ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

417

PICARD SETS FOR MEROMORPHIC FUNCTIONS

BY

SAKARI TOPPILA

HELSINKI 1967
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1968.417

Communicated 10 October 1967 by K. I. VIRTANEN and LAURI MYRBERG

KESKUSKIRJAPAINO
HELSINKI 1967

Preface

I am deeply indebted to Professor Olli Lehto and Professor K. I. Virtanen for suggesting this subject and for their kind interest and valuable advice.

I also wish to thank the E. J. Sariolan Säätiö for financial support.

Alavus, September 1967.

SAKARI TOPPILA

Contents

1. Introduction

1.1. Let us consider functions f meromorphic in the complement of a compact and totally disconnected set E in the extended complex plane. We call E an n-Picard set in the sense of Lehto if every f with at least one essential singularity in E omits at most n values in the intersection of $-E$ and an arbitrary neighbourhood of any singularity.

E is called an n-Pichard set in the sense of Matsumoto if every f with a singularity at all points of E omits at most n values in the intersection of $-E$ and an arbitrary neighbourhood of any point of E.

In both cases, a 2-Picard set is called briefly a Picard set.

An n-Picard set in Lehto's sense is of course one in Matsumoto's sense. The converse is not true: The set

$$E = \{\infty\} \cup \{2n\pi i\}_{n=0,\pm 1,\pm 2,\ldots}$$

is not a Picard set in Lehto's sense, for $e^z \neq 0, 1, \infty$ in $-E$, but it is a Picard set in Matsumoto's sense because E has isolated points in every neighbourhood of any of its points.

1.2. The term Picard set was first used by Lehto. In [2] he proved that there exist sets with an infinite number of points which are Picard sets in his sense. Carleson [1] proved that there exist 3-Picard sets in Lehto's sense which are of positive capacity. Matsumoto [4—6] extended these results and proved that there exist perfect Picard sets in his and in Lehto's sense.

In this paper we give in Section 2 a sufficient condition for a countable set with one limit point to be a Picard set in Lehto's sense. An example shows that the condition cannot be improved. In Section 3 we show that by adding points we can make any totally disconnected compact set into a perfect Picard set in the Matsumoto sense. In order to achieve monotonicity, i.e. that A a Picard set implies $B \subset A$ a Picard set, we modify Matsumoto's definition in Section 4 and study the relationship of these new Picard sets with those of Matsumoto and Lehto.

2. Picard sets in Lehto's sense

2.1. Let $\{a_n\}_{n=1,2,\ldots}$ be a point set whose points converge to infinity, and let E denote the union of $\{a_n\}_{n=1,2,\ldots}$ and the point at infinity. Lehto [2] proved that E is a Picard set in his sense if the points a_n satisfy the condition

$$(\log |a_n|)^{2+\alpha} = O(\log |a_{n+1}|) \quad (\alpha > 0) .$$

Matsumoto [6] established the same result under the condition

$$(1) \qquad |a_n|^3 = O(|a_{n+1}|) .$$

We first show by an example that the exponent in condition (1) cannot be made smaller than 2, and we then prove that it really can be replaced by 2.

2.2. We begin by presenting three lemmas which are essentially due to Carleson [1]. Let Σ be the Riemann sphere with radius 1/2 touching the w-plane at the origin. The chordal distance of the images on Σ of two points w and w' in the plane is denoted by $[w, w']$, and $C(w, \delta)$ is the spherical open disc with centre at the image of w and with chordal radius δ.

Lemma 1. Let f be analytic in an annulus $1 < |z| < e^{\mu}$ and omit the values 0 and 1. There exists a positive constant A such that the spherical diameter of the image curve of $|z| = e^{\mu/2}$ by f is not greater than $Ae^{-\mu/2}$ for all $\mu > 0$.

Proof. The lemma is proved by Matsumoto [6]. (See also Carleson [1], Matsumoto [5] and Sario-Noshiro [9].)

Lemma 2. Let f be analytic in a closed annulus $r \leq |z| \leq R$. If $|f(z)| \leq m$ on $|z| = r$ and $|f(z)| \leq M$ on $|z| = R$ then the euclidian diameter of the image curve of $|z| = \varrho$, $r < \varrho < R$, by f is dominated by

$$\frac{\pi m r}{\varrho(1 - r/\varrho)^2} + \frac{\pi M \varrho}{R(1 - \varrho/R)^2} .$$

Proof. By Cauchy's integral theorem we have

$$f'(z) = \frac{1}{2\pi i}\left\{\int\limits_{|t|=R} f(t)(t - z)^{-2}dt - \int\limits_{|t|=r} f(t)\,(t - z)^{-2}dt\right\}$$

for every z on $|z| = \varrho$, so that

$$|f'(z)| \leq mr(\varrho - r)^{-2} + MR(R - \varrho)^{-2} .$$

For any z and z_0 on $|z| = \varrho$ this implies

$$|f(z) - f(z_0)| \leq \pi m \varrho r(\varrho - r)^{-2} + \pi M \varrho R(R - \varrho)^{-2},$$

and the lemma is proved.

Let Δ be a triply connected domain with boundary components Γ_1, Γ_2 and Γ_3, and let f be analytic and omit the values 0 and 1 in $\bar{\Delta}$. We assume that the images of Γ_1, Γ_2 and Γ_3 by f are contained in the spherical discs C_1, C_2 and C_3, respectively, and give the following lemma of Matsumoto [6].

Lemma 3. Let $\delta > 0$ be so small that the spherical discs $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. If the radii of C_1, C_2 and C_3 are less than $\delta/2$, only two possibilities can occur:
(1) C_1, C_2 and C_3 contain the origin, the point $w = 1$, and the point at infinity, one by one, so that C_1, C_2 and C_3 are contained in $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively, and f takes each value outside the union of $C(0, \delta), C(1, \delta)$ and $C(\infty, \delta)$ once and only once in Δ.
(2) Of C_1, C_2 and C_3 none can be disjoint from the union of the other two, so that there is a disc with radius less than $3\delta/2$ which contains the image of Δ.

2.3. We can now construct the desired counter example:

Theorem 1. For each $\varepsilon > 0$ there exists a point set

$$E = \{a_n : n = 1, 2, \ldots\} \cup \{\infty\}$$

for which

$$|a_n|^{2-\varepsilon} = O(|a_{n+1}|)$$

but which is not a Picard set in the sense of Lehto.

