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ON THE USE OF STEP-FUNCTIONS IN EXTREMUM PROBLEMS OF THE CLASS WITH BOUNDED BOUNDARY ROTATION

BY

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§ 1. On the use of step-functions in S_k

We wish to discuss in detail a method of extremalization which is based on the use of step-functions. Let us consider a subclass of univalent functions, the class S_k, which although rather simple, displays features typical of extremum problems. This class consists of the normalized functions

$$(1) \qquad f(z) = z + a_2 z^2 + \ldots ,$$

regular in the open disc $|z| < 1$, and of which the boundary rotation is bounded by the number $k\pi$, where $2 \leq k \leq 4$. According to PAATERO, the functions of S_k are univalent [3].

The functions of the class S_k can be generated by means of functions ψ of bounded variation in the following sense:

The class S_k consists of those functions f which satisfy the Poisson-Stieltjes equation

$$(2) \qquad 1 + z \frac{f''(z)}{f'(z)} = \frac{1}{2} \int_0^{2\pi} \frac{e^{i\varphi} + z}{e^{i\varphi} - z} \, d\psi(\varphi) , \qquad |z| < 1 ,$$

where

$$(3) \qquad \int_0^{2\pi} d\psi(\varphi) = 2 , \quad \int_0^{2\pi} |d\psi(\varphi)| \leq k , \qquad 2 \leq k \leq 4 .$$

In what follows, Φ_k denotes the space of all functions ψ of bounded variation, defined on the interval $I = [0, 2\pi]$ and satisfying the conditions (3).

By using the relation (2), the coefficients a_p in series (1) of $f(z)$ can be expressed as functionals $a_p(\psi)$ of the generating function ψ. Thus, for example

$$(4) \qquad \begin{cases} 2a_2(\psi) = \displaystyle\int_0^{2\pi} e^{-i\varphi} \, d\psi(\varphi) , \\ 6a_3(\psi) = \left(\displaystyle\int_0^{2\pi} e^{-i\varphi} d\psi(\varphi) \right)^2 + \displaystyle\int_0^{2\pi} e^{-i2\varphi} d\psi(\varphi) . \end{cases}$$

For every $\psi \in \Phi_k$, there is accordingly a unique $f \in S_k$, and this mapping $\Phi_k \to S_k$ is surjective. The maximum of the absolute value of coefficient a_p in S_k (p fixed) can thus be determined by maximizing the functional $A(\psi) = |a_p(\psi)|$ in Φ_k.

From now on, Φ_k is regarded as a metric space, with the metric ϱ defined by

$$\varrho(\psi_1, \psi_2) = \sup_{\varphi \in I} |\psi_1(\varphi) - \psi_2(\varphi)| .$$

The functional $A(\psi)$ is then continuous. This is easily seen for instance by using a connection which expresses the a-coefficients by means of certain c-coefficients (cf. TAMMI [6]):

$$\text{(6)} \qquad \begin{cases} p(p+1)a_{p+1} = \sum\limits_{\nu=1}^{p} c_\nu (p - \nu + 1) a_{p-\nu+1} , \\ c_p = \int\limits_0^{2\pi} e^{-ip\varphi}\, d\psi(\varphi) \qquad (a_1 = 1;\ p = 1, 2, \ldots) . \end{cases}$$

By partial integration, we get the formula

$$\text{(7)} \qquad c_p(\psi_1) - c_p(\psi_2) = -ip \int\limits_0^{2\pi} e^{-ip\varphi} [\psi_1(\varphi) - \psi_2(\varphi)]\, d\varphi ,$$

from which, together with the fact that the a-coefficients are polynomials of the c-coefficients, there follows the continuity of the functional A.

Let now Σ_k denote the subspace of Φ_k, which consists of step-functions. Since Σ_k is dense in Φ_k, it is tempting to see whether the functional A can be maximized in Σ_k. In fact, *if there exists*

$$\max_{\sigma \in \Sigma_k} A(\sigma) = A(\hat{\sigma}) ,$$

we have

$$\max_{\psi \in \Phi} A(\psi) = A(\hat{\sigma}) ,$$

as a consequence of the continuity of A, and the fact that Σ_k is dense in Φ_k. In cases $p = 3$ and $p = 4$ we show in fact, by applying the methods of calculus, that the maximum of $A(\psi)$ in Σ_k really exists, and its value can also be found. This will be done in §§ 2 and 3. It should be noted that this also leads to a differential equation of Schiffer-type for the extremal function.

