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RADII OF UNIFORM BOUNDEDNESS AND INDETERMINATION OF HOLOMORPHIC FUNCTIONS, AND EXAMPLES IN CONFORMAL MAPPING OF JORDAN REGIONS

BY

F. BAGEMIHL and J. E. McMILLAN

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1. Introduction

Let C be the unit circle and D be the open unit disk in the complex plane, and suppose that E is a subset of C. In 1954, Bagemihl and Seidel [1] proved a general theorem on cluster sets which enabled them to derive the following result [1, p. 192, Theorem 3]:

A necessary and sufficient condition that there exist a function, holomorphic in D, that is uniformly bounded on the radii terminating in the points of E but has no radial limit whatever, is that E be nowhere dense on C.

They were then naturally led to ask the following question:

What is a necessary and sufficient condition that there exist a function, holomorphic in D, that is uniformly bounded on the radii terminating in the points of E but has no radial limit at any point of E?

They made the conjecture (unpublished) at that time that such a condition was *that E be of Lebesgue measure zero on every arc of C on which E is everywhere dense.*

That this is a reasonable conjecture is apparent from the following considerations. First of all, the necessity of the condition follows immediately from Fatou's theorem. The sufficiency of the condition is also readily established for two special cases: (i) E is of measure zero, and (ii) E is nowhere dense. In case (i), Lusin and Priwaloff have shown [9, pp. 156—159] that there exists a bounded holomorphic function in D having no radial limit at any point of E. In case (ii), if $\bar{E}$ denotes the closure of E, then it follows from [1, p. 190, Corollary 2] that there exists a function, holomorphic in D, that is uniformly bounded on the radii terminating in the points of $\bar{E}$, but whose radial cluster set at every point of $\bar{E}$ is the unit circle. By an appropriate combination of these two examples, it is possible to prove the sufficiency of the condition for more general cases, but apparently not in general.

The purpose of the present paper is to establish the truth of the conjecture in its full generality. This is accomplished by constructing a Jordan curve $J \subset C \cup D$, with $E \subset J$, in such a way that the interior Δ of J is star-shaped with respect to the origin $(0 \in \Delta)$, and under a conformal mapping of Δ onto the unit disk, the set E corresponds to a set of measure zero on the unit circle. Then essentially the aforementioned example of Lusin and Priwaloff is transplanted to Δ, and finally the desired holomorphic function is obtained in all of D by means of an approximation that is also of independent interest.

The Jordan curve J turns out to be useful for another purpose. For if we take E in particular to be a perfect nowhere dense subset of C of positive measure, and denote by Δ' the exterior of J, then if Δ' is mapped conformally onto the unit disk, it is not difficult to see that E corresponds to a set of positive measure on the unit circle, whereas we have already noted that if Δ is mapped onto the unit disk, E corresponds to a set of measure zero on the unit circle. Such an example has been given by Lohwater and Seidel [8], but in their example, no point of E is finitely (or rectifiably) accessible from Δ, whereas in our example every point of J is rectilinearly accessible from Δ as well as from Δ'.

In the final section of this paper, an example is given of a Jordan curve J, with interior domain Δ and exterior domain Δ', and a subset H of J, with the property that $0 \in \Delta$, Δ is star-shaped with respect to the origin, and in fact every point of J is radially accessible (through Δ) from the origin, and if Δ and Δ' are mapped conformally onto the unit disk, the set H corresponds to a set of measure 0 and 2π, respectively, on the unit circle.

We also give an example of a Jordan curve J, with interior domain Δ $(0 \in \Delta)$, and a subset K of J of positive area (two-dimensional Lebesgue measure), such that if Δ is mapped conformally onto the unit disk, the set K corresponds to a set of measure zero on the unit circle. Such an example has been given by Lohwater and Piranian [6], but in their example, no point of K is finitely accessible from Δ, whereas in our example every point of J is uniformly finitely accessible (through Δ) from the origin.

2. Outline of proof of conjecture

Theorem 1. *Let E be a subset of C. Then in order that there exist a function, holomorphic in D, that is uniformly bounded on the radii terminating in the points of E but has no radial limit at any point of E, it is necessary and sufficient that E be of Lebesgue measure zero on every arc on which it is everywhere dense.*

Proof. The necessity of the condition clearly follows from Fatou's theorem.

