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AREA DISTORTION UNDER QUASICONFORMAL MAPPINGS

BY

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To Rolf Nevanlinna on his 70th birthday

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Area distortion under quasiconformal mappings[1]

1. *Introduction.* Suppose that f is a sense-preserving plane quasiconformal mapping, in the sense of Ahlfors and Pfluger, of a domain D. Then it follows from the work of Mori [10] and Morrey [11] that f preserves sets of plane measure zero in D, that is, if $E \subset D$ and $m(E) = 0$, then $m(f(E)) = 0$. (See also [6] and [12].) In connection with deeper studies of the degree of regularity of quasiconformal mappings, it is of interest to investigate to what extent one can say that $m(f(E))$ is small whenever $m(E)$ is small. Obviously this problem is meaningful only if f is normalized in some way. For example, one might require that f map the unit disk U onto itself so that $f(0) = 0$. With this normalization, Bojarski [3] made the following observation. *There exist a pair of functions of K, $a(K) > 0$ and $b(K) > 0$, such that if f is K-quasiconformal, then*

$$\frac{m(f(E))}{\pi} \leq b(K) \left(\frac{m(E)}{\pi}\right)^{a(K)} \tag{1.1}$$

for each measurable set $E \subset U$.

Bojarski derived inequality (1.1) from an important theorem on the integrability of the derivatives of quasiconformal mappings. This latter result was established by him using a fundamental inequality, due to Calderón and Zygmund [4], which relates the L_p norms of a function and its Hilbert transform. In the present paper we shall use a parametric representation[2] for quasiconformal mappings, similar to Loewner's representation for conformal mappings, to study the above mentioned problem *ab ovo* and to prove the following slightly more precise form of (1.1).

Theorem 1. *There exists a constant a and a function $b(K)$, where $1 \leq a \leq 40$, $b(K) > 0$, and $b(K) = 1 + O(K-1)$ as $K \to 1$, such that, if f is a K-quasiconformal mapping of U onto itself with $f(0) = 0$, then*

$$\frac{m(f(E))}{\pi} \leq b(K) \left(\frac{m(E)}{\pi}\right)^{K^{-a}} \tag{1.2}$$

for each measurable set $E \subset U$.

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In a recent paper [8], Lehto has studied the integrability question for plane quasiconformal mappings. Using his results, one also can show that there exist a pair of functions $a(K)$ and $b(K)$ for which (1.1) holds, where $a(K) = K^{-a}$ and a is a constant, $a \geqq 1$. However, this method does not yield an explicit upper bound for the best possible a, nor does it give much information about the function $b(K)$.

Now suppose that f is a K-quasiconformal mapping of U onto itself, normalized so that $f(0) = 0$ and $f(1) = 1$. Then we can extend f by reflection in ∂U to obtain a K-quasiconformal mapping of the extended plane $\overline{\Omega}$ with $f(\infty) = \infty$. The complex dilatation $\mu = f_{\bar{z}}/f_z$ will satisfy the symmetry condition

$$\overline{\mu(z)} = \mu(1/\bar{z})\bar{z}^2/z^2 \tag{1.3}$$

a.e. in the finite plane Ω. For convenience of notation, we let S_K denote the class of all such mappings f. In the proof of Theorem 1, it is obviously sufficient to consider only $f \in S_K$.

2. *The parametric representation.* We shall establish Theorem 1 using a parametric representation, first derived by Shah Dao-shing in [5]. However, since this paper is relatively inaccessible, we first show how this representation can be obtained from some recent results due to Ahlfors and Bers [2].

Suppose that f is in S_K and has complex dilatation μ, and set

$$\nu(z, t) = (\operatorname{sgn} \mu(z)) \tanh \left(\frac{t}{T} \operatorname{arctanh} | \mu(z) |\right), \qquad T = \log K, \tag{2.1}$$

where $\operatorname{sgn} w = \frac{w}{|w|}$ if $w \neq 0, \infty$ and $\operatorname{sgn} w = 0$ if $w = 0$ or ∞. Next let $g = g(z, t)$ be the quasiconformal mapping of $\overline{\Omega}$ which has ν as its complex dilatation and is normalized so that $g(0, t) = 0$, $g(1, t) = 1$, and $g(\infty, t) = \infty$. Since $\nu(z, 0) = 0$ and $\nu(z, T) = \mu(z)$, we see that $g(z, 0) = z$ and $g(z, T) = f(z)$. Moreover from (2.1) it is obvious that ν satisfies a symmetry condition like (1.3), and hence g maps U onto itself for each t. From (2.1) we have

