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SOME THEOREMS ON CLUSTER SETS

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Since about fifty years ago, Japanese mathematicians have obtained many valuable suggestions from the brilliant school of Helsinki in the geometric theory of functions.

First of all, I should like to extend my hearty gratitude to our respected teachers Prof. R. Nevanlinna, Prof. P. J. Myrberg, as well as Prof. Ahlfors, and our esteemed colleagues Prof. Lehto, Prof. L. Myrberg, Prof. Tammi, Dr. Virtanen and all other teachers and colleagues in Helsinki.

It is a great honour for me to speak at the Mathematical Society of Finland. I shall start with historical backgrounds.

§ 1. Localization of Picard's theorem.

Let $w = f(z)$ be a non-rational meromorphic function in $|z| < \infty$. Suppose that the inverse $z = \varphi(w)$ has a transcendental singularity Ω at $w = \omega$. Map a ϱ-neighbourhood Φ_ϱ of Ω, which is a covering surface of the disk (c): $|w - \omega| < \varrho$, by $z = \varphi(w)$ into $|z| < \infty$. Then, the image Δ of Φ_ϱ is a (relatively non-compact region) peninsula with boundary Γ consisting of countably many analytic curves and $z = \infty$, because there exists a path Λ in Δ terminating at $z = \infty$ along which ω is an asymptotic value. Suppose that Δ is simply connected. Then, $w = f(z)$ assumes every value of (c) infinitely often in Δ except for at most one value of (c) (Noshiro [13]). The proof is based on the Ahlfors theory of covering surfaces. Kunugui [6] proved: Without imposing any restriction on the connectivity of Δ, the number of such exceptional values is at most 2.

Corollary. The set of direct transcendental singularities of the inverse function $z = \varphi(w)$ of a non-rational meromorphic function $w = f(z)$ in the finite z-plane: $|z| < +\infty$ is at most countable.

§ 2. Single-valued meromorphic functions in general plane regions.

Let E be a compact set of logarithmic capacity 0 in a region D. Suppose that $w = f(z)$ is a single-valued meromorphic function in $D - E$, which has an essential singularity at every point $z_0 \in E$. Then, $w = f(z)$

assumes every value infinitely often in any neighbourhood of z_0 with a possible set of exceptional values of capacity zero (an extension of Nevanlinna's theorem) (af Hällström [3], Kametani [5]).

Nevanlinna's theory of meromorphic functions has been extended by Hällström [3] and Tsuji [19], [21] independently to such meromorphic functions.

In their works, the following theorem plays a fundamental role.

The Evans — Selberg theorem. Let E be a compact set of capacity zero. Then, there exists a positive mass-distribution μ on E with total mass unity, such that the logarithmic potential $U_\mu(z) = \int_E \log 1/|z - \zeta|\, d\mu(\zeta)$ is $+\infty$ at every point of E and at no other points (Evans [2], H. Selberg [18]).

Remark. Nakai [12] succeeded in extending the Evans — Selberg theorem to the case of open Riemann surfaces belonging to O_G.

§ 3. Cluster sets of functions meromorphic in general plane regions.

Let D be an arbitrary region, Γ its boundary, E a compact set of capacity zero contained in Γ, and z_0 a point of E. Suppose that $w = f(z)$ is a single-valued meromorphic function in D. We define:

Cluster set $C_D(f, z_0)$: $\alpha \in C_D(f, z_0)$ if there exists $\{z_n\}$ such that $z_n \in D$, $z_n \to z_0$, $f(z_n) \to \alpha$.

Boundary cluster set $C_{\Gamma-E}(f, z_0)$: $\alpha \in C_{\Gamma-E}(f, z_0)$ if there exists $\{\zeta_n\}$ of $\Gamma - E$ such that $w_n \in C_D(f, \zeta_n)$ for each n, $\zeta_n \to z_0$ and $w_n \to \alpha$.

Range of values $R_D(f, z_0)$: $\alpha \in R_D(f, z_0)$ if there exists $\{z_n\}$ such that

$$z_n \in D\,,\ z_n \to z_0\,,\ f(z_n) = \alpha\,.$$

Then,

(i) if z_0 is an accumulation point of $\Gamma - E$, i.e., $z_0 \in \overline{\Gamma - E}$, then $\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is an open set (Tsuji [20]);

(ii) if $z_0 \in \overline{\Gamma - E}$ and if Ω is not empty, then $\Omega - R_D(f, z_0)$ is at most of capacity zero (Tsuji [20]);

(iii) if D is a simply connected region of hyperbolic type, and if Ω is not empty, and further if $f(z)$ is bounded in the intersection of D and some neighbourhood of z_0, then $w = f(z)$ takes every value belonging to each component Ω_n of Ω infinitely often in any neighbourhood of z_0 with one possible exception belonging to Ω_n (Noshiro [15]);

(iv) if each point of E belongs to a boundary component of D which consists of a non-degenerate continuum, and if Ω is not empty, then

$w = f(z)$ assumes every value belonging to each component Ω_n of Ω infinitely often in any neighbourhood of z_0 with two possible exceptions (Noshiro [16] in the particular case where E is contained in a single boundary component; Hervé [4]).

Remark. In the particular case where E consists of a single point z_0, the condition that z_0 belongs to a boundary component consisting of a nondegenerate continuum is unnecessary. We have only to assume that z_0 is a nonisolated boundary point. The proofs of (iii), (iv) are based upon the Ahlfors theory of covering surfaces and the Evans-Selberg theorem.

