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DISTORTION THEOREMS FOR QUASICONFORMAL MAPPINGS

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DISTORTION THEOREMS FOR QUASICONFORMAL MAPPINGS*

The idea that quasiconformal mappings transform infinitesimal circles into infinitesimal ellipses with bounded eccentricity is quite familiar. It is certainly a consequence of Mori's estimate, [7], for a K-quasiconformal mapping $w = f(z)$ of a plane domain D onto a plane domain D': if $|\xi - \zeta| = |\eta - \zeta|$, if the disk $\{z: |z - \zeta| \leqq |\xi - \zeta|\}$ lies in D, and if the disk $\{w: |w - f(\zeta)| \leqq |f(\xi) - f(\zeta)|\}$ lies in D', then

$$\left|\frac{f(\xi) - f(\zeta)}{f(\eta) - f(\zeta)}\right| \leqq e^{\pi K}.$$

Gehring, [4], has shown that a definition of quasiconformality can be based on these notions. An orientation preserving homeomorphism f of a plane domain D is K-quasiconformal, $1 \leqq K$, if and only if

$$H_f(\zeta) = \limsup_{\substack{|\xi - \zeta| = r \\ |\eta - \zeta| = r \\ r \to 0}} \left|\frac{f(\xi) - f(\zeta)}{f(\eta) - f(\zeta)}\right|$$

is bounded in D, and a.e. $\leqq K$.

A substantially different approach to quasiconformal mappings is through the Beltrami equation

$$f_{\bar{z}} = \chi f_z, \tag{1.1}$$

satisfied weakly by each K-quasiconformal mapping f, with χ measurable, $|\chi(z)| \leq k < 1$ a.e. in D, $\dfrac{1+k}{1-k} = K$. Conversely, [2], [5], given such χ, there exists a weak solution f of (1.1), which is K-quasiconformal and unique in the sense that if g is another solution, $f \circ g^{-1}$ is conformal in $g(D)$. If we assume that D is the finite plane, then the image $f(D)$ will also be the finite plane, and the allowable normalization $f(0) = 0$, $f(1) = 1$, assures that f is unique. We will denote this unique, normalized solution of (1.1) by f^{χ}.

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Various investigations have centered on determining the maximum for the critical ratio

$$R = \left| \frac{f(\xi) - f(\zeta)}{f(\eta) - f(\zeta)} \right| , \tag{1.2}$$

although it is clear that without some normalization the ratios will be unbounded even in the class of conformal (1-quasiconformal) mappings. For example, we may denote by N_1 the following problem: *For* $a \geq 1$, $K \geq 1$, *in the class* Q_1 *of* K*-quasiconformal mappings* φ *of the unit disk onto itself, with* $\varphi(0) = 0$, *find*

$$P_1 = P_1(a\,,K) = \sup \left\{ \left| \frac{\varphi(\xi)}{\varphi(\eta)} \right| : \left| \frac{\xi}{\eta} \right| = a \right\} .$$

Shah and Fan, [10], solved problem N_1 by the method of parametric representation, [9], in the following implicit form: If η is defined by

$$\eta(x) = \frac{(1+x)}{2\pi} \iint\limits_{\xi\text{-plane}} \frac{d\sigma(\xi)}{|\xi||1-\xi||\xi+x|} , \tag{1.3}$$

then $y = P_1(a\,,K)$ is the solution to the equation

$$\int_a^y \frac{dx}{x\eta(x)} = \log K . \tag{1.4}$$

A related problem was to determine the supremum of the numbers $H_f(\zeta)$, among all K-quasiconformal mappings f. For this purpose, Lehto, Virtanen, and Väisälä, [6], solved (for $a = 1$) the following normalized problem, which we denote by N_2: *For* $a \geq 1$, $K \geq 1$, *in the class* Q_2 *of* K*-quasiconformal mappings* φ *of the finite plane onto itself, with* $\varphi(0) = 0$, $\varphi(1) = 1$, *find*

$$P_2 = P_2(a\,,K) = \sup \{ |\varphi(\xi)| : |\xi| = a \} .$$

This problem is of course equivalent to the problem of maximizing the ratios R in (1.2), for f K-quasiconformal in the finite plane, with

$$\left| \frac{\xi - \zeta}{\eta - \zeta} \right| = a .$$

It comes as only a mild surprise that the solutions to problems N_1 and N_2 are the same. In this note, we calculate $P_2(a\,,K)$ in a reasonably explicit form. We then establish the equivalence of the problems from an abstract viewpoint, and finally show that $y = P_2(a\,,K)$ also solves (1.4).

