3-t}(0),\ i = 1, \dots, k,\ \frac{\eta|y_i - y_j|}{\varepsilon} \ge R_1,\ i \ne j\Big\},$$

where $B_\tau^l(0) = \{y \in \mathbb{R}^l\colon |y| \le \tau\}$, $R_1 > 0$ is a large constant, and

$$K^c = \big\{Y\colon Y \in \Omega_{\varepsilon^\alpha},\ K(Y) \le c\big\}.$$

Lemma 4.1. *For any $\varphi \in E_{\varepsilon,Y}^k$, there holds*

$$\langle \mathcal{L}_{\varepsilon,Y}\varphi, \varphi\rangle_\varepsilon = O(\|\varphi\|_\varepsilon^2). \tag{4.1}$$

Proof. By the definition of $\mathcal{L}_{\varepsilon,Y}$, we have

$$\begin{aligned}\langle \mathcal{L}_{\varepsilon,Y}\varphi, \varphi\rangle_\varepsilon &= \langle \varphi, \varphi\rangle_\varepsilon + \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int |\nabla\varphi|^2 \\ &\quad + 2\varepsilon b \Big(\int \nabla W_{\varepsilon,Y}\nabla\varphi\Big)^2 - (q-1)\int Q(x) W_{\varepsilon,Y}^{q-2}\varphi^2.\end{aligned} \tag{4.2}$$

Calculating directly yields

$$\varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int |\nabla\varphi|^2 \le \varepsilon b k \int \sum_{i=1}^k |\nabla w_{\varepsilon,y_i}|^2 \int |\nabla\varphi|^2 \le C\|\varphi\|_\varepsilon^2. \tag{4.3}$$

By Hölder inequality, we obtain

$$\varepsilon b \Big(\int \nabla W_{\varepsilon,Y}\nabla\varphi\Big)^2 \le \varepsilon b \int |\nabla W_{\varepsilon,Y}|^2 \int |\nabla\varphi|^2 \le C\|\varphi\|_\varepsilon^2. \tag{4.4}$$

Finally, as $Q(x)$ is bounded, we have

$$\int Q(x) W_{\varepsilon,Y}^{q-2}\varphi^2 \le C\Big(\int W_{\varepsilon,Y}^q\Big)^{\frac{q-2}{q}} \Big(\int |\varphi|^q\Big)^{\frac{2}{q}} \le C\|\varphi\|_\varepsilon^2. \tag{4.5}$$

Combining (4.2)–(4.5) yields (4.1). □

Lemma 4.2. *There exist constants $\varepsilon_0, \delta_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$ and $\delta \in (0, \delta_0]$, (Y, φ) is a critical point of J_ε on $D_{\varepsilon,\delta}^k \times E_{\varepsilon,Y}^k$ is equivalent to*

$$u = \sum_{i=1}^k w_{\varepsilon,y_i} + \varphi$$

is a critical point of I_ε.

Proof. This lemma can be proved by the same arguments as that of [3, 6] with minor modifications. We omit the details. □

Lemma 4.3. *For every $Y \in \partial\Omega_{\varepsilon^\alpha}$, we have either $K(Y) < c_{\varepsilon,1}$ or $\frac{\partial K(Y)}{\partial n} > 0$, where n is the outward unit normal of $\partial\Omega_{\varepsilon^\alpha}$ at Y.*

Proof. We divide the proof of this lemma into two steps.

Step 1: Suppose that $\frac{\eta|y_i - y_j|}{\varepsilon} = R_1$ for some $i \ne j$, or $y_i \in \partial B_\delta^t(0) \times B_{\varepsilon^\alpha}^{3-t}(0)$ for some $i \in \{1, \dots, k\}$. We claim that $Y \in K^{c_{\varepsilon,1}}$.

In fact, since $\varphi_{\varepsilon,Y} \in N_\varepsilon$, by Lemma 3.2, we obtain

$$|l_{\varepsilon,Y}(\varphi_{\varepsilon,Y})| = O(\|\varphi_{\varepsilon,Y}\|_\varepsilon^2). \tag{4.6}$$

And by Lemma 3.3, we have

$$|R_{\varepsilon,Y}(\varphi_{\varepsilon,Y})| = o_\varepsilon(1)\|\varphi_{\varepsilon,Y}\|_\varepsilon^2. \tag{4.7}$$

Combining (3.1), (4.1), (4.6) and (4.7) yields

$$J_\varepsilon(Y,\varphi_{\varepsilon,Y}) = J_\varepsilon(Y,0) + O(\|\varphi_{\varepsilon,Y}\|_\varepsilon^2). \tag{4.8}$$

Then combining Lemma A.1, (4.8) and (3.41) yields

$$\begin{aligned}
K(Y) &= \varepsilon^3\left(kA - k^2B\right) - \frac{2}{q-2}A\varepsilon^3\sum_{i=1}^k\left(Q(y_i)-1\right) \\
&\quad - \int\sum_{i=1}^{k-1} w_{\varepsilon,y_i}\Big(\sum_{j=i+1}^k w_{\varepsilon,y_j}\Big)^{q-1} + O\big(\varepsilon^{4+[h]}\big) \\
&\quad + O\Big(\sum_{i=1}^k\sum_{m=1}^{[h]}\varepsilon^{3+m}|D^mQ(y_i)| + \varepsilon^3\sum_{i\neq j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big) \\
&\quad + \varepsilon^3 O\Big(\sum_{i=1}^k|Q(y_i)-1|^{2(1-\tau)} + \sum_{i=1}^k\sum_{m=1}^{[h]}\varepsilon^{2(m-\tau)}|D^mQ(y_i)|^{2(1-\tau)}\Big) \\
&\quad + \varepsilon^3 O\Big(\varepsilon^{2([h]+1-\tau)} + \sum_{i\neq j}e^{-2(\bar\theta-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).
\end{aligned} \tag{4.9}$$

