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A GENERALIZATION OF LAPLACE'S TRANSFORMATION

BY

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To Rolf Nevanlinna on his 70th birthday

HELSINKI 1966
SUOMALAINEN TIEDEAKATEMIA

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A Generalization of Laplace's Transformation

In this paper we seek to deal with a question regarding elementary solutions of second-order partial differential equations in two independent variables, namely that of solutions of the form

$$z = A(x\,,y\,,\varphi(x)\,,\varphi'(x)\,,\varphi''(x)\,,\ldots,\varphi^{(n)}(x))\,, \tag{1}$$

where φ is an arbitrary function.

The term «elementary solution» is here used, not with ıeference to any particular class of functions called elementary functions, but simply to mean a solution involving one or more arbitrary functions and expressed in terms of them by means of particular functions (not necessarily elementary) and the operations of differentiation and indefinite integration.

In place of (1), our results will apply in part also to solutions of the somewhat more general form

$$z = A(x\,,y\,,\varphi(x)\,,\varphi_1(x)\,,\varphi_2(x)\,,\ldots,\varphi_n(x))\,, \tag{2}$$

where φ is an arbitrary function, and $\varphi_1\,,\varphi_2\,,\ldots,\varphi_n$ are functions that depend on the function φ in any way at all, subject to the restriction that no relation of the form

$$\begin{aligned}\varphi_n'(x) = B(x\,,\varphi(x)\,,\varphi_1(x)\,,\varphi_2(x)\,,\ldots,\varphi_n(x)\,,\varphi'(x)\,,\varphi_1'(x)\,,\\ \varphi_2'(x)\,,\ldots,\varphi_{n-1}'(x))\end{aligned} \tag{3}$$

shall hold for arbitrary choice of the function φ.

The forms of solution (1) and (2) are of course special, both in restricting the argument of the arbitrary function φ to be x and in the restriction that no relation of the form (3) shall hold, if not also otherwise, and in a concluding section we shall give some indications in regard to the possibility of extending our methods so as to remove the two restrictions named.

Following conventions of notation that are standard we shall use z as dependent variable and x and y as independent variables. And then the letters p and q are used to stand for the two first-order partial derivatives (of z with respect to x and to y respectively), and the letters r, s, and t

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are used to stand for the three second-order partial derivatives. Similarly, if new variables Z, X, and Y are introduced by a transformation, we shall use P and Q to stand for the two first-order partial derivatives (of Z with respect to X and to Y), and R, S, and T to stand for the three second-order partial derivatives. And if new variables ζ, ξ, η are introduced, we shall use π, φ for the two first-order partial derivatives, and ϱ, σ, τ for the three second-order partial derivatives.

We shall also employ in connection with n-ary functions a notation, analogous to the standard use of the prime in connection with singulary functions, by which the derived functions are denoted by placing a numerical subscript after the function letter. For example, if the letter f denotes a ternary function, then f_1, f_2, f_3 denote the three derived functions obtained by taking the partial derivative with respect to the first argument, the second argument, and the third argument respectively. And thus, e.g., $f_3(x, y, z)$ expresses what would more usually be expressed by

$$\frac{\partial}{\partial z} f(x, y, z)$$

or by $f_z(x, y, z)$, while $f_3(z, x, y)$ corresponds rather to

$$\frac{\partial}{\partial y} f(z, x, y)$$

or to $f_y(z, x, y)$. Similarly f_{12} denotes the function obtained from f by taking the second partial derivative, with respect to the first argument and then with respect to the second argument; and f_{22} denotes the function obtained from f by taking the second partial derivative, with respect to the second argument twice.

Properly the standard notations such as

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial^2}{\partial y \partial z} \tag{4}$$

are applicable to forms or to letters which stand for forms, while the numerical-subscript notation is applicable rather to letters that denote functions. The distinction is important in principle and must be maintained, although it will be somewhat obscured in the present paper by our practice of omitting the arguments after a function letter, purely as an abbreviation, when it is clear from the context what the arguments are, or when (as frequently happens in the treatment of differential equations) the arguments of a particular function letter remain the same throughout some one context.

This use of, for example, the letter f alone as an abbreviation of $f(x, y, z)$ must of course also be distinguished from the more proper use of the letter f, to denote the function itself as an abstract entity. The use of the abbreviation is justified only so far as it does not engender real confusion.

The numerical-subscript notation for the derived functions of an n-ary function has some inconveniences, among them that subscripts used for other purposes must sometimes be enclosed in parentheses to avoid confusing them with subscripts referring to partial derivatives.

However, the numerical-subscript notation avoids the well-known equivocacy of the standard notation (4) — which fails to indicate, when the partial derivative is taken with respect to a particular variable, what the other variables are that are being held constant. This makes the numerical-subscript notation a substantial aid to thought when there are distinctions to be made in this regard, and for this reason we shall tend to use it in preference to the notation (4) whenever it seems to be easily and conveniently possible to do so.

In the statement of results in the following sections there are certain more or less evident conditions which will generally be left tacit. These include the existence of derivatives which are used, the existence of solutions of certain differential equations (as shown by the context to be required), the existence of certain implicit functions, and the restriction of results to an appropriate neighborhood as may be necessary to secure the foregoing. In general these are conditions which we expect could be secured by imposing appropriate ordinary conditions of regularity on the coefficients of the differential equation for which solutions are sought, and a more thorough account than is attempted in the present paper should spell this out in detail.

1. Preliminary cases

1.1. *If a partial differential equation*

$$f(x, y, z, p, q, r, s, t) = 0 \tag{5}$$

has a solution of the form (1) *or of the form* (2), *then this solution must satisfy the partial differential equation* (5) *identically in* r.

By satisfying (5) «identically in r» it is meant that the solution in fact satisfies

$$f(x, y, z, p, q, u, s, t) = 0 \tag{6}$$

where u is a new independent variable (independent of x and y). And of course it follows as a corollary that any solution of (5) of the form (1) or (2) must be a solution also of the differential equation

$$f_6(x, y, z, p, q, r, s, t) = 0 \tag{7}$$

For a solution of the form (1), theorem 1.1 becomes immediately evident if we substitute the assumed solution in the differential equation (5). In fact substitution of (1) in (5) yields

$$f(x, y, A, A_1 + A_3 \varphi' + A_4 \varphi'' + \ldots + A_{n+3} \varphi^{(n+1)}, A_2, A_{11} + 2A_{13}\varphi' + \ldots + A_{n+3} \varphi^{(n+2)}, A_{12} + A_{23} \varphi' + A_{24} \varphi'' + \ldots + A_{2(n+3)} \varphi^{(n+1)}, A_{22}) = 0 \tag{8}$$

And since this must hold for an arbitrary function φ, it must hold identically in $x, y, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n+2)}$ as $n + 5$ independent variables. We may of course assume that the function A is not independent of its last argument, and hence that A_{n+3} does not vanish identically. Theorem 1.1 then follows because, in (8), $\varphi^{(n+2)}$ occurs in the sixth argument of f but nowhere else.

On substituting (2) in (5), we see that the condition that no relation of the form (3) shall hold is dispensable as far as 1.1 is concerned. For we may argue that not all of $\varphi'', \varphi_1'', \varphi_2'', \ldots, \varphi_n''$ can be expressed each as a function of $x, \varphi, \varphi_1, \varphi_2, \ldots, \varphi_n, \varphi', \varphi_1', \varphi_2', \ldots, \varphi_n'$, as this would not be compatible with the arbitrariness of the function φ. Then if a solution of the form (2) should satisfy (5) otherwise than identically in r, we would have an equation of the form

$$\begin{gathered} A_3 \varphi'' + A_4 \varphi_1'' + A_5 \varphi_2'' + \ldots + A_{n+3} \varphi_n'' = \\ \theta(x, y, \varphi, \varphi_1, \varphi_2, \ldots, \varphi_n, \varphi', \varphi_1', \varphi_2', \ldots, \varphi_n') \end{gathered} \tag{9}$$

holding identically. From this equation by taking n times in succession the partial derivative with respect to y we get the n equations:

$$\begin{aligned} &A_{23} \varphi'' + A_{24} \varphi_1'' + A_{25} \varphi_2'' + \ldots + A_{2(n+3)} \varphi_n'' = \theta_2 \\ &A_{223} \varphi'' + A_{224} \varphi_1'' + A_{225} \varphi_2'' + \ldots + A_{22(n+3)} \varphi_n'' = \theta_{22} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &A_{22\ldots 23} \varphi'' + A_{22\ldots 24} \varphi_1'' + A_{22\ldots 25} \varphi_2'' + \ldots + A_{22\ldots 2(n+3)} \varphi_n'' = \theta_{22\ldots 2} \end{aligned}$$

These $n + 1$ equations, regarded as linear algebraic equations in φ'', $\varphi_1'', \varphi_2'', \ldots, \varphi_n''$, must not be all independent, i.e., the determinant

$$\begin{vmatrix} A_3 & A_4 & A_5 & \ldots & A_{n+3} \\ A_{23} & A_{24} & A_{25} & \ldots & A_{2(n+3)} \\ A_{223} & A_{224} & A_{225} & \ldots & A_{22(n+3)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{22\ldots 23} & A_{22\ldots 24} & A_{22\ldots 25} & \ldots & A_{22\ldots 2(n+3)} \end{vmatrix}$$

must vanish identically. From this it follows that there must be a linear relation

$$(10) \qquad a_0 A_3 + a_{(1)} A_4 + a_{(2)} A_5 + \ldots + a_{(n)} A_{n+3} = 0$$

which holds identically and whose coefficients $a_0, a_{(1)}, a_{(2)}, \ldots, a_{(n)}$ are functions of $x, \varphi, \varphi_1, \varphi_2, \ldots, \varphi_n$. By treating (10) as a partial differential equation to be solved for A and considering the form of its solution, we see that it must be possible to rewrite the solution (2) of (5) in such a way that the number n is reduced by 1. This can evidently be iterated until n is reduced to 0, at which point the solution (2) has been reduced to a special case of (1).