Proof. We construct the desired set E with the aid of the function

$$f(z) = z \prod_{n=1}^{\infty} \left(\frac{1 - ze^{-M^{2n}}}{1 - ze^{-M^{2n-1}}} \right)^2,$$

where we first take $M \geq 5$.

Let z, $|z| > e^M$, be an 1-point of f. We choose n such that $z \in \bar{S}_n$, where

$$S_n = \{z : \exp[(M^n + M^{n-1})/2] < |z| < \exp[(M^n + M^{n+1})/2]\}.$$

If $n = 2p$, we note

$$\log\left| z \prod_{s=1}^{p-1} \left(\frac{1 - ze^{-M^{2s}}}{1 - ze^{-M^{2s-1}}} \right)^2 \right| = \log|z| - \frac{2M^{2p-1}}{M+1} + O(1)$$

where $O(1)$ is bounded when $n \to \infty$, and

$$\log \prod_{s=p+1}^{\infty} \left| \frac{1 - ze^{-M^{2s}}}{1 - ze^{-M^{2s-1}}} \right|^2 = O(1) .$$

Further, we have

$$\log\left| \frac{1 - ze^{-M^{2p}}}{1 - ze^{-M^{2p-1}}} \right|^2 = -2 \log|z| + 2M^{2p-1} + 2 \log|1 - ze^{-M^{2p}}| + O(1) .$$

Combining these results we get

$$- \log|z| + \frac{2M^{2p}}{M+1} + 2 \log |1 - ze^{-M^{2p}}| + O(1) = 0 .$$

This is possible only for $\log |z| > M^{2p} + O(1)$ and we get the estimate

$$|z| = \exp\left\{ 2M^{2p} - \frac{2M^{2p}}{M+1} + O(1) \right\} .$$

In the same manner we get for $n = 2p + 1$ a similar estimate, so that an arbitrary 1-point of f, $z \in \bar{S}_n$, satisfies the condition

$$\text{(i)} \qquad |z| = \exp\left\{ 2M^n - \frac{2M^n}{M+1} + O(1) \right\} .$$

On the other hand, since $f(z) \geq 0$ on the positive real axis, $f(e^{M^{2n}}) = 0$ and $f(e^{M^{2n-1}}) = \infty$, $n = 1, 2, \ldots$, we see that f has at least one 1-point in each annulus $e^{M^n} < |z| < e^{M^{n+1}}$.

We take a $\mu > 0$, such that $\delta = Ae^{-\mu/2}$, where A is the constant of Lemma 1, is so small that the spherical discs $C(0, 8\delta)$, $C(1, 8\delta)$ and $C(\infty, 8\delta)$ are mutually disjoint. From condition (i) it then follows that the annulus $\exp(-\mu + M^n) < |z| < \exp(\mu + M^n)$ contains no 1-point of f for sufficiently large values of n, say $n \geq n_1$.

Let

$$t_n = \{z : |z| = \exp(-\mu/2 + M^n)\} ,$$
$$T_n = \{z : |z| = \exp(\mu/2 + M^n)\} ,$$

and denote by R_n be the triply connected domain buonded by t_n, $\{e^{M^n}\}$, and T_n. We conclude from Lemma 1 that for $n \geq n_1$, the curves $f(t_n)$ and $f(T_n)$ are contained in some spherical discs c_n and C_n with radius δ.

2.4. Let us suppose that the boundary components of some R_{2n+1}, $2n \geq n_1$, are mapped by f into $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$, respectively. By Lemma 3 we see that f takes each value outside the union of $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ exactly once in R_{2n+1}. Since $e^{M^{2n+1}}$ is a pole of order two, f takes a value w outside the union of $\{\infty\}$, $C(0, 2\delta)$ and $C(1, 2\delta)$ at two points z' and z'' of R_{2n+1}. We join w to $C(0, 2\delta)$ with a curve Λ which lies outside this union and does not pass through any point which is the projection of a branch point of the Riemann surface $f(R_{2n+1})$. The elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along Λ to its end point and, since $f(\bar{R}_{2n+1} - R_{2n+1})$ is contained in the union of $\{\infty\}$, $C(0, 2\delta)$, and $C(1, 2\delta)$, we see that every value on Λ is taken at two points of R_{2n+1}. This is not possible for $w \in \Lambda$, $[w, \infty] > 2\delta$. By means of the linear transformation $1/f$ we get the same contradiction if R_{2n+1}, $2n \geq n_1$, is replaced by R_{2n}, $2n \geq n_1$.

2.5. In view of Lemma 3, it follows from the considerations in 2.4 that the discs c_{2n} and C_{2n}, $2n \geq n_1$, are contained in $C(0, 4\delta)$. We see now in the same manner as in 2.4, since $e^{M^{2n+1}}$ is a pole of order two that f takes each value outside $C(0, 4\delta)$ exactly twice in the annulus bounded by T_{2n} and t_{2n+2}, and we note that f has exactly one 1-point in the closed region $e^{M^n} \leq |z| \leq e^{M^{n+1}}$ for $n \geq n_1$.

Let now $\{a_n\}_{n=1,2,\ldots}$, $|a_1| \leq |a_2| \leq \ldots$, be the set of the zeros, 1-points, and poles of f. It follows from the above and (i) that for any $\varepsilon < 0$, we can take M so large that the numbers a_n satisfy the condition

$$|a_n|^{2-\varepsilon} = O(|a_{n+1}|) .$$

The set $E = \{\infty\} \cup \{a_n\}_{n=1,2\ldots}$ thus provides the desired example, and Theorem 1 is proved.

2.6. In view of Theorem 1 it is of interest to show that Matsumoto's condition (1), quarantering E to be a Picard set, can be improved to $|a_n|^2 = O(|a_{n+1}|)$. In order to prove this result we need an estimate for the modulus of a ring domain.

Lemma 4. Let a and b be two points such that $|a| < |b|$, and A a ring domain such that one component of its complement contains the point 0 and a and the other the points b and ∞. Then

$$\operatorname{mod} A \leq \log (32 |b/a|) .$$

Proof. The modulus of A is majorized by the modulus of the Teichmüller ring T with the boundary components $\{x : -|a| \leq x \leq 0\}$ and $\{x : x \geq |b|\}$. Since

$$\mathrm{mod}\, T \leq \log (16(|b/a| + 1)) ,$$

the lemma follows. (For the details we refer to Lehto—Virtanen [3] pp. 58 and 64.)