Suppose now that E satisfies the condition. Let $\{V_n\}$ be an enumeration of the (possibly only finitely many) components of $C - \bar{E}$. For each V_n, let G_n be an open arc of C such that $\bar{G}_n \subset V_n$, and set $G = \bigcup G_n$. For an open arc γ of C and a real number r $(0 < r < 1)$, let

$$S(\gamma\,, r) = \{\varrho e^{i\theta} : r < \varrho < 1\,, e^{i\theta} \in \gamma\}\,,\ \ \gamma(r) = \{re^{i\theta} : e^{i\theta} \in \gamma\}\,.$$

Choose a sequence $\{r_m\}$ such that $0 < r_m < r_{m+1} < 1$ $(m \geqq 1)$ and $\lim r_m = 1$.

We shall construct a domain $\varDelta$, which will be D minus certain of the sets $\bar{S}(G_n, r_m)$ and which will be bounded by a Jordan curve J, such that under a conformal mapping φ of $\varDelta$ onto $\{|w'| < 1\}$ $(= D')$, the subset E of J will correspond to a set E' on $\{|w'| = 1\}$ $(= C')$ of measure zero. Suppose for the moment that this has been done, and let h be a function, bounded and holomorphic in D', such that for each $\zeta \in E'$ and for each arc σ at ζ $(\sigma \subset D'$ and $\sigma \cup \{\zeta\}$ is a Jordan arc), the diameter of the cluster set of h on σ at ζ is greater than 2 (the existence of such an h can be established by means of a simple elaboration of the Lohwater-Piranian version [7, p. 11, Theorem 4] of the Lusin-Priwaloff example [9, pp. 156—159], or most simply and directly by referring to an elegant version of this example given by W. Schneider [10]). Let $g(z) = h(\varphi(z))$ $(z \in \varDelta)$. Then for each $\zeta \in E$, the oscillation of g at ζ on the radius terminating at ζ is greater than 2. We shall construct a function f, holomorphic in D, such that

$$|f(z) - g(z)| < 1 \quad \text{if} \quad z \in \{re^{i\theta} : 0 \leqq r < 1\,,\ \ e^{i\theta} \in E\}\,.$$

It is clear that f will then have the desired properties.

3. Construction of $\varDelta$

We repeatedly use the following lemma of Löwner-Montel (see [3, p. 36]): *Suppose that $\varDelta^{(j)}$ is the interior of a Jordan curve $J^{(j)}$ $(j = 1, 2)$ and that $0 \in \varDelta^{(1)} \subset \varDelta^{(2)}$. Let $\varphi^{(j)}$ be a conformal mapping of $\varDelta^{(j)}$ onto D with $\varphi^{(j)}(0) = 0$, B be a Borel subset of $J^{(1)} \cap J^{(2)}$, and $B^{(j)}$ be the subset of C that corresponds to B under $\varphi^{(j)}$. Then $m(B^{(1)}) \leqq m(B^{(2)})$ (where m denotes Lebesgue measure).*

Let U be the interior (relative to C) of the closure $\bar{E}$ of E. Then $m(E \cap U) = 0$. Thus $E \cap U$ corresponds to a set of measure zero under a conformal mapping onto D' of any domain which is D minus certain of the sets $\bar{S}(G_n, r_m)$ and which is bounded by a Jordan curve. Hence we need to require of $\varDelta$ only that $E - U$ correspond under φ to a set of measure zero.