This completes the proof of 1.1.

Now for present purposes the problem of finding all the solutions of the form (1) or the form (2) for a given second-order partial differential equation will be dismissed as solved if we have found at least one additional partial differential equation in the same dependent and independent variables that has to be satisfied — provided that the additional partial differential equation is independent of the first one and is of not higher than the second order. This dismissal might be thought too summary, in the absence of a definitive treatment of the question of simultaneous solutions of two second-order partial differential equations in one dependent variable z and the same independent variables x and y. But it will serve to separate this rather different question from the main topic of the present paper. And there is moreover no real difficulty over the matter in the present context, because when simultaneous partial differential equations arise we are concerned, not with all their common solutions, but only with their common solutions of the form (1) or (2), and the theorems of the present section, especially 1.2, may therefore be used to make further reductions.

We shall therefore regard the problem of finding all solutions of the form (1) or (2) for a partial differential equation (5) as solved by 1.1, except in the case in which f_6 vanishes identically, i.e., the case in which (5) is independent of r. In fact we shall think of 1.1 as meaning that, although there do exist cases in which a differential equation (5) not independent of r has a solution of the form (1) or (2), such solutions are in a sense exceptional, and the main case we have to consider is that of a differential equation

$$(11) \qquad f(x, y, z, p, q, s, t) = 0$$

However, we go on immediately to a number of theorems analogous to 1.1, which we regard as showing that (11) is still too general, and that, in a certain sense which we do not attempt to make definite, the main case has to be regarded as consisting only of certain subcases of the case (11).

1.2. *If a partial differential equation*

$$f(x, y, z, p, q, t) = 0 \tag{12}$$

has a solution of the form (1) *or of the form* (2), *then this solution must satisfy the partial differential equation* (12) *identically in* p.

The proof of 1.2 is exactly analogous to that of 1.1.

1.3. *If a partial differential equation*

$$s = f(x, y, z, p, q, t) \tag{13}$$

has a solution of the form (1), *or a solution of the form* (2) *such that no relation* (3) *holds for arbitrary* φ, *then this solution must satisfy*

$$f_{44}(x, y, z, p, q, t) = 0 \tag{14}$$

identically in p.

For if we substitute in (13) an assumed solution of the form (1), we get

$$\begin{aligned} A_{12} + A_{23}\varphi' + A_{24}\varphi'' + \ldots + A_{2(n+3)}\varphi^{(n+1)} = \\ f(x, y, A, A_1 + A_3\varphi' + A_4\varphi'' + \ldots + A_{n+3}\varphi^{(n+1)}, A_2, A_{22}) \end{aligned} \tag{15}$$

This must hold identically in $x, y, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n+1)}$ as $n+4$ independent variables. We may assume that A_{n+3} does not vanish identically, and hence by taking in (15) the second partial derivative with respect to $\varphi^{(n+1)}$,

$$f_{44}(x, y, A, A_1 + A_3\varphi' + A_4\varphi'' + \ldots + A_{n+3}\varphi^{(n+1)}, A_2, A_{22}) = 0 \tag{16}$$

Equation (16) means that (1) satisfies (14), and from 1.2 it then follows that (1) satisfies (14) identically in p.

The argument is analogous in the case of a solution of the form (2), but the condition that there is no relation of the form (3) holding for arbitrary φ is evidently essential.

At this point it remains to consider differential equations of the two forms

$$f(x, y, z, q, t) = 0 \tag{17}$$

$$s = \mu(x, y, z, q, t) + \nu(x, y, z, q, t)p \tag{18}$$

as being the only ones not yet covered. That is, for a second-order differential equation of any form other than (17) or (18), theorems 1.1 — 1.3 provide the means to find all solutions of the form (1), and at least all those solutions of the form (2) which obey the condition that no relation (3) holds for arbitrary φ.

We shall ignore (17) as being properly an ordinary rather than a partial differential equation, and our main concern will therefore be with the case of differential equations of the form (18). This is the case to which the generalization of Laplace's transformation applies, and we might now turn immediately to consideration of this transformation. But since the method of finding a second differential equation which the required solution must satisfy can as a matter of fact be pressed somewhat further, at least in the case in which the required solution is of the form (1), we proceed to develop this first.

In connection with (18) we shall need to use functions $\varkappa$ and λ determined as follows:

$$\lambda(x\,,y\,,z\,,q\,,t) = \mu_3 + \mu_4\nu - \mu\nu_4 - \nu_1 + (\nu_2 + \nu_3 q + \nu_4 t + \nu^2)\mu_5 - (\mu_2 + \mu_3 q + \mu_4 t + \mu\nu)\nu_5 \tag{19}$$

$$\varkappa(x\,,y\,,z\,,q\,,t) = \lambda_3 + \lambda_4\nu - \lambda\nu_4 + (\nu_2 + \nu_3 q + \nu_4 t + \nu^2)\lambda_5 - (\lambda_2 + \lambda_3 q + \lambda_4 t + 2\lambda\nu)\nu_5 \tag{20}$$

If we assume for (18) a solution of the form (1), we get, by substituting the assumed solution in the differential equation (18),

$$A_{12} + A_{23}\,\varphi' + A_{24}\,\varphi'' + \ldots + A_{2(n+3)}\,\varphi^{(n+1)} = \mu(x\,,y\,,A\,,A_2\,,A_{22}) + \nu(x\,,y\,,A\,,A_2\,,A_{22})\,[A_1 + A_3\,\varphi' + A_4\,\varphi'' + \ldots + A_{n+3}\,\varphi^{(n+1)}] \tag{21}$$

This must hold identically in $x\,,y\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots,\varphi^{(n+1)}$ as independent variables, and hence we must have separately the two following equations (in which the arguments of μ and ν are $x\,,y\,,A\,,A_2\,,A_{22}$):

$$A_{12} + A_{23}\,\varphi' + \ldots + A_{2(n+1)}\,\varphi^{(n-1)} + A_{2(n+2)}\,\varphi^{(n)} = \mu + \nu[A_1 + A_3\,\varphi' + \ldots + A_{n+1}\,\varphi^{(n-1)} + A_{n+2}\,\varphi^{(n)}] \tag{22}$$

$$A_{2(n+3)} = \nu A_{n+3} \tag{23}$$

From equation (22), by taking the partial derivative with respect to y, and also with respect to $\varphi^{(n)}$, we get the two equations:

$$\begin{aligned} &A_{122} + A_{223}\,\varphi' + \ldots + A_{22(n+1)}\,\varphi^{(n-1)} + A_{22(n+2)}\,\varphi^{(n)} = \\ &\mu_2 + \mu_3 A_2 + \mu_4 A_{22} + \mu_5 A_{222} + (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu_5 A_{222}) \\ &[A_1 + A_3\varphi' + \ldots + A_{n+1}\,\varphi^{(n-1)} + A_{n+2}\,\varphi^{(n)}] \\ &+ \nu[A_{12} + A_{23}\varphi' + \ldots + A_{2(n+1)}\,\varphi^{(n-1)} + A_{2(n+2)}\,\varphi^{(n)}] \end{aligned} \tag{24}$$

$$\begin{aligned} &A_{12(n+3)} + A_{23(n+3)}\,\varphi' + \ldots + A_{2(n+1)\,(n+3)}\,\varphi^{(n-1)} + A_{2(n+2)(n+3)}\varphi^{(n)} \\ &+ A_{2(n+2)} = \mu_3 A_{n+3} + \mu_4 A_{2(n+3)} + \mu_5 A_{22(n+3)} + (\nu_3 A_{n+3} + \nu_4 A_{2(n+3)} \\ &+ \nu_5 A_{22(n+3)})\,[A_1 + A_3\varphi' + \ldots + A_{n+1}\,\varphi^{(n-1)} + A_{n+2}\,\varphi^{(n)}] + \nu[A_{1(n+3)} \\ &+ A_{3(n+3)}\,\varphi' + \ldots + A_{(n+1)\,(n+3)}\,\varphi^{(n-1)} + A_{(n+2)\,(n+3)}\,\varphi^{(n)} + A_{n+2}] \end{aligned} \tag{25}$$

From equation (23), by applying the operator

$$\frac{\partial}{\partial x} + \varphi' \frac{\partial}{\partial \varphi} + \varphi'' \frac{\partial}{\partial \varphi'} + \ldots + \varphi^{(n-1)} \frac{\partial}{\partial \varphi^{(n-2)}}$$

and by taking the partial derivative with respect to y and with respect to $\varphi^{(n-1)}$, we get the three equations:

$$\begin{aligned}(26)\quad & A_{12(n+3)} + A_{23(n+3)}\,\varphi' + \ldots + A_{2(n+1)\,(n+3)}\,\varphi^{(n-1)} = \\ & [\nu_1 + \nu_3(A_1 + A_3\,\varphi' + \ldots + A_{n+1}\,\varphi^{(n-1)}) + \nu_4(A_{12} + A_{23}\,\varphi' + \ldots \\ & + A_{2(n+1)}\,\varphi^{(n-1)}) + \nu_5(A_{122} + A_{223}\,\varphi' + \ldots + A_{22(n+1)}\,\varphi^{(n-1)}]\,A_{n+3} \\ & + \nu[A_{1(n+3)} + A_{3(n+3)}\,\varphi' + \ldots + A_{(n+1)\,(n+3)}\,\varphi^{(n-1)}]\end{aligned}$$

$$(27)\qquad A_{22(n+3)} = (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu_5 A_{222})A_{n+3} + \nu A_{2(n+3)}$$

$$(28)\qquad A_{2(n+2)\,(n+3)} = (\nu_3 A_{n+2} + \nu_4 A_{2(n+2)} + \nu_5 A_{22(n+2)})A_{n+3} + \nu A_{(n+2)\,(n+3)}$$