2.7. We can now give the above mentioned complement for Theorem 1:

Theorem 2. If the points a_n satisfy the condition

$$|a_n|^2 = O(|a_{n+1}|) , \tag{2}$$

then $E = \{a_n : n = 1, 2, \ldots\} \cup \{\infty\}$ is a Picard set in Lehto's sense.

2.8. *Proof.* It is obviously sufficient to prove that the assumption of the existence of a function f, meromorphic and non-rational for $z \neq \infty$, and different from 0, 1 and ∞ outside of E, leads to a contradiction. There is no loss of generality to assume that the set $\{a_n\}$ consists only of the zeros, 1-points and poles of f, for we can delete from $\{a_n\}$ all other points and the remaining points also satisfy the condition (2).

Applying Lemma 1 to the annulus $|a_n| \leq |z| \leq |a_{n+1}|$, we conclude that the diameter of the image of $\Gamma_n = \{z : |z| = |a_n a_{n+1}|^{1/2}\}$ by f is dominated by $\delta_n = A\,|a_n/a_{n+1}|^{1/2}$ for all n. Hence there exists a spherical disc C_n with radius less than δ_n which contains this image.

We take $\delta > 0$ so small that the discs $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. By the condition (2) there exists an $M > 0$ such that $|a_{n+2}|^2 < M|a_{n+3}|$ for any n. Therefore

$$\frac{|a_{n+2}|}{|a_{n+3}|} < \frac{|a_{n+1}|}{|a_{n+2}|}\,\frac{M}{|a_{n+1}|} .$$

We choose an n_0 so large that

$$12 \cdot 2400\, \pi\, A\, |a_{n+2}/a_{n+3}|^{1/4} < |a_{n+1}/a_{n+2}|^{1/4} \tag{a}$$

and $\delta_n < \delta/8$ for any $n \geq n_0$.

2.9. Let Δ_n be the domain with boundary components Γ_n, Γ_{n+1} and $\{a_{n+1}\}$. Suppose that there exists no Δ_n, $n \geq n_0$, whose boundary components are mapped into $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively. It follows from Lemma 3 that $f(\Gamma_{n_0} \cup \Gamma_{n_0+1})$ is contained in one of the spherical discs $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, say in $C(0, \delta)$. Lemma 3 applied to the region Δ_{n_0+1} gives $f(a_{n_0+1}) = f(a_{n_0+2}) = 0$, and we conclude by induction that $f(a_{n_0+p}) = 0$ for every $p \geq 1$. Then f is bounded in

$|z| \geq |a_{n_0+1}|$, and the point at infinity is no essential singularity of f. This is a contradiction, and so there is a $\Delta_n, n \geq n_0$, whose boundary components Γ_n, $\{a_{n+1}\}$ and Γ_{n+1} are mapped into $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively. We may assume that $C_n \subset C(0, \delta)$, $f(a_{n+1}) = 1$ and $C_{n+1} \subset C(\infty, \delta)$.

Let $\Delta = \Delta_n \cup \Delta_{n+1} \cup \Gamma_{n+1}$. Since the image of the boundary component Γ_{n+2} of Δ is contained in the spherical disc C_{n+2} with radius less than $\delta_{n+2} < \delta/8$ we see, by applying Lemma 3 to f in Δ_{n+1} and the maximum principle to f in Δ, that f has a pole at a_{n+2} and $C_{n+1} \cup C_{n+2}$ is contained in $C(\infty, 4\delta_{n+1})$.

2.10. We now modify the proof given by Matsumoto in [6] by considering the circles $\gamma_n = \{z : |z| = \varrho_n\}$, where $\varrho_n = |a_n a_{n+1}^2 a_{n+2}|^{1/4}$. By virtue of the condition (a) we obtain $\gamma_{n+1} \subset S_{n+2} \cap \Delta_{n+1}$, where $S_n = \{z : |a_n| < |z| < |a_{n+1}|\}$. Since $C_{n+1} \cup C_{n+2} \subset C(\infty, 2\delta)$, it follows from the maximum principle that $f(\gamma_{n+1})$ is contained in a $C(\infty, d)$ with radius $d = \sup_{z \in \gamma_{n+1}} [f(z), \infty] < 2\delta$.

Next we shall prove that f takes each value outside the union of the three discs $C(0, \delta), C(1, \delta)$ and $C(\infty, d)$ exactly once in the region G bounded by Γ_n and γ_{n+1}. By Lemma 3, f takes each value outside the union of $C(0, \delta), C(1, \delta)$ and $C(\infty, \delta)$ once and only once in Δ_n. Now suppose that f takes a value w_0 outside the union of $C(0, \delta), C(1,\delta)$ and $C(\infty, d)$ at two points z' and z'' in G. We join w_0 to $C(0, \delta)$ with a curve Λ which lies outside this union and does not pass through any projection of the branch points of the Riemann surface $f(G)$. The elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along Λ to its end point, and since $C_n \subset C(0, \delta)$, $f(a_{n+1}) \in C(1, \delta)$ and $f(\gamma_{n+1}) \subset C(\infty, d)$ we see that every value on Λ is taken by f at least twice in G. Therefore we can assume that w_0 lies outside $C(\infty, 2\delta)$. Then one of the points z' and z'', say z', must lie in the domain G_0 bounded by Γ_{n+1} and γ_{n+1}. When we apply the maximum principle to the region G_0 we are led to a contradiction with the fact that w_0 lies outside $C(\infty, 2\delta)$.

