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ON THE NULL-SETS FOR EXTREMAL DISTANCES

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1. *Introduction.* Let $\bar{R}^n$ be the n-dimensional Möbius space, i.e. the one point compactification of the Euclidean n-space R^n. Let E be a closed proper subset of $\bar{R}^n$. Suppose that F_1 and F_2 are two disjoint continua in the complement of E. Denote by Γ the family of all arcs in $\bar{R}^n$ which join F_1 and F_2 and let Γ_E be the subfamily of Γ whose members do not meet E. Consider the modules[1]) $M(\Gamma)$ and $M(\Gamma_E)$. Since $\Gamma_E \subset \Gamma$, $M(\Gamma_E) \leqq M(\Gamma)$.

We say that E is a *null-set for extremal distances* (with respect to $\bar{R}^n$) if $M(\Gamma_E) = M(\Gamma)$ for all pairs of sets F_1, F_2. In other words, the removal of E does not change the extremal distance between two continua in its complement. We abbreviate this by saying that E is an NED-set or simply E is NED.

In the case $n = 2$, Ahlfors and Beurling [1] have proved that E is NED if and only if it is an O_{AD}-set, i.e. every non-constant analytic function in the complement of E has an infinite Dirichlet integral.

The purpose of this paper is to study the NED-sets in higher dimensions. We establish for a closed $E \subset \bar{R}^n$ the following *measure-theoretic* conditions:

(a) If E is NED, then the n-dimensional Lebesgue measure of E is zero.

(b) If the $n-1$-dimensional Hausdorff measure of E is zero, then E is NED.

In addition we prove the *topological* condition

(c) If E is NED, then $\dim E \leqq n-2$.

For $n = 2$, these are well-known properties of the O_{AD}-sets.

2. *Terminology.* The points of R^n are treated as vectors. We denote by B^n the n-ball $|x| < 1$, where $|x|$ is the norm of the vector x. If $x \in R^n$, $A \subset R^n$, $C \subset R^n$ and r is a real number, we let

$$A + x = \{a + x : a \in A\},$$
$$A \pm C = \{a \pm c : a \in A, c \in C\},$$
$$rA = \{ra : a \in A\}.$$

[1]) For the definition of the module, see Section 2.

With this notation, we have for instance

$$A + rB^n = \{x: d(x , A) < r\} ,$$

where d is the Euclidean distance. The complement of A with respect to C is denoted by $C \sim A$.

For $A \subset R^n$, we let $m_n(A)$ be its n-dimensional Lebesgue outer measure. Put $\Omega_n = m_n(B^n)$. The p-dimensional[2]) Hausdorff outer measure $m_p(A)$ of A is defined as follows: Let $\varepsilon > 0$ and let $B_1 , B_2 ,\ldots$ be a countable covering of A by open n-balls with radii $r_1 , r_2 ,\ldots$ such that $r_i < \varepsilon$. Set

$$m_p^\varepsilon(A) = \inf \sum_{i=1}^{\infty} \Omega_p r_i^p$$

over all such coverings. Then

$$m_p(A) = \lim_{\varepsilon \to 0+} m_p^\varepsilon(A) = \sup_{\varepsilon > 0} m_p^\varepsilon(A) .$$

The measure of a set A which contains the point at infinity is defined as the measure of $A \sim \{\infty\}$.

Let Γ be a family of curves in $\bar{R}^n$. We define $F(\Gamma)$ as the family of all non-negative Borel-measurable (= Baire) functions, defined in R^n and satisfying the condition

$$\int_\gamma \varrho \, ds \geqq 1 \tag{1}$$

for every $\gamma \in \Gamma$. The greatest lower bound

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^n \, d\tau$$

is the *module* of Γ . Here $d\tau$ is the n-dimensional volume element. We will usually omit the domain of integration if it is the whole R^n . For the properties of the module of a curve family, see [8].

In this paper, we will only consider curve families of the following type. Let G be an open set in $\bar{R}^n$ and let F_1 , F_2 be two disjoint continua in G . Then Γ is the family of all rectifiable[3]) arcs which join F_1 and F_2 in G . We say that Γ is the *family joining* F_1 *and* F_2 *in* G . The number $1/M(\Gamma)$ is called the *extremal distance* between F_1 and F_2 in G .