Now we multiply equations (24) — (28) by $-\nu_5 A_{n+3}$, $+1$, -1 , $\mu_5 + \nu_5[A_1 + A_3\varphi' + \ldots + A_{n+2}\,\varphi^{(n)}]$, $\varphi^{(n)}$ respectively, and add. Then use (22) to replace $A_{12} + A_{23}\,\varphi' + \ldots + A_{2(n+2)}\,\varphi^{(n)}$ by $\mu + \nu[A_1 + A_3\,\varphi' + \ldots + A_{n+2}\,\varphi^{(n)}]$, and (23) to replace $A_{2(n+3)}$ by νA_{n+3} . The result is

$$(29)\qquad A_{2(n+2)} = \nu A_{n+2} + \lambda(x\,, y\,, A\,, A_2\,, A_{22})A_{n+3}$$

We need to look separately at the special case in which n is 0, i.e., the case in which the solution (1) reduces to

$$(30)\qquad z = A(x\,, y\,, \varphi(x))$$

In this special case equations (22) — (25) become:

$$\begin{aligned}& A_{12} = \mu + \nu A_1\,, \qquad A_{23} = \nu A_3 \\ & A_{122} = \mu_2 + \mu_3 A_2 + \mu_4 A_{22} + \mu_5 A_{222} + (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu_5 A_{222})A_1 + \nu A_{12} \\ & A_{123} = \mu_3 A_3 + \mu_4 A_{23} + \mu_5 A_{223} + (\nu_3 A_3 + \nu_4 A_{23} + \nu_5 A_{223})A_1 + \nu A_{13}\end{aligned}$$

For (26) and (27) we use the two equations obtained from $A_{23} = \nu A_3$ by taking the partial derivative with respect to x and with respect to y:

$$\begin{aligned}A_{123} &= (\nu_1 + \nu_3 A_1 + \nu_4 A_{12} + \nu_5 A_{122})A_3 + \nu A_{13} \\ A_{223} &= (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu A_{222})A_3 + \nu A_{23}\end{aligned}$$

There is no equation (28) in the special case. If we multiply the four last equations, corresponding to (24) — (27), by $-\nu_5 A_3$, $+1$, -1 , $\mu_5 + \nu_5 A_1$ respectively and add, and then replace A_{12} by $\mu + \nu A_1$ and A_{23} by νA_3 , we get in place of (29):

$$0 = \lambda(x\,, y\,, A\,, A_2\,, A_{22})A_3$$

As the intention of (30) is of course that A_3 does not vanish identically, i.e., that we actually have a solution depending on an arbitrary function, it follows for any solution of (18) of the form (30) that

$$\lambda(x, y, A, A_2, A_{22}) = 0 \tag{31}$$

Now by taking in (29) the partial derivative with respect to y and with respect to $\varphi^{(n)}$, we get the two equations:

$$\begin{aligned} A_{22(n+2)} &= (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu_5 A_{222} + \nu^2) A_{n+2} \\ &+ (\lambda_2 + \lambda_3 A_2 + \lambda_4 A_{22} + \lambda_5 A_{222} + 2\lambda\nu) A_{n+3} \end{aligned} \tag{32}$$

$$\begin{aligned} A_{2(n+2)(n+3)} &= (\nu_3 A_{n+3} + \nu_4 A_{2(n+3)} + \nu_5 A_{22(n+3)}) A_{n+2} \\ &+ \nu A_{(n+2)(n+3)} + (\lambda_3 A_{n+3} + \lambda_4 A_{2(n+3)} + \\ &\lambda_5 A_{22(n+3)})\, A_{n+3} + \lambda A_{(n+3)(n+3)} \end{aligned} \tag{33}$$

And upon multiplying equations (27), (28), (32), (33) by $\nu_5 A_{n+2} + \lambda_5 A_{n+3}$, -1, $-\nu_5 A_{n+3}$, $+1$ respectively and adding, and then using (23) and (29) to replace $A_{2(n+3)}$ by νA_{n+3} and $A_{2(n+2)}$ by $\nu A_{n+2} + \lambda A_{n+3}$, we get

$$\lambda(x, y, A, A_2, A_{22}) A_{(n+3)(n+3)} + \varkappa(x, y, A, A_2, A_{22}) A_{n+3}^2 = 0 \tag{34}$$

From (34), by taking the partial derivative with respect to y and using (23), there results

$$\begin{aligned} &\lambda A_{2(n+3)(n+3)} + (\lambda_2 + \lambda_3 A_2 + \lambda_4 A_{22} + \lambda_5 A_{222}) A_{(n+3)(n+3)} \\ &+ (\varkappa_2 + \varkappa_3 A_2 + \varkappa_4 A_{22} + \varkappa_5 A_{222} + 2\varkappa\nu) A_{n+3}^2 = 0 \end{aligned} \tag{35}$$

Also from (23), by taking the partial derivative with respect to $\varphi^{(n)}$, we get

$$A_{2(n+3)(n+3)} = (\nu_3 A_{n+3} + \nu_4 A_{2(n+3)} + \nu_5 A_{22(n+3)}) A_{n+3} + \nu A_{(n+3)(n+3)} \tag{36}$$

Then multiply equations (27), (34) — (36) by $\lambda^2 \nu_5 A_{n+3}$, $\lambda_2 + \lambda_3 A_2 + \lambda_4 A_{22} + \lambda_5 A_{222} + \lambda\nu$, $-\lambda$, λ^2 respectively, and add. Use (23) to replace $A_{2(n+3)}$ by νA_{n+3}, then divide out A_{n+3}^2 (since we may assume that A_{n+3} does not vanish identically). The result is

$$\begin{aligned} &\lambda^2 \nu_3 + \lambda^2 \nu \nu_4 + (\nu_2 + \nu_3 A_2 + \nu_4 A_{22} + \nu_5 A_{222} + \nu^2) \lambda^2 \nu_5 \\ &+ \varkappa\lambda\nu + \varkappa_2 \lambda - \varkappa\lambda_2 + (\varkappa_3 \lambda - \varkappa\lambda_3) A_2 + (\varkappa_4 \lambda - \varkappa\lambda_4) A_{22} \\ &+ (\varkappa_5 \lambda - \varkappa\lambda_5) A_{222} = 0 \end{aligned} \tag{37}$$

Hence the theorems:

1.4. *If a partial differential equation* (18) *has a solution of the form* (30), *this solution must satisfy also the differential equation*

$$\lambda(x, y, z, q, t) = 0 \tag{38}$$

or must satisfy a relation among x, y, z, q, t *for which one of* $\mu, \mu_2, \mu_3, \mu_4, \mu_5, \nu, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ *is singular.*

1.5. *If a partial differential equation* (18) *has a solution of the form* (1), *this solution must satisfy also the differential equation*

$$\begin{aligned} &\lambda^2\nu_3 + \lambda^2\nu\nu_4 + (\nu_2 + \nu_3 q + \nu_4 t + \nu^2)\lambda^2\nu_5 + \varkappa\lambda\nu + \varkappa_2\lambda \\ &- \varkappa\lambda_2 + (\varkappa_3\lambda - \varkappa\lambda_3)q + (\varkappa_4\lambda - \varkappa\lambda_4)t + (\lambda^2\nu_5^2 + \varkappa_5\lambda - \varkappa\lambda_5)\frac{\partial t}{\partial y} = 0 \end{aligned} \tag{39}$$

or must satisfy a relation among x, y, z, q, t *for which one of* $\mu, \mu_2, \mu_3, \mu_4, \mu_5, \nu, \nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5$ *is singular.*

The alternative which enters in 1.4 and 1.5 — that at $z = A, q = A_2, t = A_{22}$ either μ or ν or one (or more) of the partial derivatives which are listed has a singularity — arises because the argument above has tacitly assumed that they are non-singular. The necessity of including this alternative may be shown by examples. For instance the differential equation

$$s = 1 + 3pt^{\frac{1}{3}}$$

has a solution $z = (x + c)y + \varphi(x)$ which, since $\lambda = -3t^{-\frac{1}{3}}$, corresponds to a singularity of ν_5 rather than a zero of λ. Again the differential equation

$$(z + yq)s = (z + yq)^2 + 2pq + ypt$$

has a solution $z = y^{-1}\varphi(x)$ which, since $\lambda = -1$, corresponds to a singularity of ν rather than a zero of λ.

As a consequence of 1.4 and 1.5, we may expect generally that if a partial differential equation (18) has a two-arbitrary-function solution which involves, besides $\varphi(x)$, one other arbitrary function ψ and which, for all or almost all particular choices of ψ, reduces to the form (30), or the form (1), then $\lambda(x, y, z, q, t)$ will vanish identically, or

$$\begin{aligned} &\lambda^2\nu_3 + \lambda^2\nu\nu_4 + (\nu_2 + \nu_3 q + \nu_4 t + \nu^2)\lambda^2\nu_5 + \varkappa\lambda\nu \\ &+ \varkappa_2\lambda - \varkappa\lambda_2 + (\varkappa_3\lambda - \varkappa\lambda_3)q + (\varkappa_4\lambda - \varkappa\lambda_4)t \end{aligned} \tag{40}$$

$$\lambda^2\nu_5^2 + \varkappa_5\lambda - \varkappa\lambda_5 \tag{41}$$

will both vanish identically. But it seems to be difficult to find an exact statement of such a result without imposing some undesirable restriction on the form of the two-arbitrary-function solution (as e.g. that it shall be elementary). And it may be better instead to seek related results about the existence of intermediate integrals of the sort which is illustrated by theorem 2.2 below.

As a special case we notice that (40) and (41) both vanish identically when the differential equation (18) is linear.