Naturally, there can be extremal generating functions also in the complement $\Phi_k - \Sigma_k$, although our method does not give them. In the determination of extremal $\hat{\sigma} \in \Sigma_k$, we use a special variation, applicable only to step-functions (cf. 2 §, n:o 1). Consequently our elementary necessary condition for the extremum is not applicable for characterization of possible extremal generating functions which are not step-functions.

§ 2. A necessary condition for the step-function maximizing $|a_3|$

1. *Variation of the points of discontinuity*

Let $\psi_N \in \Sigma_k$ be the step-function which has non-zero jumps $\Delta_1, \Delta_2, \ldots, \Delta_N$ at points $\varphi_1, \varphi_2, \ldots, \varphi_N$. By (3),

$$\sum_{\nu=1}^{N} \Delta_\nu = 2, \quad \sum_{\nu=1}^{N} |\Delta_\nu| \leq k, \tag{8}$$

where $2 \leq k \leq 4$. The variation used in the following is effected by shifting the points φ_ν and keeping jumps Δ_ν fixed.

We turn to the case of a_3. It is well known that no restriction is involved in assuming a_3 to be real and positive. We introduce the notations

$$t_\nu = e^{i\varphi_\nu} \qquad (\nu = 1, 2, \ldots, N). \tag{9}$$

Then, by applying the formulae (4) to step-function ψ_N, we get

$$\begin{cases} 2\, a_2(\psi_N) = \sum_{\nu=1}^{N} \dfrac{\Delta_\nu}{t_\nu}, \\ 0 < 6\, a_3(\psi_N) = \left(\sum_{\nu=1}^{N} \dfrac{\Delta_\nu}{t_\nu}\right)^2 + \sum_{\nu=1}^{N} \dfrac{\Delta_\nu}{t_\nu^2}. \end{cases} \tag{10}$$

By use of the abbreviation

$$H(t_1, t_2, \ldots, t_N) = \left(\sum_{\nu=1}^{N} \Delta_\nu t_\nu\right)^2 + \sum_{\nu=1}^{N} \Delta_\nu t_\nu^2, \tag{11}$$

the function to be maximized is

$$\begin{aligned} A_3(t_1, t_2, \ldots, t_N) &= 12 a_3(\psi_N) \\ &= H(t_1, t_2, \ldots, t_N) + H(t_1^{-1}, t_2^{-1}, \ldots, t_N^{-1}). \end{aligned} \tag{12}$$

For maximal $\psi_N \in \Sigma_k$ the conditions

$$\frac{\partial A_3}{\partial \varphi_\mu} = i\, t_\mu \frac{\partial A_3}{\partial t_\mu} = 0 \qquad (\mu = 1, 2, \ldots, N)$$

necessarily hold. In view of (11), (12) and the assumption $\Delta_\mu \neq 0$, these conditions are easily reduced to the form

$$2\bar{a}_2 t_\mu + t_\mu^2 - \frac{2a_2}{t_\mu} - \frac{1}{t_\mu^2} = 0 \qquad (\mu = 1, \ldots, N) .$$

These equations are of the fourth degree with respect to t_μ. The extremal step-function has thus at most four points of discontinuity φ_μ, i.e.

$$N \leq 4 .$$

On the other hand

$$2 \leq N \qquad \text{for} \quad k > 2 .$$

This is seen as follows. Let Δ_ν^+ denote the positive jumps and Δ_ν^- the absolute values of the negative jumps of ψ_N. By (8) we have

$$\sum \Delta_\nu^+ - \sum \Delta_\nu^- = 2, \qquad \sum \Delta_\nu^+ + \sum \Delta_\nu^- \leq k ,$$

from which

$$\sum \Delta_\nu^+ \leq \frac{k}{2} + 1, \qquad \sum \Delta_\nu^- \leq \frac{k}{2} - 1 . \tag{13}$$

If $k = 2$, $\Sigma \Delta_\nu^- \leq 0$. Then every $\Delta_\nu^- = 0$, and we are dealing with the convex case, which has been completely studied by Löwner [2]. We are here interested solely in the non-convex case, where at least one $\Delta_\nu^- > 0$ and thus $k > 2$. Both of the sets $\{\Delta_\nu^+\}$ and $\{\Delta_\nu^-\}$ are then non-empty, and hence $N \geq 2$.