3. We first state some general remarks on NED-sets. It is clear that a closed subset of an NED-set is also NED. Furthermore, since the module

[2]) We shall consider only the case $p = n - 1$.

[3]) This restriction is unessential, because the non-rectifiable curves have no influence on the module of a curve family. See [8], p. 8.

of a curve family is a conformal invariant, the image of an NED-set under a conformal mapping of $\bar{R}^n$ is also NED. If E disconnects $\bar{R}^n$, it cannot be NED. For, then we can choose two non-degenerate continua F_1, F_2 from different components of $\bar{R}^n \sim E$. The $M(\Gamma) > 0$[4]) while $M(\Gamma_E) = M(\emptyset) = 0$.

4. We next prove the proposition (a) mentioned in the introduction.

Theorem 1. *If* $E \subset \bar{R}^n$ *is* NED, *then* $m_n(E) = 0$.

Proof. Choose two distinct points a, b from the complement of E and map $\bar{R}^n$ conformally onto itself so that a, b are mapped into $0, \infty$, respectively. The image E' of E is still NED. Since E' is closed, we can find positive numbers r_1, r_2 such that E' is contained in the spherical ring

$$A = \{x: r_1 < |x| < r_2\}.$$

Let F_1, F_2 be the components of $\bar{R}^n \sim A$. If Γ is the family joining F_1 and F_2, we have

$$M(\Gamma) = \int \varrho^n \, d\tau ,$$

where $\varrho \in F(\Gamma)$ is defined by

$$\varrho(x) = \frac{1}{|x| \log \frac{r_2}{r_1}} \quad \text{for } x \in A ,$$

$$\varrho(x) = 0 \quad \text{for } x \in R^n \sim A ,$$

(see [8], p. 8). We define a function ϱ_1 by

$$\varrho_1(x) = \varrho(x) \quad \text{for} \quad x \in R^n \sim E' ,$$

$$\varrho_1(x) = 0 \quad \text{for} \quad x \in E' .$$

Then $\varrho_1 \in F(\Gamma_{E'})$, whence

$$M(\Gamma_{E'}) \leqq \int \varrho_1^n \, d\tau \leqq \int \varrho^n \, d\tau = M(\Gamma) .$$

Because E' is NED, we have $M(\Gamma_{E'}) = M(\Gamma)$. Thus

$$0 = \int_{R^n} (\varrho^n - \varrho_1^n) \, d\tau = \int_{E'} \varrho^n \, d\tau .$$

Since $\varrho(x) > 0$ for $x \in E'$, this implies $m_n(E') = 0$. Hence, $m_n(E) = 0$, q.e.d.

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[4]) See Loewner [6].

5. In order to prove the proposition (b) in the introduction we need five lemmas.

Lemma 1. *Let* $A \subset R^n$ *be the spherical ring* $r_1 < |x| < r_2$ *and let* F_1, F_2 *be two disjoint subsets of* A *such that every sphere* $|x| = r$, $r_1 < r < r_2$, *meets both* F_1 *and* F_2. *If* Γ *is the family joining* F_1 *and* F_2 *in* A, *then*

$$M(\Gamma) \geqq c_n \log \frac{r_2}{r_1} ,$$

where c_n *is a constant depending only on* n.

In the case $n = 3$, this was proved in [8] (Theorem 3.9), with $c_3 = 1/200$. The general case can be proved in an analogous manner.

6. Let G be a domain in $\bar{R}^n$ and let F_1, F_2 be two disjoint non-degenerate bounded continua in G. Denote by Γ the family which joins F_1 and F_2 in G. Let δ be the smallest of the numbers $d(F_1, \bar{R}^n \sim G)$, $d(F_2, \bar{R}^n \sim G)$ and $\frac{1}{2}d(F_1, F_2)$. For each $0 < r < \delta$ denote

$$F_i^r = F_i + r\bar{B}^n ,$$

$i = 1, 2$. Furthermore, let Γ^r be the family joining F_1^r and F_2^r in G. For each $\varrho \in F(\Gamma)$ put

$$L(r, \varrho) = \inf_{\gamma \in \Gamma^r} \int_\gamma \varrho \, ds .$$

As r decreases, $L(r, \varrho)$ increases. Thus the limit $\lim_{r \to 0+} L(r, \varrho)$ exists.