2. Generalization of Laplace's transformation, first case

Treating (23) as a partial differential equation to be solved for A, we find that it has an intermediate integral involving an arbitrary function. In fact if we assume an intermediate integral

$$A_2 = \theta(x, y, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n-1)}, A) \tag{42}$$

we get as condition on θ

$$\theta_{n+3} = \nu(x, y, A, \theta, \theta_2 + \theta\theta_{n+3}) \tag{43}$$

And solution of the first-order differential equation (43) for θ yields the intermediate integral. In this section we deal with the special case in which the function ν is of not higher than first degree in its last argument,

$$\nu(x, y, z, q, t) = \beta(x, y, z, q) + \delta(x, y, z, q)t \tag{44}$$

— the intermediate integral of (23) being in this case of the Monge form.

When ν has the form (44), the condition (43) becomes

$$\theta_{n+3} = \beta(x, y, A, \theta) + \delta(x, y, A, \theta)[\theta_2 + \theta\theta_{n+3}] \tag{45}$$

The general solution of the differential equation (45) may be written as

$$G(x, y, A, \theta) = \Phi(x, F(x, y, A, \theta), \varphi, \varphi', \varphi'', \ldots, \varphi^{(n-1)}) \tag{46}$$

where Φ is an arbitrary function and F and G are particular functions satisfying the conditions

$$G_3 = -\beta G_4 + \delta[G_2 + \theta G_3] \tag{47}$$

$$F_3 = -\beta F_4 + \delta[F_2 + \theta F_3] \tag{48}$$

— as (47), (48) are in fact the conditions for $G(x, y, A, \theta) = \text{constant}$, $F(x, y, A, \theta) = \text{constant}$ to be solutions of (45), and where of course we must so choose F and G that these are independent solutions of (45), if (46) is to be the general solution of (45).

The partial differential equation (18) has in the present case the special form

$$s = \mu(x, y, z, q, t) + \beta(x, y, z, q)p + \delta(x, y, z, q)pt \tag{49}$$

Generally, for solutions of (49) of the form (1), it follows from (42) and (46) that

$$G(x, y, A, A_2) = \Phi(x, F(x, y, A, A_2), \varphi, \varphi', \varphi'', \ldots, \varphi^{(n-1)}) \tag{50}$$

must hold identically in $x, y, \varphi, \varphi', \varphi'', \ldots, \varphi^{(n)}$ for some choice of the function Φ. Hence if we make a transformation of the differential equation (49) by letting

$$X = x\,, \quad Y = F(x, y, z, q)\,, \quad Z = G(x, y, z, q) \tag{51}$$

the solution of the transformed differential equation which corresponds to the solution $z = A$ of (49) may be expected to be

$$Z = \Phi(X, Y, \varphi(X), \varphi'(X), \varphi''(X), \ldots, \varphi^{(n-1)}(X)) \tag{52}$$

That is, the effect of the transformation will be to replace any solution of (49) of the form (1) by a solution which, in terms of the new variables X, Y, Z, has the same form with the number n decreased by at least 1.

To find the transformation (51) when the differential equation (49) is given, we have the differential equations (47) and (48) to solve for F and G. As (47) and (48) are conditions on the functions F and G, not on their arguments, we may rewrite (47) and (48) with x, y, z, q as the arguments, in place of x, y, A, θ. The subsidiary equations then are

$$\frac{dx}{0} = \frac{dy}{\delta(x, y, z, q)} = \frac{dz}{q\delta(x, y, z, q) - 1} = -\frac{dq}{\beta(x, y, z, q)} \tag{53}$$

And we must find three independent integrals, $x =$ constant, $F(x, y, z, q) =$ constant, $G(x, y, z, q) =$ constant, of the subsidiary equations (53).

Using (47) and (48) in the forms

$$G_3 + \beta G_4 = \delta(G_2 + G_3 q)\,, \quad F_3 + \beta F_4 = \delta(F_2 + F_3 q) \tag{54}$$

we may work out details cf the transformation (51) as follows:

$$P = \frac{\partial(Z, Y)}{\partial(x, y)} \Big/ \frac{\partial Y}{\partial y} = G_1 + G_3 p + G_4 s - \frac{(F_1 + F_3 p + F_4 s)(G_2 + G_3 q + G_4 t)}{F_2 + F_3 q + F_4 t}$$

$$= G_1 + G_3 p + G_4(\mu + \beta p + \delta p t) - \frac{[F_1 + F_3 p + F_4(\mu + \beta p + \delta p t)]\,(G_2 + G_3 q + G_4 t)}{F_2 + F_3 q + F_4 t}$$

$$= G_1 + G_4 \mu + \delta p(G_2 + G_3 q + G_4 t) - \frac{[F_1 + F_4 \mu + \delta p(F_2 + F_3 q + F_4 t)]\,(G_2 + G_3 q + G_4 t)}{F_2 + F_3 q + F_4 t}$$

$$P = G_1 + G_4 \mu - (F_1 + F_4 \mu)\,\frac{G_2 + G_3 q + G_4 t}{F_2 + F_3 q + F_4 t} \tag{55}$$

$$Q = \frac{\partial Z}{\partial y} \Big/ \frac{\partial Y}{\partial y} = \frac{G_2 + G_3 q + G_4 t}{F_2 + F_3 q + F_4 t} \tag{56}$$

$$t = \frac{G_2 + G_3 q - (F_2 + F_3 q)Q}{F_4 Q - G_4} \tag{57}$$

$$P = G_1 - F_1 Q + (G_4 - F_4 Q)\mu\left(x, y, z, q, \frac{G_2 + G_3 q - (F_2 + F_3 q)Q}{F_4 Q - G_4}\right) \tag{58}$$

Then letting

$$\Delta(x, y, z, q) = F_4 G_2 - F_2 G_4 + q(F_4 G_3 - F_3 G_4) = \frac{F_4 G_3 - F_3 G_4}{\delta} \tag{59}$$

we get:

$$F_2 + F_3 q + F_4 t = \frac{\Delta}{F_4 Q - G_4} \tag{60}$$

$$\frac{\partial y}{\partial Y} = 1 \Big/ \frac{\partial Y}{\partial y} = \frac{F_4 Q - G_4}{\Delta} \tag{61}$$

$$\frac{\partial z}{\partial Y} = \frac{\partial z}{\partial y} \Big/ \frac{\partial Y}{\partial y} = \frac{(F_4 Q - G_4)q}{\Delta} \tag{62}$$

$$\frac{\partial q}{\partial Y} = \frac{\partial q}{\partial y} \Big/ \frac{\partial Y}{\partial y} = \frac{(F_4 Q - G_4)t}{\Delta} = \frac{G_2 + G_3 q - (F_2 + F_3 q)Q}{\Delta} \tag{63}$$

Unless the Jacobian

$$J = \frac{\partial(Y, Z, P, Q)}{\partial(y, z, q, t)} \tag{64}$$

vanishes identically, the five equations (51), (55), (56) can be solved to express x, y, z, q, t each as a function of X, Y, Z, P, Q. Then the transformed differential equation (that results from (49) by the transformation (51)) can be obtained by taking the partial derivative with respect to Y in (58), using the expressions (61) — (63) for the partial derivatives of y, z, and q with respect to Y, and finally replacing x by X and y, z, q by the expressions just found for them as functions of X, Y, Z, P, Q.

The transformed differential equation will evidently be of the second order and in fact will have the form

$$S = D(X, Y, Z, P, Q) + E(X, Y, Z, P, Q)T \tag{65}$$

It will be possible to iterate the transformation, i.e., to apply the generalized Laplace's transformation again to (65), only if D and E are both of the first degree in P. But if there exist solutions of (49) that are of the form (1), it must be possible to iterate the transformation often enough to find them in accordance with theorem 2.3 below.

By a straightforward computation, expanding the determinant and making use of (54), we find the following expression for the Jacobian, in which λ and Δ are of course the λ and Δ of equations (19) and (59) respectively:

$$J = \frac{\lambda \Delta^3}{(F_2 + F_3 q + F_4 t)^3} \tag{66}$$

That neither Δ nor $F_2 + F_3 q + F_4 t$ can vanish identically follows from the condition that $x =$ constant, $F =$ constant, $G =$ constant are independent integrals of (53). Hence J vanishes identically if and only if λ vanishes identically.

If J, or λ, vanishes identically, a relation of the form

$$f(X, Y, Z, P, Q) = 0 \tag{67}$$

must hold as an identity in x, y, z, q, t. We may regard (67) as a first-order differential equation to be solved for Z; and if in its general solution we replace X by x, Y by $F(x, y, z, q)$, and Z by $G(x, y, z, q)$, in accordance with (51), the result will be an intermediate integral of (49). We rely on the methods of Lagrange and Charpit, not so much as a means of finding an expression of the general solution of (67) in particular cases, but for a proof of the existence of the general solution of (67), in one or other of two forms involving an arbitrary function, according to whether or not (67) is of the first degree in P and Q. And hence we have:

2.1. *If λ vanishes identically, the differential equation* (49) *has a first-order intermediate integral which involves an arbitrary function ψ and which has either the form*

$$H(x, y, z, q, \psi(\iota(x, y, z, q))) = 0 \tag{68}$$

or the parametric form:

$$H(x, y, z, q, u, \psi(u)) = 0 \tag{69}$$

$$H_5(x, y, z, q, u, \psi(u)) + \psi'(u) H_6(x, y, z, q, u, \psi(u)) = 0 \tag{70}$$

(In the case of the parametric form we must add separately the intermediate integral

$$H(x, y, z, q, a, b) = 0 \tag{71}$$

obtained from the complete integral of (67), *a and b being arbitrary constants and H the same function as in* (69).)

The converse of 2.1 is:

2.2. *If the differential equation* (49) *has a first-order intermediate integral which involves an arbitrary function* ψ *and has one of the forms* (68) *or* (69) — (70), *then* λ *vanishes identically.*

For the proof of 2.2 we notice that, for any particular choice of ψ, the general solution of the intermediate integral (68) or (69) — (70) will ordinarily be of the form (30), so that 1.4 applies. This fails only for those choices of ψ for which (68) or the equation obtained by eliminating u between (69) and (70) is independent of q, and we shall exclude such choices of ψ simply as «exceptional».