$$| \nu(z, t + \Delta t) - \nu(z, t) - \frac{\partial \nu}{\partial t}(z, t) \Delta t | \leq \tfrac{1}{2} | \Delta t |^2,$$

where

$$\frac{\partial \nu}{\partial t} = (\operatorname{sgn} \mu) \frac{\operatorname{arctanh} |\mu|}{T} (1 - |\nu|^2). \tag{2.2}$$

Since $\frac{\partial \nu}{\partial t}$ is continuous in t and $\left|\frac{\partial \nu}{\partial t}\right| \leq \frac{1}{2}$, Theorem 10 of [2] implies that

$$\frac{\partial g}{\partial t}(z\,,t) = \lim_{\Delta t \to 0} \frac{g(z\,,t+\Delta t) - g(z\,,t)}{\Delta t}$$

uniformly on each compact set in Ω, that $\frac{\partial g}{\partial t}$ has generalized derivatives with respect to z and $\bar{z}$ which are locally L^2-integrable, that

$$\text{(2.3)}\qquad \begin{aligned} &\lim_{\Delta t \to 0} \iint_E \left| \frac{g_z(z\,,t+\Delta t) - g_z(z\,,t)}{\Delta t} - \left(\frac{\partial g}{\partial t}\right)_z \right|^2 d\sigma = 0\,, \\ &\lim_{\Delta t \to 0} \iint_E \left| \frac{g_{\bar{z}}(z\,,t+\Delta t) - g_{\bar{z}}(z\,,t)}{\Delta t} - \left(\frac{\partial g}{\partial t}\right)_{\bar{z}} \right|^2 d\sigma = 0\,, \end{aligned}$$

for each compact set $E \subset \Omega$, and that

$$\text{(2.4)}\qquad \left(\frac{\partial g}{\partial t}\right)_{\bar{z}} = \nu \left(\frac{\partial g}{\partial t}\right)_z + \frac{\partial \nu}{\partial t} g_z\,.$$

Now set $\zeta = \frac{\partial g}{\partial t} \circ g^{-1} = \frac{\partial g}{\partial t}(g^{-1}\,,t)$. Then ζ has generalized derivatives which are locally L^2-integrable, $\frac{\partial g}{\partial t} = \zeta \circ g = \zeta(g\,,t)$, and by the chain rule

$$\text{(2.5)}\qquad \left(\frac{\partial g}{\partial t}\right)_z = (\zeta_z \circ g) g_z + (\zeta_{\bar{z}} \circ g) \overline{g_{\bar{z}}}\,, \quad \left(\frac{\partial g}{\partial t}\right)_{\bar{z}} = (\zeta_z \circ g) g_{\bar{z}} + (\zeta_{\bar{z}} \circ g) \overline{g_z}\,.$$

(See, for example, Lemma 10 of [2].) Since $g_{\bar{z}} = \nu g_z$, combining (2.2), (2.4), and (2.5) yields

$$\text{(2.6)}\qquad \zeta_{\bar{z}} \circ g = \frac{\partial \nu}{\partial t} \frac{1}{1 - |\nu|^2} \operatorname{sgn}(g_z)^2 = (\operatorname{sgn} \mu) \frac{\operatorname{arctanh} |\mu|}{T} \operatorname{sgn}(g_z)^2\,.$$

In particular, we see that $|\zeta_{\bar{z}}| \leq \frac{1}{2}$ in Ω. Also since $g(0\,,t) = 0$ and $g(1\,,t) = 1$, we have $\zeta(0\,,t) = \zeta(1\,,t) = 0$, while the fact that g maps U onto itself implies that

$$\text{(2.7)}\qquad \operatorname{Re}\left(\frac{\zeta(z\,,t)}{z}\right) = 0$$

for $z \in \partial U$.