Lemma 2. *If* $\varrho \in F(\Gamma)$ *and* ϱ *is* L^n*-integrable over* R^n, *then* $\lim_{r \to 0+} L(r, \varrho) \geqq 1$[5].

Proof. Let Γ_1^r be the family joining F_1^r and F_2 in G and let

$$L_1(r, \varrho) = \inf_{\gamma \in \Gamma_1^r} \int_\gamma \varrho \, ds .$$

We first prove that $\lim_{r \to 0+} L_1(r, \varrho) \geqq 1$.

Suppose $\lim_{r \to 0+} L_1(r, \varrho) < q < 1$. Set $R = \min(\delta, \frac{1}{2}d(F_1))$ and let $0 < r < R$. Then there exists an arc γ in Γ_1^r such that

$$\int_\gamma \varrho \, ds < q .$$

Let a be the endpoint of γ which belongs to F_1^r. Then there exists a point b in F_1 such that $|a - b| \leqq r$. For each s such that $r < s < R$,

[5]) A similar lemma is established in a recent paper [5] of Gehring.

the sphere $|x - b| = s$ meets both F_1 and γ. Let Γ_r' be the family joining F_1 and γ in G. Because the ring $r < |x - b| < R$ lies in G, Lemma 1 implies

$$(2) \qquad M(\Gamma_r') \geqq c_n \log \frac{R}{r} .$$

Let $\gamma' \in \Gamma_r'$. Because the continuum $\gamma \cup \gamma'$ joins F_1 and F_2 in G, there exists an arc γ'' in Γ such that $\gamma'' \subset \gamma \cup \gamma'$. Thus,

$$1 \leqq \int_{\gamma''} \varrho \, ds \leqq \int_{\gamma} \varrho \, ds + \int_{\gamma'} \varrho \, ds < q + \int_{\gamma'} \varrho \, ds$$

for each $\gamma' \in \Gamma_r'$. Hence, the function $\varrho/(1 - q)$ belongs to $F(\Gamma_r')$ so that

$$M(\Gamma_r') \leqq \frac{1}{(1 - q)^n} \int \varrho^n \, d\tau .$$

Together with (2) this yields

$$(1 - q)^n c_n \log \frac{R}{r} \leqq \int \varrho^n \, d\tau .$$

Finally, if we let $r \to 0$, we obtain a contradiction. Thus $\lim_{r \to 0+} L_1(r, \varrho) \geqq 1$.

Now let $0 < \varepsilon < 1$. By the above, there exists a positive number r_1 such that

$$L_1(r_1, \varrho) > 1 - \varepsilon .$$

We apply the first part of the above proof replacing F_1 by F_2, F_2 by $F_1^{r_1}$ and ϱ by $\varrho/(1 - \varepsilon)$. We thus find an $r_2 > 0$ such that

$$\int_{\gamma} \frac{\varrho}{1 - \varepsilon} \, ds > 1 - \varepsilon$$

for each γ joining $F_1^{r_1}$ and $F_2^{r_2}$. Thus,

$$\int_{\gamma} \varrho \, ds > (1 - \varepsilon)^2$$

whenever $\gamma \in \Gamma^r$ and $r < \min(r_1, r_2)$. This completes the proof of the lemma.

7. Next we require some estimates for the measure of sets $E + \gamma$ where γ is a rectifiable arc.

Lemma 3. *Let $\gamma \subset R^n$ be a rectifiable arc of length l and let $r > 0$. Then*

$$m_n(\gamma + rB^n) \leqq r^{n-1}(\Omega_n r + \Omega_{n-1} l) . \tag{3}$$

If γ is a segment of line, (3) *holds with equality.*

For $n = 2$, this is proved in Apostol [2], p. 285. The proof for the general case is similar.