If the intermediate integral is (68), the exceptional ψ's are only those, if any, which satisfy $H_4 + \iota_4 H_5 \psi' = 0$ identically in x , y , z , q. And the possibility that H_4 and $\iota_4 H_5$ might be both identically 0 independently of ψ is excluded by the hypothesis that (68) is of first order. If the intermediate integral is (69) — (70), the exceptional ψ's are only those, if any, which satisfy $H_4 = 0$, identically in x , y , z , q when u is treated as a function of x , y , z , q determined by (70). And in this case the possibility that H_4 is identically 0, independently of ψ, is again excluded by the hypothesis that the intermediate integral is of fiist order.

Thus the exceptional ψ's can be at most only those which satisfy a fixed first-order differential equation.

Now by 1.4, if there are no singularities of μ , ν , and their first partial derivatives (other than μ_1), and if λ is not identically 0, the relation $\lambda(x , y , z , q , t) = 0$ holds for all solutions of the intermediate integral for all non-exceptional ψ. That λ is not identically 0 implies that J is not identically 0. Hence the expressions for x , y , z , q , t as functions of X , Y , Z , P , Q exist, and we may use them to express $\lambda(x , y , z , q , t)$ as a function of X , Y , Z , P , Q. Thus we get an equation

$$\lambda(x , y , z , q , t) = \Lambda(X , Y , Z , P , Q) \tag{72}$$

Where λ does not vanish it follows that Λ does not vanish. Hence Λ does not vanish identically. Also J can be expressed as a function of X , Y , Z , P , Q. For any solution of the transformed differential equation (65) that does not correspond to a zero of Λ or a zero or singularity of J we can argue from (72) that λ is non-vanishing for the corresponding solution of (49), and hence that the solution must be one that is obtained from an exceptional ψ. The solutions of (49) arising from exceptional ψ's can evidently be covered by an equation

$$v(x , y , z , c) = 0 \tag{73}$$

obtained from either (68) or (69) — (70) and involving at most one arbitrary constant c. From this by eliminating c we get a first-order differential equation, involving say x , y , z , q , which can be reexpressed in terms of X , Y , Z , P , Q:

$$\Upsilon (X , Y , Z , P , Q) = 0 \tag{74}$$

In a suitably restricted neighborhood within which both Λ and J (as functions of the five variables X , Y , Z , P , Q) are non-zero and non-singular, the solutions of the definitely second-order differential equation (65) are thus included among those of the fixed first-order differential equation (74) — which is impossible.

In the case in which there exist solutions of (49) of the form (30) corresponding to singularities of μ , ν , and their first partial derivatives, the above argument may be modified as follows. In addition to the relation $\lambda(x , y , z , q , t) = 0$ we have also a number of relations

$$\lambda_{(i)}(x , y , z , q , t) = 0 \tag{75}$$

which represent the relevant singularities of μ , ν , and their first partial derivatives. These may be reexpressed in terms of X , Y , Z , P , Q as

$$\Lambda_{(i)}(X , Y , Z , P , Q) = 0 \tag{76}$$

And we then consider a neighborhood within which $\Lambda , \Lambda_{(i)} , J$ are non-zero and non-singular.

Now returning to the point which was made above in connection with equation (50), that the transformation (51) can be expected to transform a solution of the differential equation (49) of the form (1) into a solution which has the same form with the number n decreased by at least 1, we remark that our proof of this is not yet conclusive, because of the possibility that equation (46) may not include quite all of the solutions of (45). Indeed one class of solutions of (45) which is definitely not included in (46) is given by the equation

$$F(x , y , A , \theta) = \Omega(x , \varphi , \varphi' , \varphi'' , \dots , \varphi^{(n-1)}) \tag{77}$$

where Ω is an arbitrary function. And correspondingly equation (50) must be supplemented by the equation

$$F(x , y , A , A_2) = \Omega(x , \varphi, \varphi' , \varphi'' , \dots , \varphi^{(n-1)}) \tag{78}$$

To repair the defect we proceed by finding the following Jacobian, where F stands for $F(x , y , A , A_2)$ and G stands for $G(x , y , A , A_2)$:

$$\frac{\partial(F\,,G)}{\partial(y\,,\varphi^{(n)})} = (F_2 + F_3A_2 + F_4A_{22})(G_3A_{n+3} + G_4A_{2(n+3)})$$
$$- (G_2 + G_3A_2 + G_4A_{22})(F_3A_{n+3} + F_4A_{2(n+3)})$$
$$= [F_2G_3 - F_3G_2 + (F_4G_3 - F_3G_4)A_{22}]A_{n+3}$$
$$- [F_4G_2 - F_2G_4 + (F_4G_3 - F_3G_4)A_2]A_{2(n+3)}$$
$$= [(\beta(x\,,y\,,A\,,A_2) + \delta(x\,,y\,,A\,,A_2)A_{22})A_{n+3}$$
$$- A_{2(n+3)}]\Delta(x\,,y\,,A\,,A_2)$$

In this the factor $(\beta + \delta A_{22})A_{n+3} - A_{2(n+3)}$ vanishes by (23); and the Jacobian $\partial(F\,,G)/\partial(y\,,\varphi^{(n)})$ therefore vanishes, with a possible exception if Δ is singular for $z = A\,, q = A_2$. With this exception we do have, as a consequence of the vanishing of the Jacobian, that either equation (50) must hold for some function Φ or equation (78) must hold for some function Ω.

The exceptional case in which (78) holds is the case in which $F(x\,,y\,,A\,,A_2)$ is independent of y, and hence, taking $x\,,y\,,z\,,q$ as the arguments of F, we may describe it also as the case in which $F_2 + F_3q + F_4t$ vanishes for $z = A$.

The foregoing argument applies as well to the case $n = 0$, i.e., the case of a solution of the form (30), as it does to larger values of n. Thus if the transformation (51) is applied to a differential equation (49) that has a solution $z = A(x\,,y\,,\varphi(x))$, and if Δ is not singular and $F_2 + F_3q + F_4t$ does not vanish for $z = A(x\,,y\,,\varphi(x))$, the corresponding solution of the transformed differential equation has the form $Z = \Phi(X\,,Y)$, no longer depending on an arbitrary function $\varphi(x)$ (or $\varphi(X)$). Generally, and within an appropriately restricted region, the transformation (51) effects a one-to-one correspondence between particular solutions of (49) and particular solutions of the transformed differential equation, as is clear from the fact which we found above, that there exist not only the equations (51) expressing $X\,,Y\,,Z$ as functions of $x\,,y\,,z\,,q$ but also inverse equations expressing $x\,,y\,,z$ as functions of $X\,,Y\,,Z\,,P\,,Q$. If this fails in a particular case, as in the reduction of a solution $z = A(x\,,y\,,\varphi(x))$ to a solution not involving an arbitrary function, it follows that the Jacobian J has a zero or a singularity.

This suggests as a means of finding solutions of the differential equation (49) of the form (1) that we use iterated application of the transformation (51), examining at each stage the zeros and singularities of the Jacobian. However, we must take into account the exceptional cases that we found, that Δ is singular and that $F_2 + F_3q + F_4t$ vanishes. (For example if we transform $s = zt^3$ by $X = x$, $Y = q$, $Z = y$, we find $J = 1$, and the obvious solution $z = cy + \varphi(x)$ corresponds not to any zero or singularity of J but to a zero of $F_2 + F_3q + F_4t$.) Hence in view of the expression (66) for J it will be better not to use J itself at all but to examine at each stage the zeros and singularities of λ, Δ, and $F_2 + F_3q + F_4t$ separately.

To find all the solutions of the form (30) at each stage we might as an alternative, by 1.4, just examine at each stage the zeros of λ and the singularities of μ, ν, and their partial derivatives. But this will not do as a general procedure because there is a possibility that a transformation (51) of a differential equation (49) might transform away not only all the solutions of the form (30) but also a solution of the form (1) with $n = 1$. To illustrate this we may cite the differential equation

$$2zs = -2q + (1 - y)t + 4pq$$

which has a solution

$$z = \frac{x + \varphi(x)}{y + \varphi'(x)}$$

This solution is transformed away — there is in fact no corresponding solution at all of the transformed differential equation — if we apply the transformation $X = x$, $Y = z^2/q$, $Z = y$. To be sure this accident might have been avoided if we had made a different choice of F and G in applying the transformation (51); but this does not destroy the force of the example.

Hence we state our result in the following form:

2.3. *In order to find all solutions of the differential equation* (49) *that are of the form* (1), *for all* n *not greater than a fixed* n_0, *it is sufficient to proceed as follows. Apply the transformation* (51) *to the given differential equation* (49), *either* n_0 *times in succession if this is possible, or else until no further iteration of the transformation is possible* (*either because* λ *vanishes identically or because the final transformed differential equation is no longer of the form* (49)). *At each stage in the iteration of the transformation examine the zeros and singularities of* λ, Δ, *and* $F_2 + F_2q + F_4t$, *in the sense of writing the differential equations which represent these zeros and singularities and determining the solutions which they have in common with the differential equation which, at that stage, has been obtained by transformation of the originally given differential equation* (49). *Then examine the final transformed differential equation which has resulted at the end of the process of iterated application of* (51), *in the following way. If it is of the form* (49) *and* λ *vanishes identically, solve the corresponding differential equation* (67) *to find an intermediate integral. If it is of the form* (49) *and* λ *does not vanish identically, make use of* 1.4 *to find its solutions of the form* (30). *If it is not of the form* (49), *make use of* 1.3 *to find its solutions of the form* (1).