Choose τ sufficiently small such that

$$2([h]+1-\tau) > [h]+1,\ 2(m-\tau) > m,\ 2(\bar\theta-\tau) > 1.$$

Then by (4.9), we have

$$\begin{aligned}
K(Y) &= \varepsilon^3\left(kA - k^2B\right) - \frac{2}{q-2}A\varepsilon^3\sum_{i=1}^k\left(Q(y_i)-1\right) \\
&\quad - \int\sum_{i=1}^{k-1} w_{\varepsilon,y_i}\Big(\sum_{j=i+1}^k w_{\varepsilon,y_j}\Big)^{q-1} + O\big(\varepsilon^{4+[h]}\big) \\
&\quad + O\Big(\sum_{i=1}^k\sum_{m=1}^{[h]}\varepsilon^{3+m}|D^mQ(y_i)| + \varepsilon^3\sum_{i\neq j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).
\end{aligned} \tag{4.10}$$

Combining the above equality and the condition (Q_3) yields

$$K(Y) \le \varepsilon^3\Big(kA - k^2B - C\sum_{i=1}^k P_1(y_i') - C\sum_{i<j}e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big) + O(\varepsilon^4). \tag{4.11}$$

If $\frac{\eta|y_i-y_j|}{\varepsilon} = R_1$ for some $i\neq j$, taking $R_1 = \frac{\alpha h_1 ln\frac{1}{\varepsilon} - lnT}{2}$, by (4.11) we obtain

$$K(Y) \le \varepsilon^3\left(kA - k^2B - T\varepsilon^{\alpha h_1}\right) - C\varepsilon^3\sum_{i=1}^k P_1(y_i') + O(\varepsilon^4) < c_{\varepsilon,1}. \tag{4.12}$$

If $y_i \in \partial B^t_\delta(0) \times B^{3-t}_{\varepsilon^\alpha}(0)$ for some $i \in \{1, \dots, k\}$, combining (4.11) and (1.11) yields

$$\begin{aligned}K(Y) &\le \varepsilon^3\left(kA - k^2B - C\lambda\sum_{i=1}^k |y_i'|^{h_1}\right) - C\varepsilon^3\sum_{i<j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}} + O(\varepsilon^4)\\ &\le \varepsilon^3\left(kA - k^2B - C\lambda\varepsilon^{\alpha h_1}\right) - C\varepsilon^3\sum_{i<j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}} + O(\varepsilon^4).\end{aligned}$$

Let T be sufficiently small such that $T < C\lambda$, then we have $K(Y) < c_{\varepsilon,1}$.

Step 2: Suppose $t \in \{1,2\}$ and $y_j \in B^t_\delta(0) \times \partial B^{3-t}_{\varepsilon^\alpha}(0)$ for some $j \in \{1,\dots,k\}$. Without loss of generality, we assume $j = 1$. We claim that either $K(Y) < c_{\varepsilon,1}$ or $\frac{\partial K(Y)}{\partial n} > 0$, where n is the outward unit normal of $B^t_\delta(0) \times \partial B^{3-t}_{\varepsilon^\alpha}(0)$ at y_1.

In fact, for any $y_i \in B^t_\delta(0) \times B^{3-t}_{\varepsilon^\alpha}(0)$ and $m \ge 1$, we have

$$\begin{aligned}\varepsilon^m|D^mQ(y_i)| &= O\left(\varepsilon^m|y_i'|^{h_1-m} + \varepsilon^m|y_i''|^{h-m}\right)\\ &= O\left(\varepsilon^m|y_i'|^{h_1-m} + \varepsilon^{\alpha h+m(1-\alpha)}\right).\end{aligned} \tag{4.13}$$

By Lemma A.2, we obtain

$$\begin{aligned}\frac{\partial K}{\partial y_{1l}} &= -C\varepsilon^3 D_lQ(y_1) - (q-1)\sum_{i=2}^k \int w^{q-2}_{\varepsilon,y_1} w_{\varepsilon,y_i}\frac{\partial w_{\varepsilon,y_1}}{\partial y_{1l}}\\ &\quad + O\left(\sum_{i=1}^k\sum_{m=2}^{[h]} \varepsilon^{2+m-\tau}|D^mQ(y_i)|^{1-\tau} + \varepsilon^{3+[h]-\tau}\right)\\ &\quad + O\left(\varepsilon^2\sum_{i=1}^k |Q(y_i)-1|^{1-\tau}\right) + O\left(\varepsilon^2\sum_{i\ne j} e^{-(\bar\theta-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\right).\end{aligned} \tag{4.14}$$

Denote $\bar\eta = \min_{i\ne j}\eta|y_i - y_j|$. We divide it into two cases.

(i) Suppose that $e^{-\frac{\bar\eta}{\varepsilon}} > L\varepsilon^{\alpha h}$ or $|y_i'| > L\varepsilon^{\alpha h/h_1}$ for some $i \in \{1,\dots,k\}$, where $L > T$ is a large constant. We claim that $K(Y) < c_{\varepsilon,1}$.

In fact, combining (4.10) and (1.11) yields

$$\begin{aligned}K(Y) &\le \varepsilon^3\left(kA - k^2B\right) - C_1\varepsilon^3\sum_{i=1}^k |y_i'|^{h_1} - C_1\varepsilon^3 e^{-\frac{\bar\eta}{\varepsilon}}\\ &\quad + O\left(\sum_{i=1}^k\sum_{m=1}^{[h]} \varepsilon^{3+m}|y_i'|^{h_1-m} + \varepsilon^{3+\alpha h}\right)\\ &\le \varepsilon^3\left(kA - k^2B\right) - (C_1-\tau')\varepsilon^3\sum_{i=1}^k |y_i'|^{h_1} - C_1\varepsilon^3 e^{-\frac{\bar\eta}{\varepsilon}} + C_{\tau'}\varepsilon^{3+\alpha h},\end{aligned} \tag{4.15}$$

where $\tau' > 0$ is a constant. When $L > T$ is large enough, we have $K(Y) < c_{\varepsilon,1}$.