Application. As an application of (iii), we can give a simple proof to P. J. Myrberg's theorem [11]: Let e be a closed set of capacity zero, lying completely inside the unit disk (c): $|w| < 1$ and let Φ denote the region obtained by excluding the set e from the disk (c). Let $w = f(z)$ be a function which maps the unit disk D: $|z| < 1$ conformally onto the universal covering surface $\tilde{\Phi}$ of Φ in a one-to-one fashion. Suppose further that e contains at least 2 points. Then, the perfect set E, on Γ: $|z| = 1$, of essential singularities of $w = f(z)$ is of linear measure 0, but the capacity of E must be positive (Noshiro [14]). In fact $C_D(f, z_0) = \{|w| \leqq 1\}$, $C_{\Gamma-E}(f, z_0) = \{|w| = 1\}$, $\Omega = \{|w| < 1\}$, and (iii) shows that if $\text{cap}(E) = 0$, then $\Omega - R_D(f, z_0)$ contains at most one point, because $\Omega = \Omega_n$ in the present case.

Hervé, T. Kuroda and myself tried to exclude the assumption in (iv) that each point of E belongs to a boundary componentc onsisting of a non-degenerate continuum. I conjectured that it might be possible to improve the Hällström-Kametani theorem; more precisely, the exceptional set $\mathscr{C} R_{D-E}(f, z_0)$ might be finite or at most countable[Noshiro: Amer. Math. Soc. Translation 8, Ser. 2 (1958) 1—12].

However, very recently this conjecture has been answered by Matsumoto in the negative [8], by a skillful construction of a planar covering surface of the w-plane belonging to O_G.

Theorem (Matsumoto). For every compact set K of capacity zero in the w-plane, we can find a compact set E of capacity zero in the z-plane and a single-valued meromorphic function $f(z)$ in the complementary region D of E such that $f(z)$ has an essential singularity at each point z_0 of E, and the set of exceptional values $\mathscr{C} R_D(f, z_0)$ at each singularity z_0 coincides with the given set K.

Remark. By virtue of the Matsumoto theorem, the Hällström-Kametani theorem and Tsuji's result (ii) are best possible. The Hervé result (iv) is also sharp. However, the conjecture stated above is partly true, if we impose some restrictions on the shape of the set E. (Matsumoto [9], [10], Carleson [1]). In a letter of May 26, 1965 Matsumoto communicated the following result: Let E be a linear Cantor set, and ξ_n the successive ratios of the total lengths of the remaining intervals. If $\xi_{n+1} = O(\xi_n^2)$, then for every single-valued meromorphic function $f(z)$ in $\mathcal{C} E$ which possesses E as the set of essential singularities, $f(z)$ has at most 2 exceptional values at each essential singularity. This result will be published in Nagoya Math. J. (Nakayama Commemoration Volume).

In my Minnesota lecture, I tried to extend main results to the case where E is a totally disconnected compact set of analytic capacity zero (Painlevé null-set, i.e. a null-set of the class N_B) or a related null-set of the class N_B^0. Let E be a totally disconnected compact set and D be the complementary region of E. Let Δ be a relatively non-compact or compact region in D whose relative boundary β (with respect to D) consists of at most countably many analytic curves (compact or non-compact) clustering nowhere in D. For the sake of simplicity, we call such a region Δ a »subregion». Suppose that for any subregion Δ there exists no non-constant single-valued bounded analytic function on $\Delta \cup \beta$, whose real part vanishes on the relative boundary β of Δ. Then, we say that D belongs to O_{AB}^0. This class was introduced by T. Kuroda [7]. If $D = \mathcal{C} E$ belongs to O_{AB}^0, then E is said to belong to N_B^0. $N_B^0 \subset N_B$ (Kuroda [7], p. 38).

Kuroda has proved that if E is a totally disconnected compact set of class N_B^0 and if $w = f(z)$ is a non-constant single-valued meromorphic function in $D = \mathcal{C} E$, then, the inverse $z = \varphi(w)$ has the Iversen property [Noshiro [17], p. 96].

It is clear that if a compact set E is of logarithmic capacity 0, then E belongs to the class N_B^0. It is known that there exists a compact set of positive logarithmic capacity belonging to N_B^0 (Kuroda, loc. cit., p. 54). It is still open whether the inclusion $N_B^0 \subset N_B$ is proper or not.

We can prove:

Theorem 1. Let E be a compact set of analytic capacity zero contained in a region D. Suppose that $w = f(z)$ is a single-valued meromorphic function in $D - E$ which has an essential singularity at every point z_0 of E. Then, any compact subset of the complement $\mathcal{C} R_{D-E}(f, z_0)$ of $R_{D-E}(f, z_0)$ is of analytic capacity 0 (An extension of Hällström-Kametani's theorem).

Theorem 2. Let D be an arbitrary region, Γ its boundary, E a compact set of class N_B^0 contained in Γ and z_0 a point of E. Suppose that $w = f(z)$ is single-valued and meromorphic in D and $C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is not empty. If $\alpha \in C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is an exceptional value of $f(z)$ in a neighbourhood of z_0, then either α is an asymptotic value of $f(z)$ at z_0 or there exists a sequence $\zeta_n \in E$ $(n = 1, 2, \ldots)$ converging to z_0 such that α is an asymptotic value of $f(z)$ at each ζ_n. (cf. Noshiro [17], p. 14).

Theorem 3. Under the same assumption as in Theorem 2: If $z_0 \in (\overline{\Gamma - E})$, then $\Omega = C_D(f, z_0) - C_{\Gamma-E}(f, z_0)$ is an open set (an extension of (i)). If $z_0 \in \overline{\Gamma - E}$, and if Ω is not empty, then any compact subset of $\Omega - R_D(f, z_0)$ is of analytic capacity 0 (an extension of (ii)).

As a closing remark I express the hope that a systematic cluster set theory will be established on open Riemann surfaces in a near future.

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