Our first task, however, will be to derive an integral representation for the hyperbolic density, which will be needed in the computations.

2.1. *Hyperbolic densities and distances:* For a domain E consisting of the extended z-plane minus n points $\{z_1, z_2, \ldots, z_n\}$, $n \geq 3$, we represent the universal covering surface by the upper half plane $\{\mathrm{Im}(w) > 0\}$, and let $z = J(w)$ be an analytic covering. We define the hyperbolic density ϱ_E in E by

$$\varrho_E(z) = \left|\frac{df(z)}{dz}\right| / \mathrm{Im}(f(z)) , \tag{2.1}$$

where f is a local inverse for J. The right side of (2.1) is independent of both J and the branch f. The hyperbolic distance σ_E, is then defined for points Z', Z'' in E, by

$$\sigma_E(Z', Z'') = \inf \int_\gamma \varrho_E(z)|dz| ,$$

where the infimum is taken over the class of arcs γ joining Z' and Z'' in E, for which the integral has meaning.

In the special case $E_0 : \{z_1, z_2, z_3\} = \{0, 1, \infty\}, n = 3$, a suitable covering J is the familiar elliptic modular function, [8]. We calculate the hyperbolic density ϱ_{E_0}, hereafter referred to simply as ϱ, and the corresponding hyperbolic distance σ, for certain pairs of points.

2.2. *The Integral Representation:* Let D be the domain obtained by deleting from the z-plane the real slits $\{z \leq 0\}$ and $\{z \geq 1\}$. For $z = re^{i\Theta} \in D$, $-\pi < \Theta < \pi$, set $\sqrt{z} = \sqrt{r}\, e^{i\Theta/2}$. We consider the Jacobian elliptic function $\zeta = \mathrm{sn}(u , \sqrt{z})$, doubly periodic in u. In D, we may regard its primitive periods $4K(z), 2iK^*(z)$ as single valued analytic continuations of

$$K(z) = \int_0^1 \frac{dt}{\{(1-t^2)(1-zt^2)\}^{1/2}} , \quad 0 < z < 1 , \tag{2.2}$$

$$K^*(z) = K(1-z) .$$

In this section, we use the symbol $*$ to denote replacement of the argument z by the argument $1-z$, and $'$ to denote differentiation with respect to z, hence $(K^*)' = -(K')^*$.

It is well known that $\zeta = \mathrm{sn}(u , \sqrt{z})$ maps the interior of the parallelogram P, whose vertices are $\pm K \pm iK^*$, conformally onto the ζ-plane minus four analytic arcs, with

$$\left(\frac{d\zeta}{du}\right)^2 = (1 - \zeta^2)(1 - z\zeta^2)\,.$$

The area of the parallelogram P is easily seen to be $4\,\mathrm{Im}(iK^*\bar{K}) = 4\,\mathrm{Re}\,(K^*\bar{K})$. We therefore find

$$(2.3)\qquad 4\,\mathrm{Re}(K^*\bar{K}) = \iint\limits_P d\sigma(u) = \iint\limits_{\zeta\text{-plane}} \left|\frac{du}{d\zeta}\right|^2 d\sigma(\zeta)$$
$$= \iint\limits_{\zeta\text{-plane}} \frac{d\sigma(\zeta)}{|1-\zeta^2|\,|1-z\zeta^2|}$$
$$= \frac{1}{2}\iint\limits_{\xi\text{-plane}} \frac{d\sigma(\xi)}{|1-\xi|\,|\xi|\,|\xi - z|}\,.$$

For the last step it must be remembered that in the transformation $\xi = 1/\zeta^2$, each point in the ξ-plane is covered twice.