2.11. Now we estimate d from below. For this purpose we consider the annulus $R = \{w : 2 < |w| < \sqrt{1 - d^2}/d\}$ corresponding to the annulus $1/\sqrt{5} > [w, \infty] > d$ on the Riemann sphere, which separates $C(0, \delta)$ and $C(1, \delta)$ from $C(\infty, d)$. Since $f(G)$ is a schlicht covering of R, the ring domain $f^{-1}(R) \cap G$ has the same modulus as R and separates 0 and a_{n+1} from a_{n+2} and ∞. By Lemma 4 we have

$$\log (\sqrt{1 - d^2}/2d) \leq \log (32 |a_{n+2}/a_{n+1}|) .$$

Since $d \leq 2\delta \leq \pi/6$, we have the estimate

$$d \geq (\sqrt{1 - (\pi/6)^2/64})\, |a_{n+1}/a_{n+2}| = m\,.$$

2.12. When we apply Lemma 2 to the region $\Delta_{n+1} \cup \{a_{n+2}\}$ and to the function $1/f$ we see that $f(\gamma_{n+1})$ is contained in a spherical disc C'_{n+1} with radius δ'_{n+1} satisfying the condition

$$\delta'_{n+1} \leq 24\,\pi A\, |a_{n+1}/a_{n+2}|^{3/4}\, |a_{n+2}/a_{n+3}|^{1/4} = r\,.$$

Since $12r < m$ by the condition (a), C'_{n+1} cannot contain the point at infinity. Therefore applying Lemma 3 to the region with $\gamma_{n+1} \cup \{a_{n+3}\} \cup \Gamma_{n+3}$ as boundary we see that C_{n+3} is contained in $C(\infty, 6\delta_{n+1})$. By the same argument as above we conclude that $\delta'_{n+2} \leq 2r$. Since $12r \leq m$, it results from Lemma 3, applied to the triply connected region bounded by γ_{n+1}, $\{a_{n+3}\}$ and γ_{n+2}, that a_{n+3} cannot be a zero, a 1-point or a pole of f. This is a contradiction and the theorem is proved.

3. Picard sets in Matsumoto's sense

3.1. The following lemma results from the proof of the theorem given by Matsumoto in [5] (For the notations see 4.2).

Lemma 5. If the successive ratios ξ_n of a Cantor set K satisfy the condition

$$\xi_{n+1} = o(\xi_n^2)\,, \tag{5}$$

then there exist no open set V and no function f such that $K \cap V \neq \Phi$, f is meromorphic in $V - K$, f has an essential singularity at every point of $K \cap V$ and f omits three values in $V - K$.

Remark. It follows from the proofs of our theorems 4 and 5 in Section 4 that Lemma 5 remains true under the weaker condition

$$\xi_{n+1} = o(\xi_n)\,.$$

3.2. Using Lemma 5 we can enlarge an arbitrary totally disconnected compact set A so as to make it into a Picard set in Matsumoto's sense. In fact, we take Cantor sets satisfying condition (5) and let them accumulate towards all points of A. A rigorous proof for this will now be given.

Theorem 3. Let A be a totally disconnected compact set. Then there exists a perfect totally disconnected compact set $B \supset A$ which is a Picard set in Matsumoto's sense.

Proof. It does not imply any essential restriction to assume that $\infty \notin A$. Since A is compact, it is covered by a finite number $N(n)$, $n = 1, 2, \ldots,$ of discs $C_{n,k}$, $k = 1, 2, \ldots, N(n)$, with centre $b_{n,k} \in A$ and with radius $1/n$.

We define $N(0) = 1$, $K_{0,1} = \Phi$ and $K_{n,0} = \Phi$ for each n. After we have determined the sets $K_{p,s}$, $p = 1, 2, \ldots, n-1$, $s = 1, 2, \ldots,$ $N(p)$, and the sets $K_{n,s}$, $s = 1, 2, \ldots, k-1$, we define $K_{n,k}$ inductively in the following manner. Let

$$B_{n,k} = A \cup \Big(\bigcup_{p=0}^{n-1} \bigcup_{s=1}^{N(p)} K_{p,s}\Big) \cup \Big(\bigcup_{s=0}^{k-1} K_{n,s}\Big),$$

and take a point $z_{n,k} \in C_{n,k} - B_{n,k}$. Since $C_{n,k} - B_{n,k}$ is open and nonvoid, there exists an $r_{n,k} > 0$ such that $\{z : |z - z_{n,k}| < 2r_{n,k}\}$ is contained in $C_{n,k} - B_{n,k}$. We contruct the set $K_{n,k}$ as a Cantor set on the closed interval

$$I_{n,k} = \{z : |\mathrm{Re}(z - z_{n,k})| \leq r_{n,k}\,, \mathrm{Im} z = \mathrm{Im} z_{n,k}\}$$

with the successive ratios ξ_n satifying the condition (5). Since A and the Cantor sets are totally disconnected and compact, we see that $B_{n,k+1}$ (the set $B_{n+1,1}$ if $k = N(n)$) has the same properties, and the process can be continued.

We get the desired set by defining

$$B = \bigcup_{n=1}^{\infty} B_{n,1}\,.$$

B is trivially totally disconnected. Every point of B is an accumulation point of B, for the points of the Cantor sets are such since each Cantor set is perfect, and if we take a point $z_0 \in A$ and a neighbourhood $\{z : |z - z_0| < r\} = U$, then some $C_{n,k}$, $2/r \leq n < 2/r + 1$, $1 \leq k \leq N(n)$, contains z_0, and $K_{n,k} \subset U$.

In order to prove that B is closed, and hence compact, let us suppose that there exists a point $\zeta \in \bar{B} - B$. Then there is a sequence $\{z_n\}_{n=1,2,\ldots}$, $z_n \in B$, $n = 1, 2, \ldots,$ whose points converge to ζ. There is only a finite number of points of the sequence such that

$$z_n \in \bigcup_{p=1}^{n_0} \bigcup_{s=1}^{N(p)} K_{p,s}$$

for any fixed n_0, for otherwise we have a subsequence $\{z_n'\}_{n=1,2,\ldots}$ whose points converge to ζ and belong to some $K_{p,s}$, $p \leq n_0$. Then ζ must belong to $K_{p,s}$ and we are led to a contradiction. We may assume that the points of $\{z_n\}_{n=1,2,\ldots}$ satisfy the condition

$$z_n \notin \bigcup_{p=1}^{n} \bigcup_{s=1}^{N(p)} K_{p,s}\,.$$

We define a sequence $\{a_n\}_{n=1,2,\ldots}$ in the following manner: For $z_n \in A$, we set $a_n = z_n$, and for $z_n \notin A$, z_n belonging to some $C_{p,s}, p > n$, we set $a_n = b_{p,s}$. The points of $\{a_n\}_{n=1,2,\ldots}$ belong to A and they converge to ζ since $|a_n - z_n| < 1/n$, $n = 1, 2, \ldots$. Hence ζ belongs to $A \subset B$, and we have proved thet B is compact.