(ii) Suppose that $e^{-\frac{\bar\eta}{\varepsilon}} \le L\varepsilon^{\alpha h}$ and $|y_i'| \le L\varepsilon^{\alpha h/h_1}$, $i = 1,\dots,k$. We claim that $\frac{\partial K(Y)}{\partial n} > 0$. First, we can see

$$|1 - Q(y_i)| = O(\varepsilon^{\alpha h}), \tag{4.16}$$

$$|D^mQ(y_i)|\varepsilon^m = O(\varepsilon^{\alpha h(h_1-m)/h_1}\varepsilon^m + \varepsilon^{\alpha h+m(1-\alpha)}) = O(\varepsilon^{\alpha h+m(1-\alpha)}), \tag{4.17}$$

$$\varepsilon^2 e^{-\bar\theta\frac{\bar\eta}{\varepsilon}} = O(\varepsilon^{2+\bar\theta\alpha h}). \tag{4.18}$$

Since for any $i \neq 1$,

$$\int w_{\varepsilon,y_1}^{q-2} w_{\varepsilon,y_i} \frac{\partial w_{\varepsilon,y_1}}{\partial y_{1l}} = \big(C + o(1)\big)\varepsilon^2 w\Big(\frac{|y_i - y_1|}{\varepsilon}\Big)\frac{y_{il} - y_{1l}}{|y_i - y_1|}, \tag{4.19}$$

$$\Big\langle \frac{y_i - y_1}{|y_i - y_1|}, n\Big\rangle \leq 0, \ \forall\, y_i \in B_\delta^t(0) \times B_{\varepsilon^\alpha}^{3-t}(0), \tag{4.20}$$

where

$$n = \begin{cases} \Big(0, \frac{y_{1,2}}{(y_{1,2}^2 + y_{1,3}^2)^{\frac{1}{2}}}, \frac{y_{1,3}}{(y_{1,2}^2 + y_{1,3}^2)^{\frac{1}{2}}}\Big), & t = 1, \\ \Big(0, 0, \frac{y_{1,3}}{|y_{1,3}|}\Big), & t = 2, \end{cases}$$

combining (1.12), (4.14) and (4.16)–(4.20) yields

$$\begin{aligned} \frac{\partial K(Y)}{\partial n} &\geq C\varepsilon^3 \langle -DQ(y_1), n\rangle + O(\varepsilon^{3+\alpha h + \tau''}) \\ &\geq C\varepsilon^3 |y_1''|^{h-1} + O(\varepsilon^{3+\alpha h+\tau''}) \\ &> 0, \end{aligned} \tag{4.21}$$

where $\tau'' > 0$ is a constant.

Combining Steps 1 and 2 we complete the proof of this lemma. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Lemma 2.6 and Lemma 4.3 that we just need to prove $K^{c_{\varepsilon,2}}$ cannot be deformed into $K^{c_{\varepsilon,1}}$. Then K has at least one critical point in $K^{c_{\varepsilon,2}} \backslash K^{c_{\varepsilon,1}}$. Finally, by Lemma 4.2, we obtain that $u = \sum_{i=1}^k w_{\varepsilon, y_{\varepsilon,i}} + \varphi_{\varepsilon,Y}$ is a critical point of I_ε, in the sense that it is a solution of equation (1.1).

Next, we prove $K^{c_{\varepsilon,2}}$ cannot be deformed into $K^{c_{\varepsilon,1}}$. It's easy to know

$$K^{c_{\varepsilon,2}} = \Omega_{\varepsilon^\alpha}.$$

Denote

$$M = B_\delta^t(0) \times B_{\varepsilon^\alpha}^{3-t}(0), \tag{4.22}$$

$$\Gamma_\iota = \big\{(y', y'') \in M, \ |y'| \geq \iota\big\}, \tag{4.23}$$

$$T_\gamma = \cup_{i \neq j}\big\{\eta |y_i - y_j| \leq \gamma, \ y_i, \ y_j \in M\big\}, \tag{4.24}$$

$$L_{\iota,\gamma} = (\underbrace{\Gamma_\iota \times M \times \cdots \times M}_{k}) \cup (\underbrace{M \times \Gamma_\iota \times \cdots \times M}_{k}) \cup \cdots \tag{4.25}$$

$$\cup\, (\underbrace{M \times \cdots \times M \times \Gamma_\iota}_{k}) \cup T_\gamma. \tag{4.26}$$

Step 1: We claim that there exist constants C, c' with $C > c' > 0$, such that

$$L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}} \setminus T_{\varepsilon R_1} \subset K^{c_{\varepsilon,1}} \subset L_{c'\varepsilon^{\alpha h/h_1}, C\varepsilon ln \varepsilon^{-1}} \backslash T_{\varepsilon R_1}. \tag{4.27}$$

In fact, for any $Y \in K^{c_{\varepsilon,1}}$, we have $K(Y) < c_{\varepsilon,1}$. Then by (4.10), we obtain

$$\begin{aligned} c_{\varepsilon,1} &= \varepsilon^3 (kA - k^2 B - T\varepsilon^{\alpha h_1}) \\ &> K(Y) \\ &\geq \varepsilon^3\left(kA - k^2 B\right) - c'\varepsilon^3 \sum_{i=1}^k |y_i'|^{h_1} - c'\varepsilon^3 \sum_{i \neq j} e^{-\frac{\eta|y_i - y_j|}{\varepsilon}} + O\left(\varepsilon^{4+\alpha h}\right). \end{aligned} \tag{4.28}$$

Thus, $|y_i'| \geq c'\varepsilon^{\alpha h/h_1}$ or $\eta|y_i - y_j| \leq C\varepsilon ln\varepsilon^{-1}$ for some $i \neq j$. Hence,

$$K^{c_{\varepsilon,1}} \subset L_{c'\varepsilon^{\alpha h/h_1}, C\varepsilon ln\varepsilon^{-1}} \backslash T_{\varepsilon R_1}.$$

On the other hand, choose $c' > 0$ sufficiently small. When $|y_i'| \geq \frac{\delta}{2}$ or $\eta|y_i - y_j| \leq c'\varepsilon ln\varepsilon^{-1}$ for some $i \neq j$, by (4.11), we have $K(Y) < c_{\varepsilon,1}$. Then

$$L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}} \setminus T_{\varepsilon R_1} \subset K^{c_{\varepsilon,1}}.$$

So the claim follows.