If $z = J(w)$ is the elliptic modular function, then a local inverse in $\{\mathrm{Im}(z) > 0\}$ is given by

$$f(z) = iK^*(z)/K(z)\,,$$

[8], and hence in $\{\mathrm{Im}(z) > 0\}$, we have

$$(2.4)\qquad \varrho(z) = \left|\frac{d}{dz}\left(\frac{iK^*}{K}\right)\right| \Big/ \mathrm{Im}\left(\frac{iK^*}{K}\right)$$
$$= \frac{|K(K^*)' - K^*K'|/|K|^2}{\mathrm{Re}(K^*\bar{K})/|K|^2}$$
$$= |K(K')^* + K^*K'\,|\,/\mathrm{Re}(K^*\bar{K})\,.$$

We contend that

$$(2.5)\qquad K(K')^* + K^*K' = \pi/4z(1-z)\,,$$

and since both sides are analytic in D, it is sufficient to check for $0 < z < 1$, where we have the explicit representation (2.2). We use the classical formula [3] for $0 < r < 1$,

$$\frac{dK(r^2)}{dr} = \frac{E(r^2) - (1-r^2)\,K(r^2)}{r(1-r^2)}$$

where

$$E(z) = \int_0^1 \frac{(1-zt^2)^{1/2}\,dt}{(1-t^2)^{1/2}}\,.$$

Setting $z = r^2$, we find

$$K' = \frac{dK(r^2)}{dr}\frac{dr}{dz} = \frac{E(z) - (1 - z)\,K(z)}{2z(1 - z)}$$

hence

$$K(K')^* + K'K^* = \frac{K(E^* - zK^*) + K^*(E - (1 - z)\,K)}{2z(1 - z)}$$

$$= \frac{KE^* + K^*E - KK^*}{2z(1 - z)} .$$

Applying Legendre's formula, $KE^* + K^*E - KK^* = \pi/2$, (2.5) follows at once.

We thus obtain, from (2.3), (2.4), and (2.5), for $\mathrm{Im}(z) > 0$

$$\varrho(z) = \left\{\frac{|z|\,|1 - z|}{2\pi}\iint\limits_{\xi\text{-plane}} \frac{d\sigma(\xi)}{|\xi|\,|1 - \xi|\,|\xi - z|}\right\}^{-1} . \tag{2.6}$$

Since both sides of (2.6) are unchanged if z is replaced by $1-z$, the formula holds for $\mathrm{Im}(z) \neq 0$, and by continuity for $z \neq 0, 1, \infty$.

2.3. *An inequality:* For $0 < r < 1$, the real ratio μ, defined by

$$\mu(r) = \frac{-\,i\pi f(r^2)}{2} = \frac{\pi K^*(r^2)}{2K(r^2)} ,$$

is equal to the modulus of the ring domain obtained by deleting from the unit disk the real interval $[0, r]$. μ is strictly decreasing, with limits ∞, 0 at $0, 1$ respectively, and its inverse will be denoted by μ^{-1}.

The important inequality ([6], page 6),

$$\varrho(z) \geq \varrho(-\,|z|)$$

may be derived from (2.6) as follows: we first observe that for any circle C, and any complex number w, the integral

$$\int\limits_{C} \frac{|d\zeta|}{|\zeta - w|}$$

depends only on the distance from w to C, increasing as $w \to C$ from inside, and increasing as $w \to C$ from outside. If we denote by C_R the circle

$$\zeta = \frac{1}{1 - Re^{i\Theta}} \; ; \; 0 \leq \Theta \leq 2\pi ,$$

and if w is on C_r, it then follows that

$$\int_{C_R} \frac{|d\zeta|}{|\zeta - w|} \leq \int_{C_R} \frac{|d\zeta|}{\left|\zeta - \dfrac{1}{1+r}\right|}\,.$$