Contrary to ous assertion that B is a Picard set in Matsumoto's sense, let us suppose that there exist a function f, meromorphic in $-B$ with an essential singularity at every point of B, and a point $\zeta \in B$ with a neighbourhood U such that f omits three values in $U - B$. According to Lemma 5 ζ cannot belong to any $K_{n,k}$. But it follows from the construction of B that there exists a $K_{n,k} \subset U$. $V = U - (B - K_{n,k})$ is open and f omits three values in $V - K_{n,k}$. This is a contradiction to Lemma 5 and the theorem is proved.

It follows from Theorem 3 that there exist Picard sets in Matsumoto's sense which are of positive two dimensional Lebesque measure.

4. A new definition for Picard sets

4.1. If A is an n-Picard set in Lehto's sense then so is every compact subset of A. Theorem 3 shows that n-Picard sets in Matsumoto's sense have no property like this. That is why we give the following new definition.

Definition 1. A totally disconnected compact set E is an n-Picard set, (a Picard set for $n = 2$), if each compact $B \subset E$ is an n-Picard set in Matsumoto's sense.

Let f be meromorphic in the complement of a totally disconnected compact set E, and let $E_f \subset E$ denote the set of the essential singularities of f. Definition 1 can also be expressed as follows: A totally disconnected compact set E is an n-Picard set, if the meromorphic continuation of any function f meromorphic in $-E$ omits at most n values in the intersection of $-E_f$ and an arbitrary neighbourhood of any $\xi \in E_f$.

We see immediately from Definition 1 that if A is a Picard set then so is each closed subset $B \subset A$. Of course totally disconnected n-Picard sets in Lehto's sense are n-Picard sets in the sense of our definition, and these are n-Picard sets in Matsumoto's sense.

4.2. We shall give a sufficient condition for a Cantor set to be a Picard set according to Definition 1. First we introduce some notations. Let $\{\xi_n\}_{n=1,2,\ldots}$ be a sequence of positive numbers satisfying the condition $0 < \xi_n < 1/3$, $n = 1, 2, \ldots$, and $I_{0,1} = \{z = x + iy : 0 \leq x \leq 1 , y = 0\}$.

In the n^{th} subdivision we exlude an open segment of length $(1 - 2\xi_n) \prod_{p=1}^{n-1} \xi_p$ from the middle of each segment $I_{n-1,k}$, $k = 1, 2, \ldots, 2^{n-1}$. The remaining 2^n segments, which are of equal length $l_n = \prod_{p=1}^{n} \xi_p$, are denoted by $I_{n,k}$, $k = 1, 2, \ldots, 2^n$. The set

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}$$

is a Cantor set on the interval $I_{0,1}$ with the successive ratios ξ_n.

We denote by $S_{n,k}$, $n = 1, 2, \ldots, k = 1, 2, \ldots, 2^n$, the following annuli on the complementary domain $-E$ of E:

$$S_{n,k} = \{z : l_n < |z - z_{n,k}| < l_{n-1}/3\},$$

where $z_{n,k}$ is the middle point of $I_{n,k}$. The transformation $\eta = (z - z_{n,k})/l_n$ maps $S_{n,k}$ conformally on the annulus $1 < |\eta| < e^{\mu_n}$, where $\mu_n = -\log(3\xi_n)$ is the modulus of $S_{n,k}$. Let $\Gamma_{n,k}$ denote the preimage of the circle $|\eta| = e^{\mu_n/2}$ on $S_{n,k}$, $\Delta_{n,k}$ the triply connected domain bounded by the three circles $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, and $(\Gamma_{n,k})$ the bounded domain with boundary $\Gamma_{n,k}$.

We now estimate the modulus of an arbitrary ring domain $A \subset (\Gamma_{n,k})$ such that one component of its complement contains the circles $\Gamma_{n,k}$ and $\Gamma_{n+1,2k-1}$, the other the circles $\Gamma_{p+1,2s-1}$ and $\Gamma_{p+1,2s}$, and $\Delta_{p,s} \subset (\Gamma_{n+1,2k})$. In the same manner as in 2.6 we get the following estimate.

Lemma 6. mod $A \leq \log(32\, l_n/l_p)$.

4.3. The following theorem shows that there exists perfect Picard sets.

Theorem 4. If the successive ratios $\xi_n = l_n/l_{n-1}$ of a Cantor set E satisfy the condition

$$\xi_{n+1} = O(\prod_{p=1}^{n} \xi_p), \tag{3}$$

then E is a Picard set.

4.4. *Proof.* Contrary to our assertion, let us suppose that there exist a closed set $B \subset E$ and a function f, meromorphic in $-B$ with B as the set of essential singularities, such that f omits three values in a neighbourhood of a singularity $\zeta \in B$. Actually there is no loss of generality to improve the stronger antithesis that f omits the three values 0, 1 and ∞ in $-B$, since the argument below can be applied locally.

Let $\delta > 0$ be so small that the discs $C(0, 2\delta)$, $C(1, 2\delta)$ and $C(\infty, 2\delta)$ are mutually disjoint. By the condition (3) we can take n_0 so large that $\delta_n = A(3\,\xi_n)^{1/2} < \delta/16$, where A is the constant of Lemma 1, and

$$\text{(b)} \qquad \xi_{n+1} < \xi_n/4\,, \ \text{i.e.} \qquad \delta_{n+1} < \delta_n/2\,,$$

for any $n > n_0$. Since $\mu_n = -\log(3\,\xi_n)$, it follows from Lemma 1 that the image of a circle $\Gamma_{n,k}\,, n > n_0$, is contained in a spherical disc $C_{n,k}$ with radius less than $\delta_n < \delta/16$.

4.5. Let us suppose that there exists only a finite number of $\Delta_{n,k}$'s where three boundary components are mapped into $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively. By Lemma 3, for any sufficiently large n the image of $\Delta_{n,k}$ is contained in a spherical disc $D_{n,k}$ with radius less than $3\delta_n$. The union of all $D_{n,k}$, for which $\Delta_{n,k}$ is contained in a given $(\Gamma_{p,s})$, is a connected set. Thus its diameter with respect to the chordal distance is dominated by

$$6 \sum_{n=p}^{\infty} \delta_n < 1/2$$

for p large enough in view of the condition (b) and the triangle inequality. We may assume that f is bounded in $(\Gamma_{p,s}) - E$, since this can be achieved by means of a linear transformation. Hence $E \cap (\Gamma_{p,s})$ contains no essential singularity of f. Since we get the same result for each s, $s = 1, 2, \ldots, 2^p$, for sufficiently large p, we are led to a contradiction.