It should finally be noted, that a function $f \in S_k$ generated by a step-function $\psi_N \in \Sigma_k$ has a polygonal image domain. The term »extremal polygon» is employed for the image-polygon of function $f \in S_k$, generated by an extremal step-function.

The results are collected below.

Theorem. *The pre-images t_μ $(\mu = 1, 2, \ldots, N)$ of corner points $f(t_\mu)$ of an extremal polygon satisfy the equation*

$$g_3(z) = \frac{1}{z^2} + \frac{2a_2}{z} - 2\bar{a}_2 z - z^2 = 0 \tag{14}$$

or

$$-z^2 g_3(z) = z^4 + 2\bar{a}_2 z^3 - 2a_2 z - 1 = 0 . \tag{15}$$

In the non-convex cases, $2 < k \leq 4$ number N of the corners of the extremal polygon satisfies the inequalities

$$2 \leq N \leq 4 . \tag{16}$$

2. Determination of the extremal step-function

The necessary condition $g_3(z) = 0$ considerably reduces the number of alternatives for the extremum case. Let us examine the different possibilities $N = 4, 3$ or 2. The last case of these is the easiest, and will be treated later on. If $N = 4$, all the pre-images t_μ are roots of the equation $g_3(z) = 0$. If $N = 3$, this equation has also a root which does not belong to the system of pre-images. Such a root of $g_3(z) = 0$ is termed a *free root* of this equation. It should be noted, that the symmetric structure of (14) implies, that with z, also $\frac{1}{\bar{z}}$ is a root. This means that if the absolute value of the free root differs from one, then two free roots necessarily exist. Thus in the case $N = 3$, all the roots of $g_3(z) = 0$ necessarily have the absolute value one. — Apart from case $N = 2$, accordingly, we have only the case in which all the roots of $g_3(z) = 0$ have the absolute value one.

For study of the last mentioned case, the numbers $1 + \Delta_\nu$ are estimated. For $k \leq 4$ there holds $\frac{k}{2} - 1 \leq 1$, and thus, from (13)

$$\Delta_\nu^- \leq \frac{k}{2} - 1 \leq 1 ,$$

$$1 - \Delta_\nu^- \geq 0 .$$

Since obviously $1 + \Delta_\nu^+ > 0$ we have generally

$$\delta_\nu = 1 + \Delta_\nu \geq 0 \qquad (\nu = 1, 2, \ldots, N) . \tag{17}$$

Expression (10) for the coefficient a_3 can now simply be rewritten in the quantities (17).

Denote the roots of $g_3(z) = 0$ by z_1, z_2, z_3, z_4. As was stated above, in cases $N = 3$ and $N = 4$, all these numbers have the absolute value one: $|z_\nu| = 1$ $(\nu = 1, \ldots, N)$. The pre-images t_μ are among these numbers z_ν. If the coefficients of (15) are written as symmetric polynomials of the roots z_ν, we have

$$\tag{18} \begin{cases} 2\bar{a}_2 = -(z_1 + z_2 + z_3 + z_4) , \\ 0 = z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_1 + z_2 z_4 + z_3 z_1 , \\ 2a_2 = z_1 z_2 z_3 + z_2 z_3 z_4 + z_3 z_4 z_1 + z_4 z_1 z_2 , \\ -1 = z_1 z_2 z_3 z_4 . \end{cases}$$

On the other hand, for real a_3 and for a_2, we get from (10)

$$\begin{cases} 6a_3 = (\sum_1^N \Delta_\nu t_\nu)^2 + \sum_1^N \Delta_\nu t_\nu^2 , \\ 2\bar{a}_2 = \sum_1^N \Delta_\nu t_\nu . \end{cases}$$

If $N = 3$, the sum $\sum_1^N$ does not include the free root of equation $g_3(z) = 0$. It is of use to complete this sum, making the following agreement:

(19) In $\sum_1^N$ take $\Delta_\nu = 0$ if z_ν is a free root.