Finally, since $f(z) = g(z\,,\,T)$, we may think of f as being generated as a solution of the differential equation

$$\frac{\partial g}{\partial t} = \zeta(g\,,\,t)\,, \qquad z \in \Omega\,,\ 0 \le t \le T\,, \tag{2.8}$$

subject to the initial condition $g(z\,,\,0) = z$, where ζ satisfies the above listed conditions. This is essentially the parametric representation derived by Shah Dao-shing in [5] for smooth mappings in S_K .

3. *Rate of change of area.* Now suppose that E is a measurable set in U and that $f \in S_K$. Next let g be defined as in section 2 and set

$$A(t) = m(g(E\,,\,t)) = \iint\limits_E (|g_z|^2 - |g_{\bar z}|^2) d\sigma\,.$$

Then $A(0) = m(E)$, $A(T) = m(f(E))$, and our problem is to obtain an upper bound for $A(T)$ in terms of $A(0)$. From (2.3), (2.5), and the Schwarz inequality it follows that

$$\begin{aligned} \frac{dA}{dt} &= \lim_{\Delta t \to 0} \iint\limits_E \left(\frac{|g_z(z\,,t+\Delta t)|^2 - |g_z(z\,,t)|^2}{\Delta t} - \frac{|g_{\bar z}(z\,,t+\Delta t)|^2 - |g_{\bar z}(z\,,t)|^2}{\Delta t} \right) d\sigma \\ &= 2 \iint\limits_E \mathrm{Re}\left(\overline{g_z} \left(\frac{\partial g}{\partial t}\right)_z - \overline{g_{\bar z}} \left(\frac{\partial g}{\partial t}\right)_{\bar z} \right) d\sigma \\ &= 2 \iint\limits_E \mathrm{Re}(\zeta_z \circ g)\,(\,|g_z|^2 - |g_{\bar z}|^2)\, d\sigma \\ &= 2 \iint\limits_{g(E\,,\,t)} \mathrm{Re}\ \zeta_z\, d\sigma\,. \end{aligned} \tag{3.1}$$

We may interpret this formula by thinking of g as a flow on Ω which carries U onto itself. The velocity profile of this flow is given by (2.8), and hence the outward normal component of the flow across $\partial g(E\,,\,t)$ at a point $z \in \partial g(E\,,\,t)$ is equal to

$$\zeta(z\,,\,t) \cdot \frac{1}{i}\, \frac{dz}{|dz|} = \mathrm{Re}\left(\frac{1}{i}\, \overline{\zeta(z\,,\,t)}\, \frac{dz}{|dz|} \right),$$

where dz is parallel to the tangent vector to $\partial g(E\,,\,t)$ at z . Thus if ζ is sufficiently smooth and if E has a piecewise smooth boundary, the total outward flow across $\partial g(E\,,\,t)$ is given by

$$\operatorname{Re}\left(\frac{1}{i}\int_{\partial g(E,t)} \bar{\zeta}\, dz\right) = 2\operatorname{Re}\left(\iint_{g(E,t)} \zeta_z\, d\sigma\right) .$$

Since this total outward flow is just the rate of change of A, we have an alternative derivation for formula (3.1).

Now since $\zeta_{\bar z}$ is bounded in Ω, we have

$$\zeta(z, t) = \psi(z, t) - \frac{1}{\pi}\iint_U \zeta_{\bar z}(w, t)\left(\frac{1}{w-z} - \frac{1}{w}\right) d\sigma , \tag{3.2}$$

where ψ is continuous in $\bar U$ and analytic in U for $0 \le t \le T$. Since $\zeta(0, t) = 0$, it follows that $\psi(0, t) = 0$. Thus ψ/z is analytic in U, and from (2.7) we obtain

$$\operatorname{Re}\left(\frac{\psi(z, t)}{z}\right) = \operatorname{Re}\left(\frac{1}{\pi}\iint_U \overline{\zeta_{\bar z}(w, t)}\, \frac{z}{\bar w(\bar w z - 1)}\, d\sigma\right) \tag{3.3}$$

for $z \in \partial U$. The function on the right hand side of (3.3) is also analytic in U, and hence we conclude that

$$\psi(z, t) = \frac{1}{\pi}\iint_U \overline{\zeta_{\bar z}(w, t)}\, \frac{z^2}{\bar w(\bar w z - 1)}\, d\sigma + iz\,\Theta(t) \tag{3.4}$$

for $z \in U$ and $0 \le t \le T$, where Θ is real and continuous. Since $|\zeta_{\bar z}| \le \frac{1}{2}$ in U, there exists an absolute constant $c > 0$ such that

$$|\operatorname{Re} \psi_z(z, t)| \le \frac{c}{2} \tag{3.5}$$

whenever $|z| \le \frac{1}{2}$ and $0 \le t \le T$.