4.6. We may therefore assume that for any $n_1 > n_0$, there exists a $\Delta_{n,k}$, $n > n_1$, such that its three boundary components are mapped into $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively, and

$$\text{(c)} \qquad C_{n+1,2k} \subset C(\infty, \delta)\,.$$

If $(\Gamma_{n+1,2k}) \cap B = \Phi$, the maximum principle yields the estimate $|f(z)| < 2$ in $\Delta_{n,k} \cup \Gamma_{n+1,2k} \cup (\Gamma_{n+1,2k})$, which contradicts (c). Because of Picard's theorem no point of B is isolated. Thus we see that there exists a $\Delta_{p,s} \subset (\Gamma_{n+1,2k})$, $p > n$, such that $(\Gamma_{p+1,2s-1}) \cap B \neq \Phi$, $(\Gamma_{p+1,2s}) \cap B \neq \Phi$ and $(\Gamma_{p,s}) \supset (\Gamma_{n+1,2k}) \cap B$.

4.7. Lemma 3 says that any one of the discs $C_{n+1,2k}$, $C_{p+1,2s-1}$ and $C_{p+1,2s}$ meets the union of the other two. For if we suppose that $\{0, 1\} \subset C_{p+1,2s-1} \cup C_{p+1,2s}$ and apply the maximum principle to the region Δ bounded by the circles $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$, $\Gamma_{p+1,2s-1}$, and $\Gamma_{p+1,2s}$, we arrive at a contradiction with (c).

Since $\delta_{q+1} < \delta_q/2$ for $q > n_0$, we have

$$\sum_{q=p+2}^{\infty} 2\ \delta_q < 2\ \delta_{p+1}\ .$$

Let us suppose that one of the discs $C_{p+1,2s-1}$ and $C_{p+1,2s}$, say $C_{p+1,2s}$, has a common point with the disc $[w\ ,\ \infty] \geq 8\delta_{p+1}$. Since

$$2\delta_{p+1} + 4\sum_{q=p+2}^{\infty} \delta_q < 6\ \delta_{p+1}\ ,$$

we see by Lemma 3 that no one of the discs $C_{q,r}$, $\Delta_{q,r} \subset (\Gamma_{p+1,2s})$, can have a common point with $C(\infty,\ \delta_{p+1})$. Then $(\Gamma_{p+1,2s})$ cannot contain any point of B. This is a contradiction, and it follows that

(d) $$C_{p+1,2s-1} \cup C_{p+1,2s} \subset C(\infty,\ 8\delta_{p+1})\ .$$

4.8. We denote

$$\gamma_{n,k} = \{z : |z - z_{n,k}| = \varepsilon\ l_{n-1}\}\ ,$$

where $1/\varepsilon = 32^2 \cdot 96\ \pi A$, and $g = 1/f$. By the condition (3) we have $\varepsilon l_{n-1} > 8\sqrt{l_{n-1}\ l_n}$ for sufficiently large n. Let n_1 in 4.6 be chosen such that this is valid for $n > n_1$. We estimate $|g'(z)|$, $z \in \gamma_{p+1,2s-1}$, by means of Cauchy's integral. By (c) and (d), integration along the circles $\Gamma_{n+1,2k}$, $\Gamma_{p+1,2s-1}$, and $\Gamma_{p+1,2s}$ yields

$$|g'(z)| \leq 24\ A/l_n + 32\ Al_{p+1}/\varepsilon^2 l_p^2\ .$$

Thus we get for every z and z_0 on the circle $\gamma_{p+1,2s-1}$

$$\begin{aligned} |g(z)\ -\ g(z_0)| &= \left|\int_{z_0}^{z} g'(t)dt\right| \\ &\leq 24\ \pi A\varepsilon l_p/l_n + 32\ \pi A l_{p+1}/\varepsilon l_p \\ &= 64^{-2}\ l_p/l_n + N l_{p+1}/l_p = \delta'_{p+1}\ , \end{aligned}$$

where N is a constant, $N = 32\ \pi A\varepsilon^{-1}$. Since the chordal distance remains invariant under the transformarion $1/f$, we note that $f(\gamma_{p+1,2s-1})$ is contained in a spherical disc $C'_{p+1,2s-1}$ with radius less than δ'_{p+1}. Similarly, $f(\gamma_{p+1,2s})$ is contained in as pherical disc $C'_{p+1,2s}$ with radius less than δ'_{p+1}.

4.9. Let us denote $\Delta_1 = (\Gamma_{n+1,2k}) - ((\Gamma_{p+1,2s-1}) \cup (\Gamma_{p+1,2s}))$. By (c) and (d), we obtain with the help of the maximum principle $f(\Delta_1) \subset C(\infty,\ \delta)$. Since $\gamma_{p+1,2s-1} \cup \gamma_{p+1,2s} \subset \Delta_1$, it follows that $f(\gamma_{p+1,2s-1} \cup \gamma_{p+1,2s}) \subset C(\infty,\ d)$ with radius $d = \sup\{[f(z),\ \infty] : z \in \gamma_{p+1,2s-1} \cup \gamma_{p+1,2s}\} < \delta$.