For a_3 and a_2, this allows of the following presentation, which is valid in both cases $N = 3$ and $N = 4$

$$(20) \qquad \begin{cases} 6a_3 = (\sum_1^4 \Delta_\nu z_\nu)^2 + \sum_1^4 \Delta_\nu z_\nu^2 , \\ 2\bar{a}_2 = \sum_1^4 \Delta_\nu z_\nu . \end{cases}$$

Expression (20) of a_3 is now simplified by applying (18):

$$\begin{aligned} 6a_3 &= (2\bar{a}_2)^2 + \sum_1^4 \Delta_\nu z_\nu^2 \\ &= (\sum_1^4 z_\nu)^2 + \sum_1^4 \Delta_\nu z_\nu^2 \\ &= \sum_1^4 (1 + \Delta_\nu)\, z_\nu^2 ; \end{aligned}$$

$$(21) \qquad 6a_3 = \sum_1^4 \delta_\nu z_\nu^2 .$$

In view of agreement (19), condition (17) is clearly true also if z_ν is a free root; in this case $\delta_\nu = 1$. Therefore, it can be deduced from (21) that

$$\begin{aligned} 0 < 6a_3 &\le \sum_1^4 \delta_\nu |z_\nu|^2 \\ &= \sum_1^4 \delta_\nu = \sum_1^4 (1 + \Delta_\nu) \\ &= 4 + \sum_1^4 \Delta_\nu = 6 ; \end{aligned}$$

$$a_3 \le 1 .$$

From this we conclude, in view of (22), that values $N = 3$ and $N = 4$ do not give maximum for a_3.

In the remaining case $N = 2$ we have by virtue of (13)

$$\Delta_1 = \Delta_1^+ \le \frac{k}{2} + 1 \, , \; \Delta_2 = - \Delta_1^- \ge 1 - \frac{k}{2} \, .$$

We omit the easy calculations needed to give

$$\max a_3 = \frac{1}{6} (k^2 + 2) \, . \tag{22}$$

The extremal step-function, for which $a_3 > 0$, has the jumps at the points $\varphi_1 = 0$, $\varphi_2 = \pi$.

3. *Differential equation for an extremal* $f(z)$

The above procedure, starting from the inequalities (17), was first applied by SCHIFFER and TAMMI [4] for maximizing a_3. It has been repeated here as a preparation for the following considerations, in which a differential equation for the extremal $f(z)$ will be derived. As stated above, the step-function generating $f(z)$ has $N = 2$, $\Delta_1 + \Delta_2 = 2$, $t_1 = 1$, $t_2 = -1$.

We start from the Poisson-Stieltjes presentation (2). In the step-function case this assumes the form

$$1 + z \frac{f''(z)}{f'(z)} = \tfrac{1}{2} \sum_{\nu=1}^{N} \frac{t_\nu + z}{t_\nu + z} \Delta_\nu \, . \tag{23}$$

We desire to rewrite the right side of (23) by applying the roots z_ν of the equation $g_3(z) = 0$. It should be noted that although we are aware that the maximum case for a_3 is $N = 2$, simultaneous consideration is given below to all the cases $N = 2, 3, 4$, as an exercise for the study of a_4 in § 3. By application of the same technique to the case of a_4 there is obtained the result (50), from which the value of N, which belongs to the extremal polygon, can be deduced.

According to agreement (19), formula (23) can be written in all the cases $N = 2, 3, 4$ in the form

$$1 + z \frac{f''(z)}{f'(z)} = \tfrac{1}{2} \sum_{\nu=1}^{4} \frac{z_\nu + z}{z_\nu - z} \Delta_\nu \, . \tag{24}$$

We write the sum on the right of (24) as follows

$$\sum_{\nu=1}^{4} \frac{z_\nu + z}{z_\nu - z} \Delta_\nu \tag{25}$$

$$= \frac{-1}{-z^2 g_3(z)} \, [(z + z_1)(z - z_2)(z - z_3)(z - z_4) \Delta_1 + \ldots +$$

$$(z - z_1)(z - z_2)(z - z_3)(z + z_4) \Delta_4] \, .$$

Develop the first term in the brackets by using the relations (18):