The domain Δ will be defined as the intersection of a sequence $\{\Delta_m\}$. Let m denote a fixed natural number. It is easy to see, by means of the extended maximum principle for harmonic functions, that we can let δ be a positive number such that, if γ is an open arc of C with length less than 2δ, then the harmonic measure of $\gamma(r_m) \cup \gamma$ with respect to $S(\gamma, r_m)$ is less than $1/m$ at each point of $\gamma(r_{m+1})$. Let $U' = C - \bar{G}$, and let $\{U'_n\}$ be an enumeration of the (possibly only finitely many) components of U'. Since each open arc of C intersects $G \cup U'$, we can determine a natural number n_0 such that each open arc of C with length $\delta/3$ intersects either G or $\bigcup_{n=1}^{n_0} U'_n$. Choose open arcs U''_n of C $(n = 1, \ldots, n_0)$ such that $U''_n \supset \bar{U}'_n$ and

$$m(U^*) < 1/m\,, \quad \text{where} \quad U^* = \bigcup_{n=1}^{n_0} (U''_n - \bar{U}'_n)\,.$$

Choose a finite covering of $\bar{G}$ by open arcs (of C) intersecting $\bar{G}$, each with length less than $\delta/3$, and let n_1 be a natural number such that each of these open arcs intersects one of the sets $G_1, \ldots, G_{n_1}$. Let n_2 be a natural number such that $n_2 \geqq n_1$ and for each $n = 1, \ldots, n_0$, each component of $U''_n - \bar{U}'_n$ intersects one of the sets $G_1, \ldots, G_{n_2}$. Set

$$\Delta_m = D - \bigcup_{n=1}^{n_2} \bar{S}(G_n, r_m)\,,$$

and let φ_m be a conformal mapping of Δ_m onto D' with $\varphi_m(0) = 0$. We prove that under φ_m, $\bar{E} - U$ corresponds to a set on C' of measure less than $(2\pi + 1)/m$.

Let G^* be the (possibly empty) union of the components of $C - \bigcup_{n=1}^{n_2} \bar{G}_n$ with length less than 2δ. Denote by u the harmonic measure of G^* with respect to Δ_m. If γ is a component of G^*, then again by the maximum principle, at each $z \in S(\gamma, r_m)$, $u(z)$ is less than the value at z of the harmonic measure of $\gamma(r_m) \cup \gamma$ with respect to $S(\gamma, r_m)$; in particular, this holds for $z \in \gamma(r_{m+1})$, so that $u(z) < 1/m$ if $z \in \gamma(r_{m+1})$, because of the choice of δ. Thus, by the maximum principle, $u(0) < 1/m$. Therefore, under φ_m, G^* corresponds to a set on C' of measure less than $2\pi/m$.

We now prove that all points of $\bar{E} - U$, with the exception of at most enumerably many, are in $U^* \cup G^*$. Note first that the set $(\bar{E} - U) - (\bar{G} - G)$ is at most enumerable, since any accumulation point of $\bar{E} - U$ is in $\bar{G} - G$. Suppose that ζ is a point of $\bar{G} - G$ that is not an end point of any of the arcs G_n or U'_n. We prove that $\zeta \in U^* \cup G^*$. Let γ' be an open arc of C with length δ and one end point ζ. Then, because of the choice of n_0, the open arc γ'' with length $\delta/3$ that is the middle third of γ'

intersects either G or $\bigcup_{n=1}^{n_0} U'_n$. If γ'' intersects G, then one of the sets $G_1, \ldots, G_{n_1}$ intersects γ'. If γ'' intersects $U'_{n'}$ $(1 \leqq n' \leqq n_0)$, then, since $\zeta \notin \bar{U}'_{n'}$, either $\zeta \in U''_{n'} - \bar{U}'_{n'}$, or one component of $U''_{n'} - \bar{U}'_{n'}$ is contained in γ' in which case one of the sets $G_1, \ldots, G_{n_2}$ intersects γ'. Thus, either $\zeta \in U^*$, or each open arc with length δ and one end point ζ intersects one of the sets $G_1, \ldots, G_{n_2}$ (recall that $n_2 \geqq n_1$), in which case $\zeta \in G^*$, since $\zeta \notin \bigcup_{n=1}^{n_2} \bar{G}_n$. Thus we have shown that all points of $\bar{E} - U$, with the exception of at most enumerably many, are in $U^* \cup G^*$.