Lemma 4. *Let $\gamma \subset R^n$ be a rectifiable arc of length l and let E be any subset of R^n. Then*

$$m_n(E + \gamma) \leqq l\, m_{n-1}(E) . \tag{4}$$

The bound is sharp.

Proof. If $m_{n-1}(E) = \infty$, the lemma is trivial. Assume $m_{n-1}(E)$ is finite. Let $\varepsilon > 0$. Cover E with balls $B_1, B_2, \ldots$ such that their radii $r_i < \varepsilon$ and

$$\sum_{i=1}^{\infty} \Omega_{n-1} r_i^{n-1} < m_{n-1}(E) + \varepsilon .$$

Then

$$E + \gamma \subset \bigcup_{i=1}^{\infty} (B_i + \gamma) .$$

Hence,

$$m_n(E + \gamma) \leqq \sum_{i=1}^{\infty} m_n(B_i + \gamma) .$$

By Lemma 3,

$$m_n(B_i + \gamma) \leqq r_i^{n-1}(\Omega_n r_i + \Omega_{n-1} l) .$$

Thus,

$$m_n(E + \gamma) \leqq (\Omega_n \varepsilon + \Omega_{n-1} l) \sum_{i=1}^{\infty} r_i^{n-1}$$

$$\leqq (\Omega_n \varepsilon + \Omega_{n-1} l) \frac{m_{n-1}(E) + \varepsilon}{\Omega_{n-1}} .$$

As $\varepsilon \to 0$, this gives (4).

If E is contained in an $n - 1$-dimensional linear subspace T of R^n and if γ is a line segment perpendicular to T, then (4) holds with equality.

Lemma 5. *Let γ be a rectifiable arc in R^n and let $E \subset R^n$ such that $m_{n-1}(E) = 0$. Then*

$$(\gamma + x) \cap E = \emptyset$$

or almost every $x \in R^n$.

Proof. Obviously,

$$\{x: (\gamma + x) \cap E \neq \emptyset\} = E - \gamma .$$

Because $-\gamma$ is rectifiable, Lemma 4 implies that $m_n(E - \gamma) = 0$, q.e.d.

8. We are now ready to prove our main theorem.

Theorem 2. *Let* E *be a closed subset of* $\bar{R}^n$ *such that* $m_{n-1}(E) = 0$. *Then* E *is* NED.

Proof. Let F_1, F_2 be disjoint continua in $\bar{R}^n \sim E$ and let Γ be the family joining F_1 and F_2 in $\bar{R}^n$. We must prove that

$$M(\Gamma) \leqq M(\Gamma_E), \tag{5}$$

where, as before, Γ_E is the subfamily of Γ whose members do not meet E.

Performing a preliminary conformal mapping, we may assume that F_1, F_2 are bounded. We may also assume that F_1, F_2 are non-degenerate, because otherwise $M(\Gamma) = 0$ and (5) holds trivially. Let $0 < \varepsilon < 1$. Choose a function $\varrho \in F(\Gamma_E)$ such that

$$\int \varrho^n \, d\tau < M(\Gamma_E) + \varepsilon . \tag{6}$$

By Lemma 2 there exists a positive number r such that

$$L(r, \varrho) > 1 - \varepsilon .$$

Here

$$L(r, \varrho) = \inf \int_\gamma \varrho \, ds ,$$

where the infimum is taken over all rectifiable arcs γ which join $F_1 + r\bar{B}^n$ and $F_2 + r\bar{B}^n$ in $\bar{R}^n \sim E$.

We construct the spherical r-average function ϱ_1 of ϱ,

$$\varrho_1(x) = \frac{1}{\Omega_n r^n} \int_{|y| < r} \varrho(x + y) \, d\tau .$$

We next prove that $\varrho_1/(1 - \varepsilon)$ belongs to $F(\Gamma)$.