From (3.1) and (3.2) we obtain

$$\frac{dA}{dt} = \iint_{g(E,t)} \operatorname{Re} \tilde{\varphi}\, d\sigma + 2\iint_{g(E,t)} \operatorname{Re} \psi_z\, d\sigma , \tag{3.6}$$

where $\tilde{\varphi}$ is the Hilbert transform of $\varphi = 2\,\zeta_{\bar z}\,\chi_U$,

$$\tilde{\varphi}(z, t) = \lim_{r\to 0} -\frac{1}{\pi}\iint_{|w-z|\ge r} \frac{\varphi(w, t)}{(w-z)^2}\, d\sigma ,$$

and χ_U is the characteristic function of U. It is not difficult to verify that

$$\iint_\Omega \tilde{\varphi}\, h\, d\sigma = \iint_\Omega \varphi\, \tilde{h}\, d\sigma \tag{3.7}$$

whenever h is bounded and has compact support. Hence if we take $h = \chi_{g(E,t)}$, we obtain

$$\text{(3.8)} \qquad \frac{dA}{dt} \leq \iint_U |\tilde{\chi}_{g(E,t)}| \, d\sigma + 2 \iint_{g(E,t)} \operatorname{Re} \psi_z \, d\sigma$$

from (3.6) and (3.7).

4. *Proof of Theorem 1.* In order to make use of (3.8), we must appeal to the following inequality, which is implicit in the work of Calderón and Zygmund [4] and which will be established in section 5.

Theorem 2. *There exist a pair of constants* a *and* b, $1 \leq a \leq 40$ *and* $0 \leq b \leq 2$, *such that*

$$\text{(4.1)} \qquad \iint_U |\tilde{\chi}_E| \, d\sigma \leq a\, m(E) \log \frac{\pi}{m(E)} + b\, m(E)$$

for each measurable set $E \subset U$.

We now complete the proof of Theorem 1 in two steps. Suppose first that E is a measurable set in $|z| \leq 8^{-K}$. Then since g is a K-quasiconformal mapping of U onto itself with $g(0, t) = 0$, $g(E, t)$ lies in $|z| \leq \frac{1}{2}$ by a well known distortion theorem due to Hersch and Pfluger [7]. From (3.5), (3.8), and (4.1) we have

$$\frac{dA}{dt} \leq a\, A \log \frac{\pi}{A} + (b + c) A$$

for $0 \leq t \leq T$, and with a change of variables and integration we obtain

$$\text{(4.2)} \qquad \frac{A(t)}{\pi} \leq \exp\left(\frac{b + c}{a} (1 - e^{-at})\right) \left(\frac{A(0)}{\pi}\right)^{e^{-at}}.$$

Setting $t = T$ in (4.2) then gives (1.2) with $b(K) = b_0(K)$, where

$$\text{(4.3)} \quad b_0(K) = \exp\left(\frac{b + c}{a} (1 - K^{-a})\right) = 1 + (b + c)(K - 1) + o(K - 1)$$

as $K \to 1$.

Next suppose that E is any measurable set in U, and for each l, $1 < l < \infty$, let V be the disk $|z| < l$ and set

$$m = \min_{|z|=l} |f(z)|, \qquad M = \max_{|z|=l} |f(z)|.$$

Then it is easy to verify that

$$\text{(4.4)} \qquad m \leq l^K, \qquad M \geq l^{1/K},$$

while by a distortion theorem due to Lehto, Virtanen, and Väisälä [9],

$$M \leq \lambda(K)\, m\,, \tag{4.5}$$

where $\lambda(K) = 1 + O(K-1)$ as $K \to 1$. Let h map U conformally onto $f(V)$ so that $h(0) = 0$. Then the Schwarz Lemma applied to $h^{-1}(mz)$ and to $h(z)/M$ implies that

$$| h(z) | \geq m \, | z| \tag{4.6}$$

and, with (4.6), that

$$|h'(z)| \leq M \frac{1 - |h(z)/M|^2}{1 - |z|^2} \leq M + \frac{2(M-m)}{m^2 - 1} \tag{4.7}$$

for $z \in h^{-1}(U)$.