But introducing polar coordinates $\xi = Re^{i\Theta}$ in (2.6), and setting $z = re^{i\varphi}$, we find

$$\frac{1}{\varrho(z)} = \frac{|z|}{2\pi} \int_0^\infty \frac{dR}{R} \int_0^{2\pi} \left|\frac{(1-\xi)(1-z)}{(\xi - z)}\right| \frac{R d\Theta}{|1-\xi|^2}$$

$$= \frac{|z|}{2\pi} \int_0^\infty \frac{dR}{R} \int_0^{2\pi} \frac{1}{\left|\dfrac{1}{1 - Re^{i\Theta}} - \dfrac{1}{1-z}\right|} \frac{R d\Theta}{|1 - Re^{i\Theta}|^2}$$

$$= \frac{r}{2\pi} \int_0^\infty \frac{dR}{R} \int_{C_R} \frac{|d\zeta|}{\left|\zeta - \dfrac{1}{1 - re^{i\varphi}}\right|}$$

$$\leq \frac{|-r|}{2} \int_0^\infty \frac{dR}{R} \int_{C_R} \frac{|d\zeta|}{\left|\zeta - \dfrac{1}{1+r}\right|} = \frac{1}{\varrho(-r)}\,.$$

We draw two conclusions: first, that the negative real axis is a geodesic for σ, and second that if $a_i = |z_i|$,

$$\sigma(z_1\,, z_2) \geq \sigma(-a_1, -a_2) \tag{2.7}$$

Hence to obtain a formula for the right hand side of (2.7), we may integrate ϱ along α, the negative real axis between $-a_1$ and $-a_2$. On α, $-if'(z) > 0$ and $\mathrm{Re}(f(z)) = 1$, hence $|f'(z)| = -if'(z)$, and $\mathrm{Im}(f(z)) = -i(f(z) - 1)$. We find

$$\int_\alpha \varrho(z)|dz| = \int_\alpha \frac{|f'(z)|\,|dz|}{\mathrm{Im}(f(z))}$$

$$= \left|\int_{a_1}^{a_2} \frac{f'(-t)dt}{1 - f(-t)}\right|$$

$$= \left|\log(1 - f(-t))\Big]_{a_1}^{a_2}\right|$$

$$= \left|\log \frac{1 - f(-a_2)}{1 - f(-a_1)}\right|\,.$$

Since the mapping f satisfies the identity

$$\frac{1}{1-f(z)} = f\left(\frac{1}{1-z}\right),$$

we conclude

$$\sigma(-a_1, -a_2) = \left|\log \frac{\mu(\{1+a_2\}^{-1/2})}{\mu(\{1+a_1\}^{-1/2})}\right| . \tag{2.8}$$

2.4. *Teichmüller's Theorem:* A fundamental theorem of Teichmüller, [11], [1], asserts: *Given* $z_0, w_0 \in E_0$, $K \geq 1$, *there exists* $\varphi \in Q_2$ *with* $\varphi(z_0) = w_0$ *if and only if* $\sigma(z_0, w_0) \leq \log K$. As a second application of (2.6), we use the results of Ahlfors and Bers [2] to prove the »only if» part of this theorem. We suppose that $\varphi = f^{\chi}$, and there is no loss in generality in assuming that χ is continuous with compact support. For $0 \leq t \leq 1$ let $f(z, t) = f^{t\chi}(z)$. Then lemmas 19 and 21 of [2] apply, and we may assert that $f(z, t)$ is differentiable in t, and

$$\frac{\partial f(z,t)}{\partial t} = (Pb_t)(w) - w(Pb_t)(1);\ w = f(z,t) , \tag{2.9}$$

where

$$b_t(w) = \frac{\chi(z)}{1 - t^2|\chi(z)|^2} \frac{f_z(z,t)}{\bar{f}_{\bar{z}}(z,t)};\ w = f(z,t) , \tag{2.10}$$

and P is the Hilbert transform,

$$(Pg)(w) = \frac{1}{\pi} \iint_{\xi\text{-plane}} g(\xi) \left(\frac{1}{\xi} - \frac{1}{\xi - w}\right) d\sigma(\xi) . \tag{2.11}$$