Let $\gamma \in \Gamma$ and let $f: [0, l] \to R^n$ be the representation of γ parametrized with respect to arc-length. Then

$$\int_\gamma \varrho_1 \, ds = \frac{1}{\Omega_n r^n} \int_0^l \left(\int_{|y| < r} \varrho(f(s) + y) \, d\tau \right) ds .$$

The function $\varrho(f(s) + y)$ is Borel-measurable in $R^1 \times R^n$. By Fubini's theorem, we may interchange the order of integration. Thus

$$\int_\gamma \varrho_1 \, ds = \frac{1}{\Omega_n r^n} \int_{|y|<r} \left(\int_{\gamma+y} \varrho \, ds \right) d\tau . \tag{7}$$

The arc $\gamma + y$ joins $F_1 + r\bar{B}^n$ and $F_2 + r\bar{B}^n$ for every $|y| < r$. By Lemma 5, $(\gamma + y) \cap E = \emptyset$ for almost every y. Hence,

$$\int_{\gamma+y} \varrho \, ds \geqq L(r, \varrho) > 1 - \varepsilon$$

for almost all y, $|y| < r$. Consequently, (7) yields

$$\int_\gamma \varrho_1 \, ds \geqq 1 - \varepsilon .$$

This proves that $\varrho_1/(1 - \varepsilon) \in F(\Gamma)$. We thus have the estimate

$$M(\Gamma) \leqq \frac{1}{(1-\varepsilon)^n} \int \varrho_1^n \, d\tau . \tag{8}$$

An application of Hölder's inequality gives

$$\int \varrho_1^n \, d\tau \leqq \int \varrho^n \, d\tau \tag{9}$$

(cf. Morrey [7], p. 687). Combining (6), (8) and (9) we obtain

$$M(\Gamma) \leqq \frac{M(\Gamma_E) + \varepsilon}{(1-\varepsilon)^n} ,$$

and letting $\varepsilon \to 0$ yields (8).

Remark. The condition $m_{n-1}(E) = 0$ of the theorem cannot be replaced by $m_{n-1}(E) < \infty$. For instance, the $n-1$-sphere $|x| = 1$ has finite $n-1$-measure, but because it disconnects R^n, it cannot be NED.

9. We next prove the topological condition (c).

Theorem 3. *If* $E \subset \bar{R}^n$ *is* NED, *then* $\dim E \leqq n - 2$.

Proof. If $\dim E = n$, then E contains an inner point. Thus $m_n(E) > 0$, which is impossible by Theorem 1.

Assume that $\dim E = n - 1$. Then, by results due to Frankl and Pontrjagin [3, 4], there exists a domain G in $\bar{R}^n$ such that $G \sim E$ is not connected. Let $a \in G \cap E$ be a common boundary point of two components U and V of $G \sim E$. We may assume that $a \neq \infty$. Fix $R > 0$ such that the ball $a + 2RB^n$ is contained in G. Let $0 < r < R$.

Choose points x and y in $U \cap (a + rB^n)$ and $V \cap (a + rB^n)$, respectively. Because $\bar{R}^n \sim E$ is connected (see Section 3), there is an arc γ which joins x and y in $\bar{R}^n \sim E$. Let $\alpha \subset a + R\bar{B}^n$ be the subarc of γ which joins x to the boundary sphere of $a + R\bar{B}^n$, and let β be the corresponding arc for y. Consider the family Γ which joins α and β. By Lemma 1,

$$M(\Gamma) \geqq c_n \log \frac{R}{r} . \tag{10}$$

Next define a function ϱ by

$$\varrho(x) = \frac{1}{2R} \quad \text{for } x \in a + 2RB^n ,$$

$$\varrho(x) = 0 \quad \text{otherwise.}$$

Obviously, $\varrho \in F(\Gamma_E)$. Consequently,

$$M(\Gamma_E) \leqq \int \varrho^n \, d\tau = \Omega_n . \tag{11}$$

Because E is NED, we have $M(\Gamma) = M(\Gamma_E)$. Hence, (10) and (11) yield

$$c_n \log \frac{R}{r} \leqq \Omega_n .$$

Letting $r \to 0$ gives the desired contradiction.

Remark. Theorem 3 has the following consequence: Let E be a closed subset of R^{n-1} such that E contains an inner point. Then no topological imbedding of E into $\bar{R}^n$ is NED. In particular, if we consider R^{n-1} as a subset of R^n, E is not NED with respect to $\bar{R}^n$.

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