Step 2: Since $L_{c'\varepsilon^{\alpha h/h_1}, C\varepsilon ln\varepsilon^{-1}} \backslash T_{\varepsilon R_1}$ can be deformed into $L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}} \setminus T_{\varepsilon R_1}$, then $K^{c_{\varepsilon,1}}$ can be deformed into $L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}} \setminus T_{\varepsilon R_1}$. Suppose $K^{c_{\varepsilon,2}}$ can be deformed into $K^{c_{\varepsilon,1}}$, then we see that $\Omega_{\varepsilon^\alpha} = K^{c_{\varepsilon,2}}$ can be deformed into $L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}} \setminus T_{\varepsilon R_1}$. Hence, $\underbrace{M \times M \times \cdots \times M}_{k}$ can be deformed into $L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}}$. However,

$$H^t(M, \Gamma_{\delta/2}) = H^t(B_\delta^t(0), \partial B_\delta^t(0)) \neq 0.$$

By Lemma 2.7, we obtain

$$H_*(\underbrace{M \times M \times \cdots \times M}_{k}, L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}}) \neq 0.$$

Then $\underbrace{M \times M \times \cdots \times M}_{k}$ cannot be deformed into $L_{\delta/2, c'\varepsilon ln\varepsilon^{-1}}$. This is a contradiction.

□

Appendix A. Energy estimates

Lemma A.1. *For ε sufficiently small and any $Y \in D_{\varepsilon,\delta}^k$, we have*

$$\begin{aligned} J_\varepsilon(Y, 0) = \varepsilon^3 \Big(kA - k^2 B - \frac{2}{q-2} A \sum_{i=1}^{k} (Q(y_i) - 1) \Big) \\ - \int \sum_{i=1}^{k-1} w_{\varepsilon,y_i} \Big(\sum_{j=i+1}^{k} w_{\varepsilon,y_j} \Big)^{q-1} + O\Big(\sum_{i=1}^{k} \sum_{m=1}^{[h]} \varepsilon^{3+m} |D^m Q(y_i)| \Big) \\ + O\Big(\varepsilon^3 \sum_{i \neq j} e^{-\frac{\eta|y_i - y_j|}{\varepsilon}} + \varepsilon^{4+[h]} \Big), \end{aligned} \tag{A.1}$$

where $A = \frac{q-2}{2q}\|w\|_{L^q}^q$ and $B = \frac{b}{4}\|\nabla w\|_{L^2}^4$.

Proof. By the definition of $J_\varepsilon(Y, \varphi)$, we obtain

$$\begin{aligned} J_\varepsilon(Y, 0) &= I_\varepsilon(W_{\varepsilon,Y}) \\ &= \frac{1}{2}\|W_{\varepsilon,Y}\|_\varepsilon^2 + \frac{\varepsilon b}{4} \Big(\int |\nabla W_{\varepsilon,Y}|^2 \Big)^2 - \frac{1}{q} \int Q(x) W_{\varepsilon,Y}^q \\ &= \frac{1}{2} \sum_{i=1}^{k} \int \big(\varepsilon^2 a |\nabla w_{\varepsilon,y_i}|^2 + w_{\varepsilon,y_i}^2 \big) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \int \big(\varepsilon^2 a |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| + w_{\varepsilon,y_i} w_{\varepsilon,y_j} \big) \\ &\quad + \frac{\varepsilon b}{4} \Big(\sum_{i=1}^{k} \int |\nabla w_{\varepsilon,y_i}|^2 \Big)^2 + \frac{\varepsilon b}{4} \Big(\sum_{i \neq j} \int |\nabla w_{\varepsilon,y_i} \nabla w_{\varepsilon,y_j}| \Big)^2 - \frac{1}{q} \int Q(x) W_{\varepsilon,Y}^q \end{aligned} \tag{A.2}$$

$$= \frac{1}{2}\sum_{i=1}^{k}\int \left(\varepsilon^2 a|\nabla w_{\varepsilon,y_i}|^2 + w_{\varepsilon,y_i}^2\right) + \frac{\varepsilon b}{4}\Big(\sum_{i=1}^{k}\int |\nabla w_{\varepsilon,y_i}|^2\Big)^2$$
$$- \frac{1}{q}\int Q(x)W_{\varepsilon,Y}^q + O\Big(\varepsilon^3\sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).$$

As $Q(0)=1$ and w is the solution of (2.3), we obtain that $w_{\varepsilon,y_i}(1\le i\le k)$ satisfies

$$-\left(\varepsilon^2 a + \varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\right)\Delta w_{\varepsilon,y_i} + w_{\varepsilon,y_i} = w_{\varepsilon,y_i}^{q-1}, \quad j=1,\dots,k.$$

Multiplying w_{ε,y_i} on both sides of the above equality and integrating, we have

$$\left(\varepsilon^2 a + \varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\right)\int |\nabla w_{\varepsilon,y_i}|^2 + \int w_{\varepsilon,y_i}^2 = \int w_{\varepsilon,y_i}^q.$$

Summing i from 1 to k, we obtain

$$\sum_{i=1}^{k}\left(\varepsilon^2 a + \varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\right)\int |\nabla w_{\varepsilon,y_i}|^2 + \sum_{i=1}^{k}\int w_{\varepsilon,y_i}^2 = \int \sum_{i=1}^{k} w_{\varepsilon,y_i}^q.$$

Then

$$\sum_{i=1}^{k}\int \left(\varepsilon^2 a|\nabla w_{\varepsilon,y_i}|^2 + w_{\varepsilon,y_i}^2\right) = -\varepsilon bk\int |\nabla w_{\varepsilon,y_j}|^2\int \sum_{i=1}^{k}|\nabla w_{\varepsilon,y_i}|^2 + \int \sum_{i=1}^{k} w_{\varepsilon,y_i}^q.$$