Under φ_m, $U^* \cap (\bar{E} - U)$ corresponds to a set of measure less than $1/m$, and $G^* \cap (\bar{E} - U)$ corresponds to a set of measure less than $2\pi/m$. Hence $\bar{E} - U$ corresponds under φ_m to a set of measure less than $(2\pi + 1)/m$. Let $\Delta = \bigcap_{m=1}^{\infty} \Delta_m$. Then the boundary of Δ is a Jordan curve J, $\bar{E} \subset J$, and under a conformal mapping φ of Δ onto D' with $\varphi(0) = 0$, $\bar{E} - U$ corresponds to a set of measure zero. Since $m(E \cap U) = 0$, $E \cap U$ corresponds under φ to a set of measure zero. Therefore E corresponds under φ to a set of measure zero.

4. The approximation

We suppose, for the sake of our notation and without loss of generality, that we can associate with each G_n a natural number m_n such that

$$\Delta = D - \bigcup \bar{S}(G_n, r_{m_n}) .$$

For each G_n, choose an open arc γ_n of C such that $\bar{G}_n \subset \gamma_n$ and $\bar{\gamma}_n \subset V_n$, and let ϱ_n be a positive number such that

$$r_{m_n} - \frac{1}{n} < \varrho_n < r_{m_n} .$$

Set $\Delta^* = D - \bigcup \bar{S}(\gamma_n, \varrho_n)$. Then the boundary of Δ^* is a Jordan curve J^*, which we orient counterclockwise. Let

$$H = \{re^{i\vartheta} : 0 \leqq r < 1 , \ e^{i\vartheta} \in \bar{E}\} .$$

Then $H \subset \Delta^*$. Choose a sequence $\{\tau_n\}$ such that $\varrho_1 < \tau_n < \tau_{n+1} < 1$ $(n \geqq 1)$, $\lim \tau_n = 1$ and

$$\tau_n \notin \{\varrho_m : m = 1, 2, \ldots\} \quad (n \geqq 1).$$

Set $\Gamma_n = \{|z| = \tau_n\} \cap \overline{\Delta^*}$, and give the finitely many components of Γ_n the counterclockwise orientation. Define

$$C_1 = J^* \cap \{|z| \leqq \tau_1\}\,, \quad C_n = J^* \cap \{\tau_{n-1} \leqq |z| \leqq \tau_n\} \quad (n > 1)\,.$$

Let the components of C_n $(n \geqq 1)$ have the orientation induced by J^*. For any rectifiable oriented curve Γ in Δ, write

$$I(\Gamma\,; z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(\zeta)}{\zeta - z}\, d\zeta\,.$$

If we approximate $I(C_n; z)$ by means of a rational function having poles only on C_n, and apply Runge's pole-pushing lemma (see [2, p. 260]), we obtain a rational function $R_n(z)$, with poles only in the complement of D, such that

$$|R_n(z) - I(C_n\,; z)| < 1/2^{n+1} \text{ if } \begin{cases} z \in H \text{ and } n = 1\,, 2\,, \text{ or} \\ z \in H \cup \{|z| < \tau_{n-2}\} \text{ and } n > 2. \end{cases}$$

Let $f_n(z) = \sum\limits_{j=1}^{n} R_j(z) + I(\Gamma_n\,; z)\ (|z| < \tau_n)$. Then

$$|f_n(z) - g(z)| < 1/2 \quad \text{if} \quad z \in H \cap \{|z| < \tau_n\}\,.$$

Note that if n' and k are natural numbers with $n' > n$, and if $|z| < \tau_n$, then

$$\begin{aligned} |f_{n'+k}(z) - f_{n'}(z)| &= \left| \sum_{j=n'+1}^{n'+k} R_j(z) + I(\Gamma_{n'+k}\,; z) - I(\Gamma_{n'}\,; z) \right| \\ &= \left| \sum_{j=n'+1}^{n'+k} (R_j(z) - I(C_j\,; z)) \right| < 1/2^n\,. \end{aligned}$$

Set $f(z) = \lim f_n(z)$ $(z \in D)$. Then f is holomorphic in D and $|f(z) - g(z)| < 1$ if $z \in H$. This completes the proof of Theorem 1.

Remark. That this type of approximation is possible was learned by J. E. McMillan under the direction of G. R. MacLane. Stebbins [11] meanwhile obtained, independently and by a different method, a general approximation theorem which could be applied here to give the desired approximation.