Now set $l = l(K) = \max(8\,,\ 3^{\frac{1}{2}}\,\lambda(K))^K$. Then (4.4) and (4.5) imply that $m \geq 3^{\frac{1}{2}}$, and with (4.7) we conclude that

$$m(f(E)) = \iint_{h^{-1}\circ f(E)} |h'(z)|^2\, d\sigma \leq M^2 \left(\frac{M}{m}\right)^2 m(h^{-1}\circ f(E))\,. \tag{4.8}$$

If we apply what was proved earlier to the K-quasiconformal mapping $f_0(z) = h^{-1}\circ f(l\,z)$, we obtain

$$\frac{m(h^{-1}\circ f(E))}{\pi} \leq b_0\,(K) \left(\frac{m(E)}{\pi\, l^2}\right)^{K-a}, \tag{4.9}$$

where $b_0(K)$ is as in (4.3). Finally combining (4.4), (4.5), (4.8), and (4.9) yields (1.2) with

$$b(K) = b_0(K)\lambda\,(K)^4\, l^{2(K-K^{-a})}\,. \tag{4.10}$$

Since each factor on the right hand side of (4.10) is of the form $1 + O(K-1)$ as $K \to 1$, we conclude that $b(K)$ is also of this form.

We have established (1.2) with $a = 40$. If $f(z) = (\operatorname{sgn} z)\,|z|^{1/K}$ and if E is any disk in U with center at the origin, then

$$\frac{m(f(E))}{\pi} = \left(\frac{m(E)}{\pi}\right)^{1/K},$$

and hence we must have $a \geq 1$ in (1.2). Thus assuming Theorem 2, we have completed the proof of Theorem 1.

5. *Proof of Theorem 2.* It remains for us to establish Theorem 2. We begin by quoting a special case of a result used by Calderón and Zygmund (Lemma 1 of [4]).

Lemma 1. *Suppose that* E *is a measurable set with* $0 < m(E) < \infty$ *and that* $0 < t < 4$. *Then there exists a sequence of nonoverlapping squares* $\{G_k\}$ *such that*

$$\frac{t}{4} < \frac{m(E \cap G_k)}{m(G_k)} \leq t \tag{5.1}$$

for each k , *and such that* $m(E - \bigcup G_k) = 0$.

For each set E with finite measure, we let $\lambda_E(t)$ denote the distribution function for the Hilbert transform $\tilde{\chi}_E$. That is, for $0 < t < \infty$, $\lambda_E(t)$ will denote the measure of the set of z for which $|\tilde{\chi}_E(z)| \geq t$.

Lemma 2. *For* $0 < t < \infty$,

$$\lambda_E(t) \leq \frac{m(E)}{t^2} . \tag{5.2}$$

Proof. Since the Hilbert transform is an isometry with respect to the L^2-norm [1],

$$\lambda_E(t)\, t^2 \leq \iint\limits_{\Omega} |\tilde{\chi}_E|^2\, d\sigma = \iint\limits_{\Omega} |\chi_E|^2\, d\sigma = m(E) ,$$

and (5.2) follows.

Lemma 3. *There exists a constant* a , $1 \leq a \leq 40$, *such that for* $0 < t < 1$,

$$\lambda_E(t) \leq a\, \frac{m(E)}{t} . \tag{5.3}$$

Proof. We may assume that $m(E) > 0$ for otherwise (5.3) is trivial. Let $\{G_k\}$ be the sequence of squares of Lemma 1. Then by (5.1), we can choose for each k a measurable set F_k such that $E \cap G_k \subset F_k \subset G_k$ and $m(E \cap F_k) = tm\,(F_k)$. If we set $F = \bigcup F_k$ and $G = \bigcup G_k$, then

$$\frac{t}{4}\, m(G) < m(E) = tm(F) . \tag{5.4}$$

Next set $h = \frac{1}{t}\, \chi_E - \chi_F$, and for $0 < s < 1$, let H be the set where $|\tilde{h}| \geq 1 - s$. If $|\tilde{\chi}_E(z)| \geq t$, then clearly $|\tilde{\chi}_F(z)| \geq s$ or $|\tilde{h}(z)| \geq 1 - s$, and hence we obtain

$$\lambda_E(t) \leq \lambda_F(s) + m(H) \leq \frac{m(E)}{s^2 t} + m(H) \tag{5.5}$$

from (5.2) and (5.4). The rest of the argument involves getting an upper bound for $m(H)$.