Substituting it into (A.2) yields

$$\begin{aligned}
J_\varepsilon(Y,0) &= \frac{1}{2}\sum_{i=1}^{k}\int w_{\varepsilon,y_i}^q - \frac{\varepsilon b}{4}\Big(\sum_{i=1}^{k}\int |\nabla w_{\varepsilon,y_i}|^2\Big)^2 \\
&\quad - \frac{1}{q}\int Q(x)W_{\varepsilon,Y}^q + O\Big(\varepsilon^3\sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big) \\
&= \Big(\frac{1}{2}-\frac{1}{q}\Big)\sum_{i=1}^{k}\int w_{\varepsilon,y_i}^q - \frac{\varepsilon b}{4}\Big(\sum_{i=1}^{k}\int |\nabla w_{\varepsilon,y_i}|^2\Big)^2 \\
&\quad - \frac{1}{q}\int \Big(Q(x)\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^q - \sum_{i=1}^{k} w_{\varepsilon,y_i}^q\Big) + O\Big(\varepsilon^3\sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big) \qquad \text{(A.3)} \\
&= \Big(\frac{1}{2}-\frac{1}{q}\Big)\varepsilon^3 k\|w\|_{L^q}^q - \frac{1}{4}\varepsilon^3 k^2 b\|\nabla w\|_{L^2}^4 - \frac{1}{q}\int \Big(\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^q - \sum_{i=1}^{k} w_{\varepsilon,y_i}^q\Big) \\
&\quad - \frac{1}{q}\int (Q(x)-1)\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^q + O\Big(\varepsilon^3\sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).
\end{aligned}$$

Next, we estimate the third and fourth term of the right side of (A.3) respectively. First, to estimate the third term, we use the following inequalities

$$
\begin{aligned}
\big||a+b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}\big| &\le \begin{cases} C|b|^{q-2}|a|^2, & |b| \le |a|; \\ C|a|^{q-2}|b|^2, & |a| \le |b|, \end{cases} \\
&\le C|a|^{\frac{q}{2}}|b|^{\frac{q}{2}}, \quad (2 < q \le 3), \\
\big||a+b|^q - a^q - b^q - qa^{q-1}b - qab^{q-1}\big| &\le C(a^{q-2}b^2 + a^2 b^{q-2}), \quad (q > 3).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big(\sum_{i=1}^k w_{\varepsilon,y_i}\Big)^q - w_{\varepsilon,y_1}^q - \Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big)^q \\
&\quad - q w_{\varepsilon,y_1}^{q-1}\Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big) - q w_{\varepsilon,y_1}\Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big)^{q-1} \\
&\le \begin{cases} C w_{\varepsilon,y_1}^{\frac{q}{2}}\Big(\sum\limits_{i=2}^k w_{\varepsilon,y_i}\Big)^{\frac{q}{2}}, & 2 < q \le 3, \\ C w_{\varepsilon,y_1}^{q-2}\Big(\sum\limits_{i=2}^k w_{\varepsilon,y_i}\Big)^2 + C w_{\varepsilon,y_1}^2\Big(\sum\limits_{i=2}^k w_{\varepsilon,y_i}\Big)^{q-2}, & q > 3. \end{cases}
\end{aligned} \tag{A.4}
$$

If $2 < q \le 3$, by (2.8), we have

$$
\int w_{\varepsilon,y_1}^{\frac{q}{2}}\Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big)^{\frac{q}{2}} \le C\int \sum_{i=2}^k w_{\varepsilon,y_1}^{\frac{q}{2}} w_{\varepsilon,y_i}^{\frac{q}{2}} \le C\sum_{i=2}^k \varepsilon^3 e^{-\frac{q}{2}\frac{\eta|y_1-y_i|}{\varepsilon}}. \tag{A.5}
$$

If $q > 3$, by (2.8), we have

$$
\begin{aligned}
&\int \Big(w_{\varepsilon,y_1}^{q-2}\Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big)^2 + w_{\varepsilon,y_1}^2\Big(\sum_{i=2}^k w_{\varepsilon,y_i}\Big)^{q-2}\Big) \\
&\le \begin{cases} C\varepsilon^3 \sum\limits_{i=2}^k e^{-(q-2)\frac{\eta|y_1-y_i|}{\varepsilon}}, & 3 < q \le 4; \\ C\varepsilon^3 \sum\limits_{i=2}^k e^{-2\frac{\eta|y_1-y_i|}{\varepsilon}}, & q > 4. \end{cases}
\end{aligned} \tag{A.6}
$$

Denote

$$
1 + \bar{\sigma} = \begin{cases} \frac{q}{2}, & 2 < q \le 3, \\ q-2, & 3 < q \le 4, \\ 2, & q > 4. \end{cases}
$$

Combining (A.4)–(A.6) yields

$$
\begin{aligned}
&\int \Big(\Big(\sum_{i=1}^k w_{\varepsilon,y_i}\Big)^q - \sum_{i=1}^k w_{\varepsilon,y_i}^q\Big) = q\int \sum_{i<j} w_{\varepsilon,y_i}^{q-1} w_{\varepsilon,y_j} \\
&\quad + q\int \sum_{i=1}^{k-1} w_{\varepsilon,y_i}\Big(\sum_{j=i+1}^k w_{\varepsilon,y_j}\Big)^{q-1} + O\Big(\varepsilon^3 \sum_{i<j} e^{-(1+\bar{\sigma})\frac{\eta|y_i-y_j|}{\varepsilon}}\Big) \\
&= q\int \sum_{i=1}^{k-1} w_{\varepsilon,y_i}\Big(\sum_{j=i+1}^k w_{\varepsilon,y_j}\Big)^{q-1} + O\Big(\varepsilon^3 \sum_{i<j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).
\end{aligned} \tag{A.7}
$$

Secondly, to estimate the fourth term of (A.3), we have

$$\int (Q(x)-1)\Big(\sum_{i=1}^{k} w_{\varepsilon,y_i}\Big)^q = \sum_{i=1}^{k}\int (Q(x)-1)\, w^q_{\varepsilon,y_i} + O\Big(\varepsilon^3 \sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}\Big). \tag{A.8}$$