$$(z + z_1)(z - z_2)(z - z_3)(z - z_4) = z^4 - (-z_1 + z_2 + z_3 + z_4)z^3$$

$$+ (-z_1z_2 + z_2z_3 + z_3z_4 - z_4z_1 + z_2z_4 - z_3z_1)\, z^2$$

$$- (-z_1z_2z_3 + z_2z_3z_4 - z_3z_4z_1 - z_4z_1z_2)\, z - z_1z_2z_3z_4$$

$$= z^4 - (-2z_1 + z_1 + z_2 + z_3 + z_4)z^3 - 2z_1(-z_1 + z_1 + z_2 + z_3 + z_4)\, z^2$$

$$- z_1z_2z_3z_4\left(\frac{2}{z_1} - \frac{1}{z_4} - \frac{1}{z_1} - \frac{1}{z_2} - \frac{1}{z_3}\right) z - z_1z_2z_3z_4$$

$$= z^4 - (-2z_1 - 2\bar{a}_2)\, z^3 - 2z_1(-z_1 - 2\bar{a}_2)\, z^2 + \left(\frac{2}{z_1} + 2a_2\right) z + 1\,.$$

The other three terms within brackets in (25) are treated in a similar fashion, multiplied by corresponding Δ_ν and added. The [ ]-expression in (25) then assumes a form which can be further simplified, in view of (20), as follows:

$$[\ ] = \sum_1^4 \Delta_\nu \cdot z^4 + 2(\bar{a}_2 \sum_1^4 \Delta_\nu + \sum_1^4 \Delta_\nu z_\nu)\, z^3$$

$$+ 2(2\bar{a}_2 \sum_1^4 \Delta_\nu z_\nu + \sum_1^4 \Delta_\nu z_\nu^2)z^2 + 2\left(a_2 \sum_1^4 \Delta_\nu + \sum_1^4 \frac{\Delta_\nu}{z_\nu}\right) z + \sum_1^4 \Delta_\nu$$

$$= 2z^4 + 2(2\bar{a}_2 + 2\bar{a}_2)z^3$$

$$+ 2(4\bar{a}_2^2 + 6a_3 - 4\bar{a}_2^2)z^2 + 2(2a_2 + 2a_2)z + 2$$

$$= 2(z^4 + 4\bar{a}_2z^3 + 6a_3z^2 + 4a_2z + 1)\,.$$

The final form of equation (23) is thus

$$1 + z\frac{f''(z)}{f'(z)} = \frac{z^2\left(\frac{1}{z^2} + \frac{4a_2}{z} + 6a_3 + 4\bar{a}_2z + z^2\right)}{z^2 g_3(z)}\,.$$

Summary. *Let $f(z) \in S_k$ be the function generated by a the step-function $\psi_N(\varphi)$, which is extremal with respect to the coefficient $a_3 > 0$. Let the points of discontinuity of $\psi_N(\varphi)$ be $\varphi_\mu(\mu = 1, \ldots, N \leq 4)$. The pre-images*

$$z = t_\mu = e^{i\varphi_\mu}$$

of corner points $f(t_\mu)$ of the extremal polygon satisfy the necessary condition

$$g_3(z) = \frac{1}{z^2} + \frac{2a_2}{z} - 2\bar{a}_2z - z^2 = 0\,. \tag{26}$$

The Poisson-Stieltjes presentation for $f(z)$, *generated by* $\psi_N(\varphi)$, *reads*

$$1 + z\frac{f''(z)}{f'(z)} = \tfrac{1}{2}\sum_{\nu=1}^{4}\frac{z_\nu + z}{z_\nu - z}\,\Delta_\nu . \tag{27}$$

Here the numbers z_ν *are the roots of equation* (26). *If* $N < 4$ *some of the numbers* z_ν *are free roots of equation* $g_3(z) = 0$. *This means that they are not among the pre-images* t_μ $(\mu = 1, \ldots, N)$ *mentioned above. The numbers* Δ_ν *belonging to the free roots* z_ν, *are taken to be* $= 0$.

Equation (27) *can also be written in the form*

$$1 + z\frac{f''(z)}{f'(z)} = \frac{G_3(z)}{g_3(z)} , \tag{28}$$

where

$$G_3(z) = \frac{1}{z^2} + \frac{4a_2}{z} + 6a_3 + 4\bar{a}_2 z + z^2 . \tag{29}$$

It is emphasized that we are here led to differential equation (28), which also results from application of the method of interior variation (SCHIFFER-TAMMI [4]) in class S_k. The fact that the step-function-generated extremal $f \in S_k$ satisfies the differential equation (28) is thus already implied by the most trivial necessary extremum conditions of calculus.