For this choose $1 < r < \infty$, let V_k be a disk with center at z_k and radius $r\, r_k$, where z_k and $2r_k$ are the center and diameter of G_k, and let $V = \bigcup V_k$. Since $h = 0$ a.e. outside of F and since

$$\iint\limits_{F_k} h d\sigma = \frac{1}{t} \iint\limits_{F_k} \chi_E \, d\sigma - \iint\limits_{F_k} \chi_F \, d\sigma = \frac{1}{t}\, m(E \cap F_k) - m(F_k) = 0$$

for each k, we have

$$\tilde{h}(z) = -\frac{1}{\pi} \sum_k \iint\limits_{F_k} h(w) \left(\frac{1}{(w-z)^2} - \frac{1}{(z_k - z)^2} \right) d\sigma \, .$$

If $z \notin V_k$ and $w \in F_k$, then

$$\frac{1}{\pi} \left| \frac{1}{(w-z)^2} - \frac{1}{(z_k-z)^2} \right| \leq \frac{r_k}{\pi} \frac{1}{|z-z_k|^3} \left(\frac{|z-z_k|}{|z-z_k| - r_k} + \left(\frac{|z-z_k|}{|z-z_k| - r_k} \right)^2 \right) = i_k(z) \, ,$$

and hence we obtain

$$(5.6) \qquad |\tilde{h}(z)| \leq \sum_k i_k(z) \iint\limits_{F_k} |h(w)| \, d\sigma$$

for $z \notin V$. Now

$$\iint\limits_{C(V)} i_k(z) \, d\sigma \leq \iint\limits_{C(V_k)} i_k(z) \, d\sigma = 2 \left(\frac{1}{r-1} + \log \frac{r}{r-1} \right) ,$$

while since $0 < t < 1$,

$$\iint\limits_{F_k} |h(w)| \, d\sigma = 2(1-t)\, m(F_k) < 2m(F_k) \, .$$

Combining these inequalities with (5.4) and (5.6) yields

$$(1-s)\, m(H - V) \leq \iint\limits_{C(V)} |\tilde{h}| \, d\sigma < 4 \left(\frac{1}{r-1} + \log \frac{r}{r-1} \right) \frac{m(E)}{t} \, .$$

Obviously

$$m(H \cap V) \leq m(V) = \frac{\pi r^2}{2}\, m(G) < 2\, \pi\, r^2\, \frac{m(E)}{t} \, ,$$

and we conclude that

$$(5.7) \qquad m(H) < \left(\frac{4}{1-s} \left(\frac{1}{r-1} + \log \frac{r}{r-1} \right) + 2\, \pi\, r^2 \right) \frac{m(E)}{t} \, .$$

If we take $r = 1.7$ and $s = .4$, then (5.5) and (5.7) imply that (5.3) holds with $a = 40$. Next if E is any disk, then it is easy to verify that for $0 < t < 1$,

$$\lambda_E(t) = \frac{1-t}{t}\, m(E)\,.$$

Hence we must have $a \geq 1$ in (5.3), and the proof of Lemma 3 is complete.

The proof of Theorem 2 is now an immediate consequence of Lemmas 2 and 3. For let $\lambda_{E,U}(t)$ denote the measure of the set of $z \in U$ for which $|\tilde{\chi}_E(z)| \geq t$. Then $\lambda_{E,U}(t) \leq \min(\pi, \lambda_E(t))$,

$$\iint_U |\tilde{\chi}_E| d\sigma = \int_0^\infty \lambda_{E,U}(t)\, dt \leqq \int_0^{m(E)/\pi} \pi\, dt + \int_{m(E)/\pi}^1 \frac{a\, m(E)}{t}\, dt + \int_1^\infty \frac{m(E)}{t^2}\, dt$$

$$= a\, m(E) \log \frac{\pi}{m(E)} + 2m(E)\,,$$

and we obtain (4.1) with $a = 40$ and $b = 2$. If E is any disk in U with center at the origin, then

$$\iint_U |\tilde{\chi}_E\,| d\sigma = m(E) \log \frac{\pi}{m(E)}\,,$$

and hence we must have $a \geq 1$ and $b \geq 0$ in (4.1). This completes the proof of Theorem 2.