Estimating the first term of the right side of (A.8), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} (Q(x)-1)\, w^q_{\varepsilon,y_i} &= \int_{B_\delta(y_i)} (Q(x)-Q(y_i))\, w^q_{\varepsilon,y_i} \\ &\quad + \int_{B^c_\delta(y_i)} (Q(x)-Q(y_i))\, w^q_{\varepsilon,y_i} + \int_{\mathbb{R}^3} (Q(y_i)-1)\, w^q_{\varepsilon,y_i},\end{aligned} \tag{A.9}$$

$$\begin{aligned}\Big|\int_{B^c_\delta(y_i)} (Q(x)-Q(y_i))\, w^q_{\varepsilon,y_i}\Big| &\le \int_{B^c_\delta(y_i)} |Q(x)-Q(y_i)| w^q_{\varepsilon,y_i} \le C\varepsilon^3 \int_{|y|\ge\frac{\delta}{\varepsilon}} w^q(y) \\ &\le C\varepsilon^3 \int_{|y|\ge\frac{\delta}{\varepsilon}} e^{-q\eta|y|}|y|^{-q} \le C\varepsilon^3 e^{-\frac{\bar q\eta\delta}{\varepsilon}},\end{aligned} \tag{A.10}$$

where $\bar q = q - \hat\theta$ and $\hat\theta > 0$ is a small constant, and

$$\begin{aligned}&\int_{B_\delta(y_i)} (Q(x)-Q(y_i))\, w^q_{\varepsilon,y_i} \\ &\le C\varepsilon^3 \int_{|y|<\frac{\delta}{\varepsilon}} \Big(\sum_{m=1}^{[h]} \varepsilon^m |y|^m |D^m Q(y_i)| + \varepsilon^{[h]+1}|y|^{[h]+1}\Big) w^q(y) \\ &\le C\Big(\sum_{m=1}^{[h]} \varepsilon^{3+m}|D^m Q(y_i)| + \varepsilon^{4+[h]}\Big).\end{aligned} \tag{A.11}$$

Combining (A.9)–(A.11) yields

$$\begin{aligned}\int (Q(x)-1)\, w^q_{\varepsilon,y_i} &= (Q(y_i)-1)\int w^q_{\varepsilon,y_i} + \varepsilon^3 O\big(e^{-\frac{\bar q\eta\delta}{\varepsilon}} + \varepsilon^{[h]+1}\big) \\ &\quad + \varepsilon^3 O\Big(\sum_{m=1}^{[h]} \varepsilon^m |D^m Q(y_i)|\Big) \\ &= (Q(y_i)-1)\,\varepsilon^3 \|w\|^q_{L^q} + O\big(\varepsilon^{4+[h]}\big) + O\Big(\sum_{m=1}^{[h]} \varepsilon^{3+m}|D^m Q(y_i)|\Big).\end{aligned} \tag{A.12}$$

Combining (A.3), (A.7), (A.8) and (A.12) yields (A.1). □

Lemma A.2. *For any $Y \in D^k_{\varepsilon,\delta}$, there holds*

$$\begin{aligned}\frac{\partial K}{\partial y_{il}} &= -C\varepsilon^3 D_l Q(y_i) - (q-1)\sum_{j=1,j\neq i}^{k} \int w^{q-2}_{\varepsilon,y_i} w_{\varepsilon,y_j} \frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} \\ &\quad + O\Big(\sum_{i=1}^{k}\sum_{m=2}^{[h]} \varepsilon^{2+m-\tau}|D^m Q(y_i)|^{1-\tau} + \varepsilon^{3+[h]-\tau}\Big) \\ &\quad + O\Big(\varepsilon^2 \sum_{i=1}^{k} |Q(y_i)-1|^{1-\tau}\Big) + O\Big(\varepsilon^2 \sum_{i\neq j} e^{-(\bar\theta-\tau)\frac{\eta|y_i-y_j|}{\varepsilon}}\Big),\end{aligned} \tag{A.13}$$

where $i = 1, \ldots, k$ and $l = 1, 2, 3$.

Proof. By the definition of $K(Y)$, we have

$$\frac{\partial K}{\partial y_{il}} = \frac{\partial J_\varepsilon}{\partial y_{il}} + \left\langle \frac{\partial J_\varepsilon}{\partial \varphi_{\varepsilon,Y}}, \frac{\partial \varphi_{\varepsilon,Y}}{\partial y_{il}} \right\rangle. \tag{A.14}$$

First, estimating the first term of (A.14), we obtain

$$\begin{aligned}
\frac{\partial J_\varepsilon}{\partial y_{il}} &= \int \varepsilon^2 a \nabla (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \\
&\quad + \varepsilon b \int |\nabla (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y})|^2 \int \nabla (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y}) \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \\
&\quad - \int Q(x) (W_{\varepsilon,Y} + \varphi_{\varepsilon,Y})_+^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \qquad \text{(A.15)} \\
&= \left\langle I_\varepsilon{}' (W_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle I_\varepsilon{}'' (W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle R'_{\varepsilon,Y} (\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle \\
&= l_{\varepsilon,Y} \left(\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) + \left\langle I_\varepsilon{}'' (W_{\varepsilon,Y}) [\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle + \left\langle R'_{\varepsilon,Y} (\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle.
\end{aligned}$$

To estimate the first term of the right side of (A.15), since $Q(0) = 1$ and w is the solution of (2.3), we obtain $w_{\varepsilon,y_j} (1 \le j \le k)$ satisfies

$$-\left(\varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_t}|^2 \right) \Delta w_{\varepsilon,y_j} + w_{\varepsilon,y_j} = w_{\varepsilon,y_j}^{q-1}, \quad t = 1, \ldots, k.$$

Summing j from 1 to k, we obtain

$$-\left(\varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_t}|^2 \right) \Delta W_{\varepsilon,Y} + W_{\varepsilon,Y} = \sum_{j=1}^k w_{\varepsilon,y_j}^{q-1}.$$

Multiplying $\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}$ on both sides of the above equality and integrating, we have

$$\left(\varepsilon^2 a + \varepsilon b k \int |\nabla w_{\varepsilon,y_t}|^2 \right) \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int W_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} = \int \sum_{j=1}^k w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}.$$

Then

$$\begin{aligned}
\left\langle W_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right\rangle_\varepsilon &= \int \left(\varepsilon^2 a \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + W_{\varepsilon,Y} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) \\
&= -\varepsilon b k \int |\nabla w_{\varepsilon,y_t}|^2 \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} + \int \sum_{j=1}^k w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}},
\end{aligned}$$

Hence,

$$\begin{aligned}
l_{\varepsilon,Y} \left(\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) &= \varepsilon b \int \nabla W_{\varepsilon,Y} \nabla \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \left(\int |\nabla W_{\varepsilon,Y}|^2 - \int \sum_{t=1}^k |\nabla w_{\varepsilon,y_t}|^2 \right) \\
&\quad - \left(\int Q(x) W_{\varepsilon,Y}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} - \int \sum_{j=1}^k w_{\varepsilon,y_j}^{q-1} \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \right) \qquad \text{(A.16)} \\
&=: \tilde{l}_1 - \tilde{l}_2.
\end{aligned}$$