We prove now that f takes each value outside the union of the three discs $C(0, \delta), C(1, \delta)$ and $C(\infty, d)$ once and only once in the region Δ' bounded by the circles $\Gamma_{n,k}, \Gamma_{n+1,2k-1}, \gamma_{p+1,2s-1}$ and $\gamma_{p+1,2s}$. Let us suppose that f takes a value w_0 outside the union of $C(0, \delta), C(1, \delta)$ and $C(\infty, d)$ at two points z' and z'' in Δ'. We join w_0 to $C(0, \delta)$ with a curve Λ which lies outside this union and does not pass through any projection of the branch points of the Riemann surface $f(\Delta')$. The elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along Λ to its end point, and since $f(\Gamma_{n,k}) \subset C(0, \delta)$, $f(\Gamma_{n+1,2k}) \subset C(1, \delta)$ and $f(\gamma_{p+1,2s-1} \cup \gamma_{p+1,2s}) \subset C(\infty, d)$, we see that every value on Λ is taken by f at least twice in Δ'. Therefore we may assume that w_0 lies outside $C(\infty, 2\delta)$. By Lemma 3, f takes each value outside the union of $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$ exactly once in $\Delta_{n,k}$. Then one of the points z' and z'', say z', must lie in the domain Δ'' bounded by $\Gamma_{n+1,2k}$, $\gamma_{p+1,2s-1}$ and $\gamma_{p+1,2s}$. When we apply the maximum principle to the function $1/f$, we get by (c) and (d)

$$f(\Delta'' \cup \Delta_{p,s}) \subset C(\infty, \delta),$$

since $8\delta_{p+1} < \delta$. Then $f(z') = w_0 \in C(\infty, \delta)$, since $z' \in \Delta''$, and we are led to a contradiction with the assumption that w_0 lies outside $C(\infty, 2\delta)$.

4.10. We estimate d from below. To this purpose we consider the annulus $R = \{w : 2 < |w| < \sqrt{1 - d^2}/d\}$, which separates $C(0, \delta)$ and $C(1, \delta)$ from $C(\infty, d)$. Since $f(\Delta')$ is a schlicht covering of R, the ring domain $f^{-1}(R) \cap \Delta'$ has the same modulus as R and separates the boundary components $\gamma_{p+1,2s-1}$ and $\gamma_{p+1,2s}$ from the boundary components $\Gamma_{n,k}$ and $\Gamma_{n+1,2k-1}$. By Lemma 6 we have

$$\log(\sqrt{1 - d^2}/2d) \leq \log(32\, l_n/l_p).$$

Since $d \leq \delta \leq \pi/6$, we obtain the estimate

$$d \geq (l_p/64 l_n)\sqrt{1 - (\pi/6)^2} > l_p/128 l_n = m\,.$$

4.11. This implies that at least one of the discs $C'_{p+1,2s-1}$ and $C'_{p+1,2s}$, say $C'_{p+1,2s}$, must intersect the disc $[w, \infty] \geq m$. $C'_{p+1,2s}$ cannot contain the point at infinity for sufficiently large n since

$$\text{(e)} \qquad \begin{aligned} \delta'_{p+1} &= 64^{-2} l_p/l_n + N l_{p+1}/l_p \\ &= m/32 + 128 m N l_n l_{p+1}/l_p^2 \\ &= m\left(\frac{1}{32} + \frac{O(\prod_{r=1}^{p} \xi_r)}{\prod_{r=n+1}^{p} \xi_r}\right) < m/16 \end{aligned}$$

for n large enough by the condition (3). Let n_1 in 4.6 be chosen such that this is valid for $n > n_1$.

We have by (b) the estimate

$$\sum_{q=p+2}^{\infty} \delta_q \leq 2\, \delta_{p+2} \, .$$

We get by (3)

$$\begin{aligned} \delta_{p+2} &= A(3\, \xi_{p+2})^{1/2} \\ &= O(\xi_{p+1}^{1/2} \, (\prod_{q=1}^{p} \xi_q)^{1/2}) \\ &= o(\prod_{q=n+1}^{p} \xi_q) < m/32 \end{aligned}$$

for sufficiently large n. We assume that n_1 in 4.6 is sufficiently large in this sense. Then we have

$$2\delta_{p+1}' + 4 \sum_{q=p+2}^{\infty} \delta_q < m/2 \, ,$$

and see by Lemma 3 and the triangle inequality that there exists no $\Delta_{q,r} \subset (\Gamma_{p+1,2s})$ whose three boundary components are mapped into $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$, respectively. Then f is bounded in $(\Gamma_{p+1,2s})$ and cannot contain any point of B. This is a contradiction, and the theorem is proved.

4.12. By the same argument we prove the following theorem.

Theorem 5. If the successive ratios ξ_n of a Cantor set E satisfy the condition

$$\xi_{n+1} = o(\xi_n) \, . \tag{4}$$

then E is a Picard set in Matsumoto's sense.

As we remarked in the beginning of Section 3, Matsumoto has established the same result under the condition

$$\xi_{n+1} = o(\xi_n^2) \, . \tag{4'}$$

Our improvement is of interest for the following reason. A Cantor set is of positive capacity if and only if

$$\sum_{n=1}^{\infty} \frac{-\log \xi_n}{2^n} < \infty$$

(Nevanlinna [8]). Under the condition (4) it is therefore possible to choose the ratios ξ_n such that the capasity of E is positive. There are thus Picard sets in Matsumoto's sense with positive capasity. Matsumoto [7] has proved the same result but his method is different.

Proof of Theorem 5. We modify the proof of Theorem 4. Taking $B = E$ in 4.5, we get $p = n + 1$. By (e) and (4) we get

(g)
$$\begin{aligned}\delta'_{n+2} &= 64^{-2}\, l_{n+1}/l_n + N l_{n+2}/l_{n+1} \\ &= m/32 + 128\, m N l_n l_{n+2}/l_{n+1}^2 \\ &= m/32 + m\, \xi_{n+1}^{-1}\, o(\xi_{n+1}) < m/16\end{aligned}$$

for sufficiently large n ($m = l_{n+1}/128 l_n$). Let n_1 in 4.6 be chosen such that this is valid for all $n > n_1$.