5. Examples in conformal mapping

If we take E to be a perfect nowhere dense subset of C of positive measure, then it is clear from the foregoing construction of Δ and from [8, p. 138, Lemma 2] that the following is true.

Theorem 2. *There exists a Jordan curve J, with $J \subset C \cup D$, containing the origin in its interior Δ which is star-shaped with respect to the origin, and having a closed set E in common with C, such that if Δ and the exterior of J are mapped conformally onto the unit disk, then E corresponds to a set of measure zero and a set of positive measure, respectively, on the unit circle.*

Remark. In the construction of Δ_m in Section 3, the points of $E - U$ that are near U require special consideration, and it is evident how to simplify the argument considerably if E is nowhere dense. It is easy to see, and it will be evident from the proof of Theorem 3, that we can also require in Theorem 2 that $J \cap C = E$ and that each point of J be radially accessible from the origin.

A condensation process leads from Theorem 2 to the following result.

Theorem 3. *There exist a Jordan curve J containing the origin in its interior Δ such that every point of J is radially accessible from the origin, and an F_σ subset H of J, with the property that if Δ and the exterior of J are mapped conformally onto the unit disk, then H corresponds to a set of measure zero and a set of measure 2π, respectively, on the unit circle.*

Proof. Let $\{h_n\}$ be a sequence of positive real numbers such that $\Sigma h_n < 1$. Associate with each n a positive number δ_n such that if $r_2 - r_1 \geqq h_n/2$ $(0 < r_1 < r_2 \leqq 1)$ and $\Theta_2 - \Theta_1 \leqq \delta_n$ $(\Theta_1 < \Theta_2)$, then the harmonic measure of

$$\{re^{i\theta} : \ r = r_1 , r_2 ; \Theta_1 < \Theta < \Theta_2\}$$

with respect to

$$\{re^{i\theta} : \ r_1 < r < r_2 , \ \ \Theta_1 < \Theta < \Theta_2\}$$

is less than $1/n$ at each point of

$$\{re^{i\theta} : \ r = (r_1 + r_2)/2 , \ \ \Theta_1 < \Theta < \Theta_2\} .$$

We define sequences $\{J_n\}$ and $\{F_{n-1}\}$ inductively as follows. Let $J_1 = C$ and let $F_0 = \emptyset$. Suppose that a Jordan curve J_n has been defined such that

$$J_n \subset \left\{ 1 - \sum_{k=1}^{n-1} h_k \leqq |z| \leqq 1 \right\},$$

the interior Δ_n of J_n is star-shaped with respect to 0, and J_n is a finite union of non-radial rectilinear segments and arcs of circles with center 0; and suppose that F_{n-1} is a closed subset of J_n which is nowhere dense on J_n. Denote by ψ_n a conformal mapping of the exterior of J_n onto $\{|z'| < 1\}$

such that $\psi_n(\infty) = 0$, and let F_n be a perfect set on J_n such that $F_{n-1} \subset F_n$, F_n is nowhere dense on J_n, and

$$m(\psi_n(F_n)) > 2\pi - \frac{1}{n},$$

where $\psi_n(F_n)$ denotes the subset of $\{|z'| = 1\}$ that corresponds to F_n under ψ_n. Take $\{G_k\}$ to be an enumeration of the components of $J_n - F_n$, and set

$$J_n^* = \{(r - h_n)e^{i\theta} : \ re^{i\theta} \in J_n\}, \quad G_k^* = \{(r - h_n)e^{i\theta} : \ re^{i\theta} \in G_k\} \quad (k \geqq 1).$$

Let k_0 be a natural number and U_k be an open subarc of G_k^* $(k = 1, \ldots, k_0)$ such that $\bar{U}_k \subset G_k^*$ and the following conditions (i), (ii), and (iii) are satisfied:

(i) The diameter of every component of each of the sets

$$J_n^* - \bigcup_{k=1}^{k_0} \bar{U}_k \tag{1}$$

and

$$J_n - \bigcup_{k=1}^{k_0} \bar{G}_k \tag{2}$$

is less than $h_n/4$.