By similar estimates of l_1 as that of Lemma 3.2, we have

$$|\tilde{l}_1| \le C\varepsilon^2 \sum_{i\neq j} e^{-\frac{\eta|y_i-y_j|}{\varepsilon}}. \tag{A.17}$$

Next, to estimate $\tilde{l}_2$, we have

$$\begin{aligned}
\tilde{l}_2 &= \int Q(x)\Big(\sum_{j=1}^{k} w_{\varepsilon,y_j}\Big)^{q-1}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} - \int \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}}\\
&= \int \left(\left(\sum_{j=1}^{k} w_{\varepsilon,y_j}\right)^{q-1} - \sum_{j=1}^{k} w_{\varepsilon,y_j}^{q-1}\right)\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}}\\
&\quad + \int (Q(x)-1)\left(\sum_{j=1}^{k} w_{\varepsilon,y_j}\right)^{q-1}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}}\\
&=: \tilde{l}_{21} + \tilde{l}_{22}.
\end{aligned} \tag{A.18}$$

Since

$$\tilde{l}_{21} \le (q-1)\sum_{j=1,j\neq i}^{k}\int w_{\varepsilon,y_i}^{q-2} w_{\varepsilon,y_j}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big), \tag{A.19}$$

and

$$\begin{aligned}
\tilde{l}_{22} &= \sum_{j=1}^{k}\int (Q(x)-1)\, w_{\varepsilon,y_j}^{q-1}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big)\\
&= \int (Q(x)-1) w_{\varepsilon,y_i}^{q-1}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big)\\
&= \frac{1}{q}\int \frac{\partial Q(x)}{\partial x_l} w_{\varepsilon,y_i}^{q} + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big)\\
&= C\varepsilon^3 D_l Q(y_i) + O\Big(\sum_{i=1}^{k}\sum_{m=2}^{[h]}\varepsilon^{2+m}|D^m Q(y_i)| + \varepsilon^{3+[h]}\Big)\\
&\quad + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big),
\end{aligned} \tag{A.20}$$

combining (A.17)–(A.20) yields

$$\begin{aligned}
l_{\varepsilon,Y}\Big(\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\Big) &= -C\varepsilon^3 D_l Q(y_i) - (q-1)\sum_{j=1,j\neq i}^{k}\int w_{\varepsilon,y_i}^{q-2} w_{\varepsilon,y_j}\frac{\partial w_{\varepsilon,y_i}}{\partial y_{il}} + O(\varepsilon^{3+[h]})\\
&\quad + O\Big(\sum_{i=1}^{k}\sum_{m=2}^{[h]}\varepsilon^{2+m}|D^m Q(y_i)|\Big) + O\Big(\varepsilon^2\sum_{i\neq j} e^{-\bar{\theta}\frac{\eta|y_i-y_j|}{\varepsilon}}\Big).
\end{aligned} \tag{A.21}$$

Next, we estimate the second term of the right side of (A.15). We have

$$\begin{aligned}\left\langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\rangle &= \left\langle \varphi_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\rangle_\varepsilon + \varepsilon b\int |\nabla W_{\varepsilon,Y}|^2 \int \nabla\varphi_{\varepsilon,Y}\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \\ &\quad + 2\varepsilon b\int \nabla W_{\varepsilon,Y}\nabla\varphi_{\varepsilon,Y}\int \nabla W_{\varepsilon,Y}\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} \\ &\quad - (q-1)\int Q(x)W_{\varepsilon,Y}^{q-2}\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\varphi_{\varepsilon,Y}.\end{aligned}\tag{A.22}$$

Since $\varphi_{\varepsilon,Y}\in E^k_{\varepsilon,Y}$,

$$\left\langle \varphi_{\varepsilon,Y}, \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\rangle_\varepsilon = 0.\tag{A.23}$$

By Hölder inequality, we obtain

$$\begin{aligned}\varepsilon b\int |\nabla W_{\varepsilon,Y}|^2\int \nabla\varphi_{\varepsilon,Y}\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} &\le \varepsilon bk\int\sum_{i=1}^k |\nabla w_{\varepsilon,y_i}|^2\|\nabla\varphi_{\varepsilon,Y}\|_{L^2}\left\|\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\|_{L^2} \\ &\le C\varepsilon^{\frac12}\|\varphi_{\varepsilon,Y}\|_\varepsilon\end{aligned}\tag{A.24}$$

$$\begin{aligned}\varepsilon b\int \nabla W_{\varepsilon,Y}\nabla\varphi_{\varepsilon,Y}\int \nabla W_{\varepsilon,Y}\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}} &\le \varepsilon b\|\nabla W_{\varepsilon,Y}\|^2_{L^2}\|\nabla\varphi_{\varepsilon,Y}\|_{L^2}\left\|\nabla\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\|_{L^2} \\ &\le C\varepsilon^{\frac12}\|\varphi_{\varepsilon,Y}\|_\varepsilon,\end{aligned}\tag{A.25}$$

$$\begin{aligned}\int Q(x)W_{\varepsilon,Y}^{q-2}\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\varphi_{\varepsilon,Y} &\le C\Big(\int W_{\varepsilon,Y}^q\Big)^{\frac{q-2}{q}}\Big(\int |\frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}|^q\Big)^{\frac1q}\Big(\int |\varphi_{\varepsilon,Y}|^q\Big)^{\frac1q} \\ &\le C\varepsilon^{\frac12}\|\varphi_{\varepsilon,Y}\|_\varepsilon.\end{aligned}\tag{A.26}$$

Combining (A.22)–(A.26) yields

$$\left\langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi_{\varepsilon,Y}], \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\rangle = O(\varepsilon^{\frac12}\|\varphi_{\varepsilon,Y}\|_\varepsilon).\tag{A.27}$$

Besides, by Lemma 3.3, we have

$$\left\langle R'_{\varepsilon,Y}(\varphi_{\varepsilon,Y}), \frac{\partial W_{\varepsilon,Y}}{\partial y_{il}}\right\rangle = o_\varepsilon(1)\|\varphi_{\varepsilon,Y}\|_\varepsilon.\tag{A.28}$$

By Lemma 3.4, we have

$$\left\langle \frac{\partial J_\varepsilon}{\partial\varphi_{\varepsilon,Y}}, \frac{\partial\varphi_{\varepsilon,Y}}{\partial y_{il}}\right\rangle = 0.\tag{A.29}$$

Combining (A.21) and (A.27)–(A.29) yields (A.13). □

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References

[1] Ambrosetti, A., E. Colorado, and D. Ruiz; Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations. - Calc. Var. Partial Differential Equations 30, 2007, 85–112.