At least one of the discs $C'_{n+2,4k-1}$ and $C'_{n+2,4k}$, say $C'_{n+2,4k}$, has a common point with $[w, \infty] \geq m$. Since $\delta'_{n+2} < m/16$, $\infty \notin C'_{n+2,4k}$, and we see by Lemma 3 that no one of the discs $C'_{n+2,4k}$, $C_{n+3,8k-1}$ and $C_{n+3,8k}$ can be disjoint from the union of the other two. Then we see in the same manner as in 4.7—4.8 that $f(\gamma_{n+3,8k-1})$ and $f(\gamma_{n+3,8k})$ are contained in spherical discs $C'_{n+3,8k-1}$ and $C'_{n+3,8k}$, respectively, with radius less than

$$\delta'_{n+3} = 64^{-2}\, l_{n+2}/l_{n+1} + N l_{n+3}/l_{n+2} \,.$$

We get by (g)

$$\delta'_{n+3} \leq 16^{-1} \cdot 128^{-1}\, l_{n+2}/l_{n+1} < m/32 \,,$$

and inductively $\delta'_{n+2+r} < m/2^r \cdot 16$ for any $r = 1, 2, \ldots$

Since now

$$2\delta'_{n+2} + 4 \sum_{s=n+3}^{\infty} \delta'_s < m/2 < d/2$$

(see 4.10) we see by repeating the conclusion above that no one of the discs $C'_{p,s}$, $\Delta_{p,s} \subset (\Gamma_{n+2,4k})$, can have a common point with $[w, \infty] \leq d/2$. Then f is bounded in $(\Gamma_{n+2,4k})$, and $(\Gamma_{n+2,4k})$ cannot contain any essential singularity of f. This is a contradiction and the theorem is proved.

4.13. Matsumoto [6] has proved that a Cantor set E is a Picard set in Lehto's sense if its successive ratios ξ_n satisfy the condition

$$\xi_{n+1} = O(\exp(-\, 1/\prod_{p=1}^{n} \xi_p)) \,.$$

Considering the product

$$f(z) = \prod_{n=1}^{\infty} (1 - r_n(1 - z)/z)$$

we get a result in the opposite direction if the points of $\{r_n\}_{n=1,2,\ldots}$, $0 < r_n < 1/2$, tend to zero with sufficient rapidity.

Theorem 6. There exists a Cantor set E whose successive ratios ξ_n satisfy the condition

$$\xi_{n+1} = O((\prod_{p=1}^{n} \xi_p)^{(n-2)/2}) \tag{6}$$

and which is no Picard set in Lehto's sense.

Proof. Let

$$f(z) = \prod_{n=1}^{\infty} (1 - e^{-e^{e^n}} (1 - z)/z) .$$

We denote $e^{-e^{e^n}} = r_n$ and $s_n = r_n/(1 + r_n)$. We see immediately that the zeros of f are s_n, $n = 1, 2, \ldots$. Let $\zeta_n = s_n + t_n$, $n \geq 2$, be a 1-point of f on the positive real axis satisfying $z \in \bar{R}_n$ with

$$R_n = \{z : (s_n s_{n-1})^{1/2} < |z| < (s_n s_{n+1})^{1/2}\} .$$

We get immediately for $z \in \bar{R}_n$

$$\log |\prod_{p=1}^{n-1} (1 - r_p(1 - z)/z)| = \log |\prod_{p=1}^{n-1} r_p/z| + O(1)$$

and

$$\log |\prod_{p=n+1}^{\infty} (1 - r_p(1 - z)/z)| = O(1) .$$

Setting

$$f(\zeta_n) = \{ \prod_{p=1}^{n-1} (1 - r_p(1 - \zeta_n)/\zeta_n)\} (1 - r_n(1 - \zeta_n)/\zeta_n) \prod_{p=n+1}^{\infty} (1 - r_p(1 - \zeta_n)/\zeta_n) = 1$$

we get $|t_n/s_n| = o(1)$ and hence

$$\log |1 - r_n(1 - \zeta_n)/\zeta_n| = \log |t_n| - \log r_n + O(1) .$$

Combining these results we get

$$\begin{aligned} |t_n| &= (\prod_{p=1}^{n-1} \zeta_n/r_p) r_n e^{O(1)} \\ &= (s_n - |t_n|)^{n-1} r_n (\prod_{p=1}^{n-1} r_p)^{-1} \left(\frac{s_n + |t_n|}{s_n - |t_n|}\right)^{n-1} e^{O(1)} \\ &= O((s_n - |t_n|)^{n-1}) \end{aligned} \tag{h}$$

Since $f((s_{2n}s_{2n-1})^{1/2}) < 0$ and $f((s_{2n}s_{2n+1})^{1/2}) > 1$ we see that f has at least one 1-point $\zeta_n = s_n + t_n$ on the positive real axis in $\bar{R}_n$.

Since $|f(z)| > 2$ for $|z| = (s_n s_{n+1})^{1/2}$ for sufficiently large n, we see in the same manner as in 2.4 that f takes the value 1 as many times as the value 0 in $|z| > (s_n s_{n+1})^{1/2}$. Because f has in $|z| > (s_n s_{n+1})^{1/2}$ n zeros each of order one, the only 1-points of f in $|z| > (s_n s_{n+1})^{1/2}$ are $\zeta_1 = 1$ and the above mentioned $\zeta_q \in \bar{R}_q$, $q = 2, 3, \ldots, n$.

We set $l_0 = 1, l_1 = s_1$ and for $n \geq 1$ $l_{2n} = s_{n+1} + \max(0, t_{n+1})$ and $l_{2n+1} = |t_{n+1}|$. We construct a Cantor set E on the interval $\{z = x + iy :$ $0 \leq x \leq 1,\ y = 0\}$ with the successive ratios $\xi_n = l_n/l_{n-1}$, $n = 1, 2, \ldots$. We see by (h) that the ratios ξ_n satisfy (6) and the calculations above show that $f \neq 0, 1$ and ∞ in $-E$. Then E is the desired set and Theorem 6 is proved.

University of Helsinki and
University of Jyväskylä
Finland

References

[1] Carleson, L.: A remark on Picard's theorem. - Bull. Amer. Math. Soc. 67 (1961), 142–144.

[2] Lehto, O.: A generalization of Picard's theorem. - Ark. Mat. 3 (1958), 495–500.

[3] —»— and K. I. Virtanen: Quasikonforme Abbildungen. - Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[4] Matsumoto, K.: Some notes on exeptional values of meromorphic functions. - Nagoya Math. J. 22 (1963), 189–201.

[5] —»— Existence of perfect Picard sets. - Nagoya Math. J. 27 (1966), 213–222.

[6] —»— Some remarks on Picard sets. - Ann. Acad. Sci. Fenn. A. I. 403, 1967.

[7] —»— Perfect Picard set of positive capasity. - Nagoya Math. J. 29 (1967), 51–56.

[8] Nevanlinna, R.: Eindeutige analytische Funktionen. - Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953.

[9] Sario, L. and K. Noshiro: Value distribution theory. - D. van Nostrand Company, Toronto-New York-London, 1966.

Printed December 1967