This theorem solves the problem of finding all solutions of a given differential equation (49) that are of the form (1), for all n not greater than a given n_0, in the sense that the problem is reduced to the problem of solving certain ordinary (as opposed to partial) differential equations. As indicated above our interest is primarily in existence questions rather than in practical solution processes. And the advantage of ordinary differential equations from this point of view is that the elementary solutions (elementary in the sense of the present paper) are known to exist. Nevertheless the method of theorem 2.3 does work out as a practical solution process in some cases.

Parenthetically it should be added that the known existence of elementary solutions of ordinary differential equations applies only to a fixed differential equation, not to an equation-form containing an arbitrary function. For this reason, even if a partial differential equation has an intermediate integral of the form (68) or the form (69) — (70), and hence has almost all its solutions of the elementary form (30), it does not follow that it therefore necessarily has a two-arbitrary-function elementary solution.

In the special case in which the differential equation (49) is linear it is true that if any solution of the form (1) exists, it must be possible to find a two-arbitrary-function elementary solution by iterations of the transformation (51) — which in this case of course reduces to Laplace's transformation.

In connection with 2.3, it is believed that substantial auxiliary theorems can be found by means of 1.5. In particular, in at least the case in which ν is independent of t, the equations obtained by setting (40) and (41) equal to 0 can be solved explicitly when regarded as simultaneous differential equations for μ and ν; and this results in great simplification in the application of 2.3 to a particular differential equation in this case. But this is a topic which we leave to a possible future paper.

3. Generalization of Laplace's transformation, second case

Now we return to (43), to deal with the case in which $\nu(x, y, z, q, t)$ does not have the form (44). In this case (43) is a partial differential equation of first order and not of the first degree, and we rely on Charpit's method for the existence of a complete integral. With an appropriate understanding as to what is meant, we may rewrite (43) as

$$(77) \qquad \frac{\partial q}{\partial z} = \nu\left(x, y, z, q, \frac{\partial q}{\partial y} + q\,\frac{\partial q}{\partial z}\right)$$

Let a complete integral be

$$G(x\,,y\,,z\,,q\,,c_1) = c \tag{78}$$

The complete integral of (43) as originally written is then

$$G(x\,,y\,,A\,,\theta\,,c_1) = c \tag{79}$$

And the corresponding general solution is

$$G(x\,,y\,,A\,,\theta\,,u) = \Phi(x\,,u\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)}) \tag{80}$$

$$G_5(x\,,y\,,A\,,\theta\,,u) = \Phi_2(x\,,u\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)}) \tag{81}$$

We therefore expect that a solution of the form (1), of the differential equation (18), will satisfy one or other of the two following conditions. Either there exist functions E_0 and Φ_0 such that

$$G(x\,,y\,,A\,,A_2\,,E_0(x\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)})) = \Phi_0(x\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)}) \tag{82}$$

or there exist functions E and Φ such that

$$G(x\,,y\,,A\,,A_2\,,E(x\,,y\,,A\,,A_2\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)})) = \Phi(x\,,E(x\,,y\,,A\,,A_2\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)})\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)}) \tag{83}$$

$$G_5(x\,,y\,,A\,,A_2\,,E(x\,,y\,,A\,,A_2\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)})) = \Phi_2(x\,,E(x\,,y\,,A\,,A_2\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)})\,,\varphi\,,\varphi'\,,\varphi''\,,\ldots\,,\varphi^{(n-1)}) \tag{84}$$

In either case, by considering x and y as the independent variables, and taking the partial derivative with respect y, we get

$$G_2 + G_3A_2 + G_4A_{22} = 0 \tag{85}$$

Then by solving (85) for E or E_0 we get

$$E = F(x\,,y\,,A\,,A_2\,,A_{22}) \quad \text{or} \quad E_0 = F(x\,,y\,,A\,,A_2\,,A_{22}) \tag{86}$$

(the same function F in either case). This suggests that if we make a transformation of the differential equation (18) by letting

$$X = x\,,\quad Y = F(x\,,y\,,z\,,q\,,t)\,,\quad Z = G(x\,,y\,,z\,,q\,,F(x\,,y\,,z\,,q\,,t)) \tag{87}$$

the solution of the transformed differential equation which corresponds to the solution $z = A$ of (18) will be

$$Z = \Phi_0(X\,,\varphi(X)\,,\varphi'(X)\,,\varphi''(X)\,,\ldots\,,\varphi^{(n-1)}(X)) \tag{88}$$

or

$$Z = \Phi(X\,,Y\,,\varphi(X)\,,\varphi'(X)\,,\varphi''(X)\,,\ldots\,,\varphi^{(n-1)}(X)) \tag{89}$$

That is, the effect of the transformation will be to replace any solution of (18) of the form (1) by a solution which, in terms of the new variables X, Y, Z, has the same form with the number n decreased by at least 1.

To find the transformation (87) when the differential equation (18) is given, we must find a complete integral (78) of the differential equation (77). This supplies the function G. And $F(x, y, z, q, t)$ is then found by solving

$$G_2(x, y, z, q, F) + qG_3(x, y, z, q, F) + tG_4(x, y, z, q, F) = 0 \tag{90}$$

The condition that (78) shall be a complete integral of (77) assures that G_4 is not identically 0 and that $(G_2 + qG_3)/G_4$ is not independent of F. Hence (90) can be solved for F, and F so obtained will not be independent of t.

The condition that (78) is an integral of (77) can be expressed by the equation

$$G_3 + G_4\nu\left(x, y, z, q, -\frac{G_2 + qG_3}{G_4}\right) = 0 \tag{91}$$

where G stands for $G(x, y, z, q, c_1)$. Hence in consequence of (90) we have the equation

$$\begin{gathered} G_3(x, y, z, q, F(x, y, z, q, t)) + \\ G_4(x, y, z, q, F(x, y, z, q, t))\nu(x, y, z, q, t) = 0 \end{gathered} \tag{92}$$

By a computation analogous to that by which equations (55) and (56) were found, and using equations (90) and (91), we find

$$P = G_1 + G_4\mu, \quad Q = G_5 \tag{93}$$

where of course the arguments of G are x, y, z, q, F. Then unless the Jacobian

$$J = \frac{\partial(Y, Z, P, Q)}{\partial(y, z, q, t)} \tag{94}$$

vanishes identically, the five equations (87), (93) can be solved to express x, y, z, q, t each as a function of X, Y, Z, P, Q. Then the transformed differential equation (that results from (18) by the transformation (87)) can be obtained by taking the partial derivative with respect to y in equations (87), (93), and eliminating $\partial Y/\partial y$ and $\partial t/\partial y$. The result is

$$\begin{aligned} S = G_{15} + \mu G_{45} + \mu_5 \frac{G_4}{F_5} + (G_{55} - T)[F_1 + \mu F_4 + \mu\nu F_5 \\ + F_5(\mu_2 + \mu_3 q + \mu_4 t) - \mu_5(F_2 + F_3 q + F_4 t)] \end{aligned} \tag{95}$$

wherein x, y, z, q, t are to be replaced by the expressions for them as functions of X, Y, Z, P, Q.

By expanding the determinant and making use of (90) and (91) we find for J the expression

(96) $$J = -\lambda G_4^3$$

Hence J vanishes identically if and only if λ vanishes identically.

If J, or λ, vanishes identically, a relation

(97) $$f(X, Y, Z, P, Q) = 0$$

must hold identically in x, y, z, q, t. If we solve (97) as a first-order differential equation in X, Y, Z as the variables, and if in the result we replace X by x, Y by $F(x, y, z, q, t)$, and Z by $G(x, y, z, q, F(x, y, z, q, t))$, in accordance with (87), the result will be a *second-order intermediate integral* of (18). We understand the italicized phrase as implying that the general solution of (18) is to be found by considering the common solutions of (18) and the intermediate integral. However, if we consider also the equations in x, y, z, q, t which result from any extra solutions (special or singular) that (97) may possess, we can say more strongly that the disjunction of these equations and the second-order intermediate integral is a consequence of the differential equation (18).

3.1. *If* ν *does not have the form* (44), *and if* λ *vanishes identically, the differential equation* (18) *has a second-order intermediate integral which involves an arbitrary function* ψ *and which has either the form*

(98) $$H(x, y, z, q, t, \psi(\iota(x, y, z, q, t))) = 0$$

or the parametric form:

(99) $$H(x, y, z, q, t, u, \psi(u)) = 0$$

(100) $$H_6(x, y, z, q, t, u, \psi(u)) + \psi'(u) H_7(x, y, z, q, t, u, \psi(u)) = 0$$

(*In the case of the parametric form we must add separately the intermediate integral*

(101) $$H(x, y, z, q, t, a, b) = 0$$

obtained from the complete integral of (97), a *and* b *being arbitrary constants and* H *the same function as in* (99).)

Now returning to the differential equation (77), we observe that the argument which led us from this equation to the transformation (87), although it was presented heuristically, is as a matter of fact substantially complete. Any solutions of (77) that are not covered by the complete integral (78) and the corresponding general integral can only be one or more singular integrals. These will be equations that may involve x, y, z, q but do not involve any arbitrary constants or arbitrary functions, and therefore they can lead only to solutions of (18) of the form (30). The conclusion above — that solutions of (18) of the form (1) are transformed by (87) to solutions of the same form with the number n decreased by at least 1 — can therefore be escaped only at most by some exceptional solutions of the form (30). And in consequence we have:

3.2. *In the case of a differential equation which is of the form* (18), *if* v *is not of the form* (44), *it is sufficient to proceed as follows, in order to find the solutions of the form* (1). *If* λ *vanishes identically, find a second-order intermediate integral by solving the differential equation* (97). *Otherwise use* 1.4 *to find the solutions of the form* (30), *and then apply the transformation* (87). *The zeros and singularities of the Jacobian* J *as given by* (96) *must also be examined for possible additional solutions that might be transformed away. The transformed differential equation is then of a form to which either* 1.3 *or* 2.3 *will be applicable to find the solutions of the form* (1).

4. Quasi-substitutions

We turn now to a class of transformations which are closely associated with the generalized Laplace's transformations and which we shall call quasi-substitutions.