(ii) For each component γ of the set (1), there exist Θ_1 and Θ_2 such that $0 < \Theta_2 - \Theta_1 \leqq \delta_n$ and

$$\gamma \subset \{re^{i\theta} : \ 0 < r < 1, \ \Theta_1 < \Theta < \Theta_2\}.$$

(iii) For each $k = 1, \ldots, k_0$, the rectilinear segments l_k and r_k joining the left-hand (as viewed from the origin) end points of G_k and U_k and right-hand end points of G_k and U_k, respectively, intersect $J_n \cup J_n^*$ only in their end points.

Denote by D_k the interior of the Jordan curve that is the union of G_k, U_k, l_k and r_k. Set

$$\Delta_{n+1} = \Delta_n - \bigcup_{k=1}^{k_0} \bar{D}_k.$$

(Note that since F_n is a perfect set, the distance between any two distinct domains D_k is positive.) Let J_{n+1} be the Jordan curve that is the boundary of Δ_{n+1}. This completes the definition of $\{J_n\}$ and $\{F_{n-1}\}$.

We retain the notation used in the above definition, and take φ_{n+1} to be a conformal mapping of Δ_{n+1} onto $\{|z'| < 1\}$ such that $\varphi_{n+1}(0) = 0$. We now prove that

$$m(\varphi_{n+1}(F_n)) < 2\pi/n. \tag{3}$$

Denote by R an arbitrary component of the intersection of Δ_{n+1} and the exterior of J_n^*. The boundary of R is the union of a component γ_1 of the set (1), a component γ_2 of the set (2), and two rectilinear segments. Write

$$r_1 = \sup\{|z| : z \in \gamma_1\}, \quad r_2 = \inf\{|z| : z \in \gamma_2\}.$$

Then (i) implies that $r_2 - r_1 \geqq h_n/2$. Let Θ_1 and Θ_2 be such that $0 < \Theta_2 - \Theta_1 \leqq \delta_n$ and

$$\gamma_1 \subset \{re^{i\Theta} : \ 0 < r < 1, \ \Theta_1 < \Theta < \Theta_2\}.$$

Then $R \subset \{re^{i\Theta} : \ 0 < r < 1, \ \Theta_1 < \Theta < \Theta_2\}$. At each point of the circular arc

$$R \cap \{re^{i\Theta} : \ r = (r_1 + r_2)/2, \ \Theta_1 < \Theta < \Theta_2\}, \tag{4}$$

the harmonic measure u of the set (2) with respect to Δ_{n+1} is less than the harmonic measure of

$$\{re^{i\Theta} : \ r = r_1, r_2; \ \Theta_1 < \Theta < \Theta_2\}$$

with respect to

$$\{re^{i\Theta} : \ r_1 < r < r_2, \ \Theta_1 < \Theta < \Theta_2\}.$$

Thus $u(z) < 1/n$ if z is on the arc (4), and it follows that $u(0) < 1/n$. Thus, since all points of F_n, with the possible exception of only finitely many, are in the set (2), we have established (3).

Define $\Delta = \bigcap \Delta_n$, and let J be the Jordan curve that is the boundary of Δ. Then clearly every point of J is radially accessible from 0. If $H = \bigcup F_n$, then $H \subset J$. Let φ and ψ map the interior and exterior, respectively, of J conformally onto $\{|z'| < 1\}$ so that $\varphi(0) = 0$ and $\psi(\infty) = 0$. By the lemma of Löwner-Montel (stated in Section 3),

$$m(\varphi(F_n)) \leqq m(\varphi_{n+1}(F_n)) \quad (n \geqq 1).$$

Thus by (3),

$$m(\varphi(H)) = \lim m(\varphi(F_n)) = 0.$$

On the other hand, by first subjecting the plane to the transformation $1/z$ and then applying the lemma of Löwner-Montel, we see that

$$m(\psi_n(F_n)) \leqq m(\psi(F_n)) \quad (n \geqq 1).$$

Hence

$$2\pi - \frac{1}{n} < m(\psi_n(F_n)) \leqq m(\psi(F_n)) \leqq m(\psi(H)).$$

Therefore, $m(\psi(H)) = 2\pi$, and the proof of Theorem 3 is complete.