[2] Arosio, A., and S. Panizzi: On the well-posedness of the Kirchhoff string. - Trans. Amer. Math. Soc. 348, 1996, 305–330.

[3] Bartsch, T., and S. Peng: Semiclassical symmetric Schrödinger equations: existence of solutions concentrating simultaneously on several spheres. - Z. Angew. Math. Phys. 58, 2007, 778–804.

[4] Bernstein, S.: Sur une classe d'équations fonctionelles aux dérivées partielles. - Bull. Acad. Sci. URSS. Sér. Math. 4, 1940, 17–26.

[5] Cao, D., S. Li, and P. Luo: Uniqueness of positive bound states with multi-bump for nonlinear Schrödinger equations. - Calc. Var. Partial Differential Equations 54, 2015, 4037–4063.

[6] Cao, D., E. S. Noussair, and S. Yan: Solutions with multiple peaks for nonlinear elliptic equations. - Proc. Roy. Soc. Edinburgh Sect. A 129, 1999, 235–264.

[7] Cui, L., G. Li, P. Luo, and C. Wang: Existence and local uniqueness of normalized multi-peak solutions to a class of Kirchhoff type equations. - Minimax Theory Appl. 7, 2022, 207–252.

[8] Dancer, E. N., K. Y. Lam, and S. Yan: The effect of the graph topology on the existence of multipeak solutions for nonlinear Schrödinger equations. - Abstr. Appl. Anal. 3, 1998, 293–318.

[9] Dancer, E. N., and S. Yan: Interior and boundary peak solutions for a mixed boundary value problem. - Indiana Univ. Math. J. 48, 1999, 1177–1212.

[10] Dancer, E. N., and S. Yan: On the existence of multipeak solutions for nonlinear field equations on $\mathbb{R}^N$. - Discrete Contin. Dynam. Systems 6, 2000, 39–50.

[11] del Pino, M., and P. L. Felmer: Multi-peak bound states for nonlinear Schrödinger equations. - Ann. Inst. H. Poincaré C Anal. Non Linéaire 15, 1998, 127–149.

[12] Deng, Y., S. Peng, and W. Shuai: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. - J. Funct. Anal. 269, 2015, 3500–3527.

[13] Figueiredo, G. M., N. Ikoma, J. R. Santos Júnior: Existence and concentration result for the Kirchhoff type equations with general nonlinearities. - Arch. Rational Mech. Anal. 213, 2014, 931–979.

[14] Figueiredo, G. M., and J. R. Santos Júnior: Multiplicity and concentration behavior of positive solutions for a Schrödinger–Kirchhoff type problem via penalization method. - ESAIM Control Optim. Calc. Var. 20, 2014, 389–415.

[15] Floer, A., and A. Weinstein: Nonspeading wave packets for the cubic Schrödinger equation with a bounded potential. - J. Funct. Anal. 69, 1986, 397–408.

[16] Gui, C.: Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method. - Comm. Partial Differential Equations 21, 1996, 787–820.

[17] Guo, Z.: Ground states for Kirchhoff equations without compact condition. - J. Differential Equations 259, 2015, 2884–2902.

[18] He, X. M., and W. M. Zou: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. - J. Differential Equations 252, 2012, 1813–1834.

[19] He, Y.: Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity. - J. Differential Equations 261, 2016, 6178–6220.

[20] He, Y., and G. Li: Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. - Calc. Var. Partial Differential Equations 54, 2015, 3067–3106.

[21] He, Y., G. Li, and S. Peng: Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. - Adv. Nonlinear Stud. 14, 2014, 483–510.

[22] Kirchhoff, G.: Mechanik. - Teubner, Leipzig, 1883.

[23] Kwong, M. K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. - Arch. Rational Mech. Anal. 105, 1989, 243–266.

[24] Li, G., P. Luo, S. Peng, C. Wang, and C. Xiang: Uniqueness and Nondegeneracy of positive solutions to Kirchhoff equations and its applications in singular perturbation problems. - J. Differential Equations 268, 2020, 541–589.

[25] Lions, J. L.: On some questions in boundary value problems of mathematical physics. - In: Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), 284–346.

[26] Liu, W., and C. Wang: Multi-peak solutions of a nonlinear Schrödinger equation with magnetic fields. - Adv. Nonlinear Stud. 14, 2014, 951–975.

[27] Luo, P., S. Peng, C. Wang, and C. Xiang: Multi-peak positive solutions to a class of Kirchhoff equations. - Proc. Roy. Soc. Edinburgh Sect. A 149, 2019, 1097–1122.

[28] Noussair, E. S., and S. Yan: On positive multipeak solutions of a nonlinear elliptic problem. - J. London Math. Soc. (2) 62, 2000, 213–227.

[29] Oh, Y. G.: On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential. - Commun. Math. Phys. 131, 1990, 223–253.

[30] Perera, K., and Z. Zhang: Nontrivial solutions of Kirchhoff-type problems via the Yang index. - J. Differential Equations 221, 2006, 246–255.

[31] Pohozaev, S. I.: A certain class of quasilinear hyperbolic equations. - Mat. Sb. (N.S.) 96:138, 1975, 152–166, 168.

[32] Wang, J., L. Tian, J. Xu, and F. Zhang: Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. - J. Differential Equations 253, 2012, 2314–2351.

[33] Xu, S., and C. Wang: Local uniqueness of a single peak solution of a subcritical Kirchhoff problem in $\mathbb{R}^3$. - Acta Math. Sci. Ser. A 40, 2020, 432–440.

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