Some of these are obtained as resultants of a transformation (51) and the inverse of a transformation (51). For example, if the differential equation

$$qs = q^3 - pt$$

is transformed by letting

$$\xi = x, \quad \eta = q, \quad \zeta = z - 2yq$$

the transformed differential equation is

$$\sigma + \eta^2 \tau = 0$$

If on the other hand the linear differential equation

$$S + Y^2 T - 2YQ + 2Z = 0$$

is transformed by letting

$$\xi = X, \quad \eta = Y, \quad \zeta = Q$$

the result is the same differential equation $\sigma + \eta^2\tau = 0$. Both transformations are instances of (51). The resultant of the first and the inverse of the second is represented by the equations

$$X = x\,, \quad Y = q\,, \quad Z = zq - yq^2$$

And this last transformation is an example of a quasi-substitution.

Generally, a quasi-substitution is represented by a triple of equations, expressing the new variables X, Y, Z as functions of x, y, z, p, q. And from these equations alone, independently of any particular differential equation to which the transformation is applied, there follow equations expressing P and Q as functions of x, y, z, p, q, and equations expressing x, y, z, p, q as functions of X, Y, Z, P, Q. Thus the quasi-substitutions share with ordinary substitutions the important property that they may be applied to any arbitrary differential equation of first or higher order, without raising the order of the differential equation. This contrasts with the situation in regard to Laplace's transformation, and its present generalizations, that a particular transformation, represented by a particular triple of equations, is adapted to a particular differential equation and in general cannot be applied to a different differential equation without raising the order, and without destroying the feature that inverse equations exist expressing x, y, z, p, q as functions of X, Y, Z, P, Q.

(In some cases it may happen, as in the example given above, that a quasi-substitution is also a birational transformation in the five-dimensional space whose coordinates are x, y, z, p, q.)

One class of quasi-substitutions is represented by equations of the form

$$X = x\,, \quad Y = F(x, y, z, q)\,, \quad Z = G(x, y, z, F(x, y, z, q)) \tag{102}$$

where F is to be found from the equation

$$G_2(x, y, z, F) + qG_3(x, y, z, F) = 0 \tag{103}$$

and where G may be chosen arbitrarily, subject to the condition that (103) is a non-trivial equation which has a solution for F, and that F so found is not independent of q. As a consequence of (102) and (103) there follow the equations

$$P = G_1(x, y, z, F) + pG_3(x, y, z, F) \tag{104}$$

$$Q = G_4(x, y, z, F) \tag{105}$$

The resultant of a transformation (102) and an ordinary substitution is of course also a quasi-substitution. These are the only quasi-substitutions that are applicable to ordinary differential equations — in the sense that when they are applied to a partial differential equation that has the property that all derivatives occurring are derivatives with respect to y, the transformed differential equation has the same property.

It can be shown that, besides the foregoing, the only remaining quasi-substitutions are those represented by equations of the form

$$X = E(x, y, z, p, q), \quad Y = F(x, y, z, p, q), \quad Z = G(x, y, z, E, F) \tag{106}$$

where E and F are to be found from the equations

$$G_1(x, y, z, E, F) + pG_3(x, y, z, E, F) = 0 \tag{107}$$

$$G_2(x, y, z, E, F) + qG_3(x, y, z, E, F) = 0 \tag{108}$$

and where G may be chosen arbitrarily, subject to the condition that (107) and (108), regarded as equations to be solved for E and F, are independent, and that E and F as found from (107) and (108) are such that $E_4F_5 - E_5F_4$ does not vanish identically. From (106) — (108) it follows that

$$P = G_4(x, y, z, E, F), \quad Q = G_5(x, y, z, E, F) \tag{109}$$

In order to find a quasi-substitution (106) when the function F is given, not independent of both of its last two arguments, we may treat

$$F\left(x, y, z, -\frac{G_1(x, y, z, c_1, c)}{G_3(x, y, z, c_1, c)}, -\frac{G_2(x, y, z, c_1, c)}{G_3(x, y, z, c_1, c)}\right) = c \tag{110}$$

as a partial differential equation to be solved for G. We must find a solution of (110) such that G_1/G_3 and G_2/G_3 are not both independent of c_1, and we may then expect that (107) and (108) will be compatible and will have a common solution for E.

To find a quasi-substitution (102) when the function F is given, not independent of its last argument, we may express q as a function of x, y, z, F, and then treat (103) as a partial differential equation to be solved for G.

The significance of quasi-substitutions in the present context is that, when the methods of the preceding sections lead to an intermediate integral, it may often happen that the argument of the arbitrary function involves derivatives. In this case the best way to treat the intermediate integral may be to use a quasi-substitution to reduce the argument of the arbitrary function to Y. We may expect to be able to do this whenever the argument of the arbitrary function is a function of x, y, z, p, q. And it may sometimes be possible even when derivatives of higher order are involved.

Some other applications of quasi-substitutions suggest themselves, which we must leave unexplored in the present paper.

One of these is that it may happen that a given differential equation can be transformed to one that is linear by means of either a generalized Laplace's transformation or a quasi-substitution, even when this is not possible by an ordinary substitution. The example cited at the beginning of this section illustrates this point. In this example the linear differential equation obtained is one that has an elementary general solution. But evidently it may also be that the linear differential equation is not elementarily solvable. And even in this case it seems that the treatment of the given differential equation might be facilitated by the knowledge that it is in a certain sense essentially linear.

Another possible application is to the treatment of the Monge-Ampère equations. It is not immediately clear how substantial this may be. But one example is included in the next section as an indication.

5. Examples

We add some further examples illustrating various points in connection with the preceding sections.

(Ex. 1) $$(x + y)s + p^3t - p + x^2q = 0$$

Theorem 1.3 is applicable, and f_{44} is $-6pt/(x + y)$. To make this 0 identically in p we must have $t = 0$ and hence $z = y\psi(x) + \chi(x)$. Substituting this in the differential equation yields the condition $\chi'(x) = x\psi'(x) + x^2\psi(x)$. Thus we have what may be described as a solution of the form (2), with the condition not satisfied that there is no relation of the form (3). It may, however be reexpressed as a solution of the form (1) by letting

$$\psi(x) = \frac{\varphi'(x)}{x^2 - 1}$$

We then have

$$z = \frac{(x + y)\varphi'(x)}{x^2 - 1} + \varphi(x)$$

This is the only solution of the differential equation of the form (1). That it is quite special compared to the general solution is indicated by the fact that we would still get the same solution if we replaced the term p^3t in the differential equation by $tf(p)$, f being chosen as any function not of the first degree or 0 degree.

(Ex. 2) $$s + yqt = 0$$

This has the intermediate integral

$$- xq + \log y = \log \psi(q)$$

By the quasi-substitution

$$X = x\,, \quad Y = q\,, \quad Z = z - yq$$

this becomes

$$Q = - e^{XY} \psi(Y)$$

By integration of this with respect to Y

$$Z = - \int e^{XY} \psi(Y) dY + \varphi(X)$$

Hence the general solution of (Ex. 2) may be written in the following parametric form, with q as the parameter:

$$y = e^{xq} \psi(q)$$

$$z = qe^{xq} \psi(q) - \int e^{xq} \psi(q) dq + \varphi(x)$$

To this we must add separately the solution

$$z = cy + \varphi(x)$$

which was lost by the form in which the intermediate integral was written above.

In this case more familiar methods would suffice for the integration of the intermediate integral, but there is an advantage of uniformity in always using a quasi-substitution when the argument of the arbitrary function contains p or q.

(Ex. 3) $$s = (xp + yq - z)t$$

By the same quasi-substitution as was used for (Ex. 2) we get

$$S = XP - Z$$

This can be solved by Laplace's transformation, with the result

$$Z = XI(X\,, Y) - I_2(X\,, Y) + X\psi(Y) - \psi'(Y)$$

where $I(X\,, Y) = \int e^{XY} \varphi(X) dX$. Hence we get for (Ex. 3) the parametric solution:

$$y = - xI_2(x\,, q) + I_{22}(x\,, q) - x\psi'(q) + \psi''(q)$$

$$z = xI(x\,, q) - (xq + 1)I_2(x\,, q) + qI_{22}(x\,, q)$$
$$+ x\psi(q) - (xq + 1)\psi'(q) + q\psi''(q)$$

To this we must add separately the solution

$$z = cy + \varphi(x)$$

which was lost by the quasi-substitution.

In the former solution we may of course take $\varphi(x)$ to be 0 and so find for (Ex. 3) the more special solution

$$y = - x\psi'(q) + \psi''(q)$$
$$z = x\psi(q) - (xq + 1)\psi'(q) + q\psi''(q)$$

We shall think of this last as a solution of a form that is analogous to the form (1) except with q instead of x as argument of the arbitrary function. For although we must regard q as an unspecified parameter in order to give the pair of equations as solution of the differential equation, it will then follow from these equations that q is in fact $\partial z/\partial y$.

(Ex. 4) $$qs = q^3 - pt$$

Following the method of 2.3 we get from $\lambda = 0$ the solution

$$z = - \frac{y}{x + c} + \varphi(x)$$

Then the remaining solutions are found by the transformation $\xi = x$, $\eta = q$, $\zeta = z - 2yq$, which, as we have already seen in section 4, leads to $\sigma + \eta^2\tau = 0$. For this we get the intermediate integral

$$\varrho = \psi\left(\xi + \frac{1}{\eta}\right)$$

and hence the solution

$$\zeta = \int \psi\left(\xi + \frac{1}{\eta}\right) d\eta + \varphi(\xi)$$

Thus as parametric solution for (Ex. 4) we have:

$$y = - \frac{1}{2q^2} I_1(x, q) - \frac{1}{2} \psi\left(x + \frac{1}{q}\right) - \frac{1}{2q^2} \varphi'(x)$$

$$z = I(x, q) - \frac{1}{q} I_1(x, q) - q\psi\left(x + \frac{1}{q}\right) + \varphi(x) - \frac{1}{q} \varphi'(x)$$

where $I(x, q) = \int \psi\left(x + \frac{1}{q}\right) dq$.