Since $\bar{f}_{\bar{z}} = \overline{f_z}$, we obtain from (2.10) the simple inequality

$$\sup_w |b_t(w)| \leq \frac{k}{1 - t^2k^2} , \tag{2.12}$$

while (2.11) yields easily, with (2.6), the inequality

$$|(Pg)(w) - w(Pg)(1)| = \left|\frac{w(w-1)}{\pi} \iint_{\xi\text{-plane}} \frac{g(\xi)d\sigma(\xi)}{\xi(\xi - w)(\xi - 1)}\right| \leq \frac{2 \sup|g|}{\varrho(w)} . \tag{2.13}$$

As a competing path from $z = f(z, 0)$ to $\varphi(z) = f(z, 1)$, we take the trace of $f(z, t)$: $0 \leq t \leq 1$. Evidently, using (2.9), (2.13), and (2.12),

$$\sigma(z, \varphi(z)) \leq \int_0^1 \varrho(f(z\ , t)) \left| \frac{\partial f(z, t)}{\partial t} \right| dt$$

$$\leq \int_0^1 \frac{2k dt}{1 - t^2 k^2} = \log \frac{1 + tk}{1 - tk} \Big]_0^1 = \log \frac{1 + k}{1 - k} = \log K .$$

3.1. *Problem* N_2*:* We now can assert that

$$P_2(a, K) = [\mu^{-1}(K\mu(\{1 + a\}^{-1/2}))]^{-2} - 1 , \tag{3.1}$$

or equivalently, in view of (2.8),

$$\sigma(-a, -P_2) = \log K .$$

For let P_2^* be defined by the right hand side of (3.1), which is to say

$$\sigma(-a, -P_2^*) = \log K, P_2^* \geq a .$$

By the »if» part of Teichmüller's theorem, there exists $\varphi^* \in Q_2$, with

$$\varphi^*(-a) = -P_2^* .$$

Consequently,

$$P_2 \geq |\varphi^*(-a)| = |-P_2^*| = P_2^* . \tag{3.2}$$

On the other hand, given $\varphi \in Q_2$, $|\xi| = a$, we find from (2.7) and Teichmüller's theorem, $\sigma(-|\varphi(\xi)|, -a) \leq \sigma(\varphi(\xi), \xi) \leq \log K = \sigma(-P_2^*, -a)$. It follows that $|\varphi(\xi)| \leq P_2^*$, hence $P_2 \leq P_2^*$, and with (3.2), the formula is verified.

3.2. *Problem* N_1*:* For the mapping φ^* of Section 3.1, and large integers n, let φ_n be defined for $|z| \leq 1$ by

$$\varphi_n(z) = f_n(\varphi^*(nz))/a_n ,$$

where f_n is a conformal mapping of $\{\varphi^*(\zeta): |\zeta| \leq n\}$ onto $\{|f_n| \leq a_n\}$, normalized by $f_n(0) = 0$, $f_n(1) = 1$. By virtue of this normalization, the $\{f_n\}$ are a normal family in E_0, and any limit function is necessarily the identity. By construction, $\varphi_n \in Q_1$, and therefore

$$P_1 \geq \left| \frac{\varphi_n(-a/n)}{\varphi_n(1/n)} \right| = \left| \frac{f_n(\varphi^*(-a))}{f_n(\varphi^*(1))} \right| = |f_n(-P_2)| .$$

Letting $n \to \infty$, we conclude

$$P_1 \geq \lim |f_n(-P_2)| = |-P_2| = P_2 . \tag{3.3}$$

On the other hand, any $\varphi \in Q_1$ can be extended by reflection and rotation to a mapping $\varphi_0 \in Q_2$, and with corresponding ratios equal. It follows that $P_1 \leq P_2$, and with (3.3), the equivalence of problems N_1 and N_2 is established.

3.3. *Remark:* Returning to (1.4), we see from (2.6) and (1.3) that

$$1/x\eta(x) = \varrho(-x) ,$$

and hence, as expected,

$$\int_a^{P_2} \frac{dx}{x\eta(x)} = \int_a^{P_2} \varrho(-x)dx = \int_{-P_2}^{-a} \varrho(t)dt$$

$$= \sigma(-a, -P_2) = \log K .$$

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