6. *Remarks.* On the basis of examples and heuristic reasoning, we conjecture that there exists a function $b(K)$ and a constant b for which Theorems 1 and 2 hold, respectively, with $a = 1$. Unfortunately, we have not been able to prove this. However, it is perhaps worth pointing out that the lower bound of values of a for which Theorem 1 holds is equal to the corresponding lower bound for Theorem 2, and that if either of these theorems holds with a equal to this common lower bound, then so does the other. These facts are immediate consequences of the following result.

Theorem 3. *If a and b are constants for which the conclusion of Theorem 2 holds, then there exists a function $b(K)$, of the form $1 + O(K-1)$ as $K \to 1$, such that the conclusion of Theorem 1 holds for a and $b(K)$. Conversely, if a is a constant and $b(K)$ a function, of the form $1 + O(K-1)$ as $K \to 1$, for which the conclusion of Theorem 1 holds, then there exists a constant b such that the conclusion of Theorem 2 holds for a and b.*

Proof. The first part of Theorem 3 follows directly from the argument given in section 4. For the second part, assume that a is a constant and $b(K)$ a function for which the conclusions of Theorem 1 hold, and let

$$d = \limsup_{K \to 1} \frac{b(K) - 1}{K - 1} < \infty . \tag{6.1}$$

We want to exhibit a constant b such that

$$\iint_U |\tilde{\chi}_E| \, d\sigma \leqq a \, m(E) \log \frac{\pi}{m(E)} + b \, m(E) \tag{6.2}$$

for all measurable sets $E \subset U$. We do this in two steps.

Suppose first that E lies in $|z| \leq \frac{1}{2}$. Next set $\omega = \overline{\tilde{\chi}}_E$ in U, and extend ω to Ω so that it satisfies a symmetry condition like (1.3) a.e. Then for $0 \leq t < \infty$ let $g = g(z, t)$ be the e^t-quasiconformal mapping of $\overline{\Omega}$ which has

$$\nu(z, t) = (\operatorname{sgn} \omega(z)) \tanh \frac{t}{2}$$

as its complex dilatation and is normalized so that $g(0, t) = 0$, $g(1, t) = 1$, and $g(\infty, t) = \infty$. As in section 2, g satisfies (2.8), where $\zeta(0, t) = \zeta(1, t) = 0$, $\operatorname{Re}(\zeta/z) = 0$ for $z \in \partial U$, and

$$\zeta_{\bar{z}} \circ g = \tfrac{1}{2}(\operatorname{sgn} \omega) (\operatorname{sgn}(g_z)^2) . \tag{6.3}$$

If $A(t) = m(g(E, t))$, then as in section 3,

$$\frac{dA}{dt} = \iint_{g(E,t)} \operatorname{Re} \tilde{\varphi} \, d\sigma + 2 \iint_{g(E,t)} \operatorname{Re} \psi_z \, d\sigma , \tag{6.4}$$

where $\tilde{\varphi}$ is the Hilbert transform of $\varphi = 2 \zeta_{\bar{z}} \chi_U$, and $|\operatorname{Re} \psi_z| \leq \frac{c}{2}$ in $|z| \leq \frac{1}{2}$, where c is the absolute constant in (3.5).