(Ex. 5) $$rt - s^2 = \frac{q^2r - s}{2(z - xp)}$$

If we use the quasi-substitution

$$X = p\,,\quad Y = q\,,\quad Z = z - xp - yq$$

we get the transformed differential equation

$$S + Y^2T - 2YQ + 2Z = 0$$

This is a linear differential equation for which, as we have already seen in section 4, Laplace's transformation is $\xi = X$, $\eta = Y$, $\zeta = Q$, reducing it to $\sigma + \eta^2\tau = 0$. Using for this last the solution which was just obtained above, we find

$$Z = YI(X\,,\,Y) - \frac{1}{2}\,I_1(X\,,\,Y) - \frac{1}{2}\,Y^2\psi\left(X + \frac{1}{Y}\right) + Y\varphi(X) - \frac{1}{2}\,\varphi'(X)$$

where $I(X\,,\,Y) = \int \psi\left(X + \frac{1}{Y}\right) dY$. Hence for (Ex. 5) we find the parametric solution:

$$x = -\,qI_1(p\,,\,q) + \frac{1}{2}\,I_{11}(p\,,\,q) + \frac{1}{2}\,q^2\psi'\left(p + \frac{1}{q}\right) - q\varphi'(p) + \frac{1}{2}\,\varphi''(p)$$

$$y = -\,I(p\,,\,q) - \varphi(p)$$

$$z = -\left(pq + \frac{1}{2}\right)I_1(p\,,\,q) + \frac{1}{2}\,pI_{11}(p\,,\,q) - \frac{1}{2}\,q^2\psi\left(p + \frac{1}{q}\right) + \frac{1}{2}\,pq^2\psi'\left(p + \frac{1}{q}\right) - \left(pq + \frac{1}{2}\right)\varphi'(p) + \frac{1}{2}\,p\varphi''(p)$$

In addition to this parametric solution we must also look into the question of solutions satisfying either $q = f(p)$ or $p =$ constant, as there is a possibility that such solutions might be lost by the particular quasi-substitution which was used. In fact we find in this way $(c - p)q = 1$, $q = 0$, $p = c$. These first-order differential equations are to be solved by familiar methods, and the resulting solutions must be adjoined to the parametric solution above, for the full general solution of (Ex. 5).

(Ex. 6) $$s^2 = a^2p^2t$$

To avoid carrying the double sign, let a stand ambiguously for either one of the two square roots of a^2, and so write the differential equation as

$$s = apt^{\frac{1}{2}}$$

Following the method of 3.2 we find

$$G(x\,,y\,,z\,,q\,,c_1) = \log\,(q + c_1) - a^2(z + c_1 y)$$

$$F(x\,,y\,,z\,,q\,,t) = a^{-1}\,t^{\frac{1}{2}} - q$$

From (93) we find $P = 0$. Hence λ must vanish identically (as is also easily verified directly). The equation $P = 0$ is the equation (97). From it we get $Z = \Phi(Y)$, and hence the second-order intermediate integral is

$$\frac{1}{2}\log t - ayt^{\frac{1}{2}} - a^2(z - yq) = \Phi(a^{-1}\,t^{\frac{1}{2}} - q) + \log a$$

In order to solve this without first specializing the arbitrary function Φ we proceed by taking the partial derivative with respect to y, obtaining

$$\left[\frac{1}{2}\,a^{-1}\,t^{-\frac{1}{2}}\,\frac{\partial t}{\partial y} - t\right]\left[at^{-\frac{1}{2}} - a^2 y - \Phi'(a^{-1}\,t^{\frac{1}{2}} - q)\right] = 0$$

From this the differential equation obtained by setting the first factor equal to 0 is easily solved, and its common solutions with (Ex. 6) are found to be:

$$z = -\,a^{-2}\log\,(y + \varphi(x)) + by + c$$

$$z = -\,a^{-2}\log\,(y + b) + (y + b)\varphi(x) + c$$

$$z = by + \varphi(x)$$

Evidently the general solution is to be obtained from the second factor in the equation above. The differential equation obtained by setting it equal to 0 may be rewritten as

$$a^{-1}\,t^{\frac{1}{2}} - q = \Omega(a^{-1}\,t^{-\frac{1}{2}} - y)$$

(since no new solutions of (Ex. 6) are obtained by taking Φ' equal to a constant). Then we treat q as the dependent variable and on this basis make the quasi-substitution

$$\xi = x\,,\quad \eta = a^{-1}\,t^{-\frac{1}{2}} - y\,,\quad \zeta = a^{-1}\,t^{\frac{1}{2}} + q$$

This yields

$$4\varrho = -\,a^2[\zeta + \Omega(\eta)]^2$$

If we let

$$\Omega(\eta) = a^{-2}\int\left[\frac{\omega''(\eta)}{\omega'(\eta)}\right]^2 d\eta - 2a^{-2}\,\frac{\omega''(\eta)}{\omega'(\eta)}$$

we may solve this as a Riccati equation, obtaining

$$a^2\zeta + \int\left[\frac{\omega''(\eta)}{\omega'(\eta)}\right]^2 d\eta = \frac{4\omega'(\eta)}{\omega(\eta) + \varphi(\xi)}$$

Hence inverting the quasi-substitution we get the parametric equations:

$$y = \frac{\omega'(\eta)[\omega(\eta) + \varphi(x)]}{2\omega'(\eta)^2 - \omega''(\eta)[\omega(\eta) + \varphi(x)]} - \eta$$

$$q = 2a^{-2} \frac{\omega'(\eta)}{\omega(\eta) + \varphi(x)} + a^{-2} \frac{\omega''(\eta)}{\omega'(\eta)} - a^{-2} \int \left[\frac{\omega''(\eta)}{\omega'(\eta)}\right]^2 d\eta$$

We omit the remainder of the work, which is lengthy and serves no further purpose of illustration. It is necessary to find an expression for z in terms of x and η by using

$$z = \int q dy + \psi(x)$$

And as three arbitrary functions are then involved, a relation among them must be found by substituting in the differential equation, i.e., in (Ex. 6).

(Ex. 7) $$s = zq + q^2$$

(Ex. 8) $$(t + y^2)s = t^2 + 2yp - yq$$

We omit details of these last two examples. (Ex. 7) is a very simple case in which a (generalized) intermediate integral, containing an arbitrary function, results by two successive applications of a generalized Laplace's transformation. (Ex. 8) has only a one-arbitrary-function elementary solution, obtained by one application of a generalized Laplace's transformation.

6. Conclusion

The writer has urged elsewhere (*Remarks on the elementary theory of differential equations as area of research*, in **Information and prediction in science,** New York, 1965) the problem of characterizing the class of partial differential equations that have elementary or elementary-parametric general solutions. It seems quite certain that there are negative results to be obtained, both for this problem and for the problem of elementary and elementary-parametric solutions that involve one arbitrary function, and that the negative results may be expected to begin with differential equations of the second order in two independent variables. But the present paper is directed entirely towards the positive side of these questions.

It is not immediately clear whether the methods of this paper may suffice to treat the positive aspect completely, or how far in this direction they may be expected to reach. But substantial extensions of these methods would seem to be possible, and we conclude by giving some brief indications in this regard.

First, while still keeping to the case in which the argument of the arbitrary function is x, we may seek to use the same or similar methods to treat the question of solutions of the form (2) for which a relation of the form (3) does hold. Or instead of a single relation (3) we may suppose that there are two or more such relations, i.e., relations having the following forms:

$$\varphi_n'(x) = B_{(1)}(x, \varphi(x), \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \varphi'(x), \varphi_1'(x), \varphi_2'(x), \ldots, \varphi_{n-k}'(x))$$
$$\varphi_{n-1}'(x) = B_{(2)}(x, \varphi(x), \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \varphi'(x), \varphi_1'(x), \varphi_2'(x), \ldots, \varphi_{n-k}'(x))$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\varphi_{n-k+1}'(x) = B_{(k)}(x, \varphi(x), \varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x), \varphi'(x), \varphi_1'(x), \varphi_2'(x), \ldots, \varphi_{n-k}'(x))$$

A somewhat hasty preliminary survey suggests that such an extension of the methods will indeed be possible, and that it may be expected to lead to a second kind (or kinds) of generalization of Laplace's transformation.

Then a different extension is to allow the argument of the arbitrary function, in the elementary or elementary-parametric solution, to be other than x. Here we must consider not only the case that the argument of the arbitrary function is a function of x, y, z, but also the case that the argument of the arbitrary function involves derivatives of z. (For an illustration of the latter possibility see (Ex. 3) above, where there is an example of an elementary-parametric solution in which the argument of the arbitrary function is q.)

At least if the argument of the arbitrary function involves derivatives of no higher than first order, the expectation is that a substitution or transformation can be used to reduce to the case in which the argument of the arbitrary function is x (or X). Indeed it is clear that if the argument of the arbitrary function is $\iota(x, y, z)$, the required reduction is accomplished by a substitution in which $X = \iota(x, y, z)$; and if the argument of the arbitrary function is $\iota(x, y, z, p, q)$, the required reduction is accomplished by a quasi-substitution in which $X = \iota(x, y, z, p, q)$. The problem which remains is to find the requisite substitution or quasi-substitution when only the differential equation is given, and not the argument of the arbitrary function.

A large part, but not the whole, of the problem of reducing to the case in which the argument of the arbitrary function is x (or X), is included in the problem of reducing a given differential equation, by an appropriate substitution or transformation, to a differential equation in which r (or R) is not present. The latter problem is treated in the paper by the present writer which is referred to above, but the treatment there is inadequate because it fails to take into account the existence of quasi-substitutions.

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