Remark. The foregoing constructions in Sections 3 and 5 could have been accomplished, less simply and directly, however, by combining geometrical arguments with a general theorem of Lavrentieff [5, p. 822, Theorem 1].

Theorem 4. *There exist a Jordan curve J containing the origin in its interior Δ such that every point of J is uniformly rectifiably accessible through Δ from the origin, and a subset K of J of positive area, with the property that if Δ is mapped conformally onto the unit disk, then K corresponds to a set of measure zero on the unit circle.*

Proof. For an arbitrary (closed) isosceles right triangle T and a positive number ε that is less than the length of the hypotenuse H of T, consider the open strip S of width ε such that the straight lines on the boundary of S are perpendicular to H and $T - S$ is the union of two congruent isosceles right triangles. Let $S_\varepsilon(T)$ be the intersection of S with the union of H and the interior of T. We shall choose a sequence $\{\varepsilon_n\}$ of positive numbers satisfying several restrictions, each of which will require that ε_n be small. We first require $\{\varepsilon_n\}$ to be such that the following construction is possible.

Let F_1 be the (closed) isosceles right triangle that has a diameter of the unit circle as its hypotenuse and is contained in the closed upper half-plane. Suppose that a closed set F_n has been defined so that its interior is the union of interiors of congruent (closed) isosceles right triangles $T_{n,j}$ $(j = 1, \ldots, j_n; j_n = 2^{n-1})$, and set

$$F_{n+1} = F_n - \bigcup_{j=1}^{j_n} S_{\varepsilon_n}(T_{n,j}) .$$

As in Knopp's triangle construction (see [4, p. 204]), $\bigcap F_n$ is a Jordan arc J', and if the numbers ε_n are sufficiently small, then J' has positive area. Let J denote the Jordan curve

$$J' \cup \{e^{i\Theta} : \pi \leqq \Theta \leqq 2\pi\} ,$$

and let Δ denote the interior of J. It is easy to see that each point of J' is accessible through Δ from the origin by a rectifiable curve that is composed of horizontal and vertical rectilinear segments and whose length is arbitrarily close to unity, and every other point of J is radially accessible from the origin.

Define a sequence $\{\Delta_n\}$ inductively as follows:

$$\Delta_1 = \{re^{i\Theta} : 0 < r < 1 , \ \pi < \Theta < 2\pi\} \cup S_{\varepsilon_1}(T_{1,1}) ,$$

$$\Delta_{n+1} = \Delta_n \cup \left(\bigcup_{j=1}^{j_{2n+1}} S_{\varepsilon_{2n+1}}(T_{2n+1,j}) \right) .$$

Then $\Delta = \bigcup \Delta_n$. Let J_n be the Jordan curve that is the boundary of Δ_n, and let φ_n and φ be conformal mappings of Δ_n and Δ, respectively, onto $\{|z'| < 1\}$ with $\varphi_n(0) = 0$ and $\varphi(0) = 0$. Now fix the sequence $\{\varepsilon_n\}$ so that J has positive area and for each n and $k = 1, \ldots, n$, the set

$$J_k \cap \left(\bigcup_{j=1}^{j_{2n+1}} S_{\varepsilon_{2n+1}}(T_{2n+1,j}) \right)$$

corresponds under φ_k to a set on $\{|z'| = 1\}$ of measure less than $1/2^n$. Note that each component of $J_k \cap \Delta$ is the intersection of J_k and one of the sets $S_{\varepsilon_{2n+1}}(T_{2n+1,j})$ $(n \geqq k)$. Hence, under φ_k, $J_k \cap \Delta$ corresponds to a set of measure less than $1/2^{k-1}$, and $J_k \cap J$ corresponds to a set of measure greater than $2\pi - (1/2^{k-1})$. By the lemma of Löwner-Montel, $J_k \cap J$ corresponds under φ to a set of measure greater than $2\pi - (1/2^{k-1})$. Therefore, under φ, the set

$$E = \left(\bigcup_{k=1}^{\infty} J_k \right) \cap J ,$$

which has zero area, corresponds to a set of measure 2π, and the set $K = J - E$, which has positive area, corresponds to a set of measure zero. This completes the proof of Theorem 4.

University of Wisconsin-Milwaukee

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