Now if we apply Theorem 1 to g and E, we have

$$\frac{A(t)}{\pi} \leqq b(e^t) \left(\frac{m(E)}{\pi} \right)^{e^{-at}}$$

for $0 \leq t < \infty$, and letting $t \to 0$, we obtain

$$\frac{dA}{dt}(0) \leq a \, m(E) \log \frac{\pi}{m(E)} + d \, m(E) , \tag{6.5}$$

where d is as in (6.1). Setting $t = 0$ in (6.4) yields

$$\frac{dA}{dt}(0) \geq \iint_E \operatorname{Re} \tilde{\varphi}(z, 0) \, d\sigma - c \, m(E) , \tag{6.6}$$

and we conclude from (6.5) and (6.6) that

$$\iint_{\Omega} \operatorname{Re} \tilde{\varphi}\, \chi_E \, d\sigma \leq a\, m(E) \log \frac{\pi}{m(E)} + b_0 m(E) \,, \tag{6.7}$$

where $b_0 = c + d$. Since $g(z, 0) = z$, we see from (6.3) that $\varphi(z, 0) = (\operatorname{sgn} \omega(z))\, \chi_U(z)$. Hence by (3.7),

$$\iint_{U} |\tilde{\chi}_E| \, d\sigma = \iint_{\Omega} \operatorname{Re} \varphi\, \tilde{\chi}_E \, d\sigma = \iint_{\Omega} \operatorname{Re} \tilde{\varphi}\, \chi_E \, d\sigma \,, \tag{6.8}$$

and (6.2) follows from (6.7) and (6.8) with $b = b_0$.

Now suppose that E is any measurable set in U. Then we can decompose E into n disjoint measurable sets E_k so that $n \leq 8$ and each E_k lies in a disk with radius $\frac{1}{2}$ and center z_k. By what was proved above,

$$\iint_{U_k} | \tilde{\chi}_{E_k} | \, d\sigma \leq a\, m(E_k) \log \frac{\pi}{m(E_k)} + b_0 m(E_k) \,, \tag{6.9}$$

where U_k is the disk $|z - z_k| < 1$. Since clearly $|\tilde{\chi}_{E_k}| \leq \frac{4}{\pi} m(E_k)$ in $U - U_k$, we have

$$\iint_{U} |\tilde{\chi}_{E_k}| \, d\sigma \leq \iint_{U_k} | \tilde{\chi}_{E_k} | \, d\sigma + 4\, m(E_k) \,. \tag{6.10}$$

From the concavity of the function $x \log \frac{\pi}{x}$ it follows that

$$\sum_1^n m(E_k) \log \frac{\pi}{m(E_k)} \leq m(E) \log \frac{\pi}{m(E)} + (\log n)\, m(E) \,, \tag{6.11}$$

and if we now sum over k from 1 to n, we obtain (6.2) from (6.9), (6.10), and (6.11) with $b = b_0 + 4 + a(\log 8)$. This completes the proof of Theorem 3.

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References

[1] AHLFORS, L. V.: Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn. 206 (1955) pp. 1—22.

[2] —»— and BERS, L.: Riemann's mapping theorem for variable metrics, Ann. Math. 72 (1960) pp. 385—404.

[3] BOJARSKI, B. V.: Homeomorphic solutions of Beltrami systems, (Russian), Dokl. Akad. Nauk SSSR 102 (1955) pp. 661—664.

[4] CALDERÓN, A. P., and ZYGMUND, A.: On the existence of certain singular integrals, Acta Math. 88 (1952) pp. 85—139.

[5] DAO-SHING, SHAH: Parametric representation of quasiconformal mappings, (Russian), Science Record 3 (1959) pp. 400—407.

[6] GEHRING, F. W., and VÄISÄLÄ, J.: On the geometric definition for quasiconformal mappings, Comm. Math. Helv. 36 (1961) pp. 19—32.

[7] HERSCH, J.: Contribution à la théorie des fonctions pseudo-analytiques, Comm. Math. Helv. 30 (1956) pp. 1—19.

[8] LEHTO, O.: Remarks on the integrability of the derivatives of quasiconformal mappings, Ann. Acad. Sci. Fenn. 371 (1965) pp. 1—8.

[9] —»— VIRTANEN, K. I., and VÄISÄLÄ, J.: Contributions to the distortion theory of quasiconformal mappings, Ann. Acad. Sci. Fenn. 273 (1959) pp. 1—14.

[10] MORI, A.: On quasi-conformality and pseudo-analyticity, Trans. Amer. Math. Soc. 84 (1957) pp. 56—77.

[11] MORREY, C. B.: On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43 (1938) pp. 126—166.

[12] PESIN, I. N.: Metric properties of Q-quasiconformal mappings, (Russian), Mat. Sbornik 40 (82) (1956) pp. 281—294.

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