§ 3. The coefficient a_4

1. Differential equation for an extremal $f(z)$

It will now be shown that the above procedure is also applicable to the next coefficient, a_4.

From (6) it is first deduced that

$$\begin{cases} 24a_4 = c_1 - 3c_1c_2 + 2c_3 . \\ c_n = \displaystyle\int_0^{2\pi} e^{-in\varphi}\, d\psi(\varphi) = \sum_{\nu=1}^{N}\frac{\Delta_\nu}{t_\nu^n} . \end{cases} \tag{30}$$

We will also need the expressions of coefficients a_2 and a_3, and have

$$\begin{cases} 0 < 24a_4 = \left(\displaystyle\sum_1^N \frac{\Delta_\nu}{t_\nu}\right)^3 + 3\sum_1^N \frac{\Delta_\nu}{t_\nu}\cdot\sum_1^N\frac{\Delta_\nu}{t_\nu^2} + 2\sum_1^N\frac{\Delta_\nu}{t_\nu^3} , \\ 6a_3 = \left(\displaystyle\sum_1^N \frac{\Delta_\nu}{t_\nu}\right)^2 + \sum_1^N\frac{\Delta_\nu}{t_\nu^2} . \\ 2a_2 = \displaystyle\sum_1^N\frac{\Delta_\nu}{t_\nu} . \end{cases} \tag{31}$$

For further treatment, it is of value to express the different sum-expressions in terms of the coefficients. By virtue of (31),

$$(32)\qquad \begin{cases} \sum_1^N \frac{A_\nu}{t_\nu} = 2a_2\,, \\ \sum_1^N \frac{A_\nu}{t_\nu^2} = 6a_3 - 4a_2^2\,, \\ \sum_1^N \frac{A_\nu}{t_\nu^3} = 12a_4 - 18a_2a_3 + 8a_2^3\,. \end{cases}$$

The following notations similar to the former ones will be used:

$$(33)\qquad \begin{cases} H(t_1, t_2, \ldots, t_N) = (\sum_1^N A_\nu t_\nu)^3 + 3\sum_1^N A_\nu t_\nu \cdot \sum_1^N A_\nu t_\nu^2 + 2\sum_1^N A_\nu t_\nu^2\,, \\ A_4 = 2\,\mathrm{Re}\,\{24a_4\} = H(t_1, t_2, \ldots, t_N) + H(t_1^{-1}, t_2^{-1}, \ldots, t_N^{-1})\,. \end{cases}$$

A necessary condition for an extremal step-function can again be derived by use of the necessary extremal conditions of calculus. Hence, we necessarily have

$$\frac{\partial A_4}{\partial t_\mu} = \frac{6A_\mu}{t_\mu}\left(3\bar{a}_3 t_\mu + 2\bar{a}_2 t_\mu^2 + t_\mu^3 - \frac{3a_3}{t_\mu} - \frac{2a_2}{t_\mu^2} - \frac{1}{t_\mu^3}\right) = 0\,.$$

It has accordingly been found that pre-images t_μ of corner points $f(t_\mu)$ of the extremal polygon are among the roots z_ν $(\nu = 1, \ldots, 6)$ of the equation

$$(34)\qquad g_4(z) = \frac{1}{z^3} + \frac{2a_2}{z^2} + \frac{3a_3}{z} - 3\bar{a}_3 z - 2\bar{a}_2 z^2 - z^3 = 0\,;$$

or

$$(35)\qquad -z^3 g_4(z) = z^6 + 2\bar{a}_2 z^5 + 3\bar{a}_3 z^4 - 3a_3 z^2 - 2a_2 z - 1 = 0\,.$$

Consequently

$$(36)\qquad 2 \le N \le 6 \qquad\qquad (2 < k \le 4)\,.$$

If $N < 6$, then (34) has also free roots z_ν which do not belong to the pre-images, denoted by t_μ. If for a free root z_ν there holds $|z_\nu| \neq 1$, then the symmetric structure of (34) indicates that together with z_ν the number $\frac{1}{\bar{z}_\nu}$ is also a free root.

Again, it is desired to make use of the symmetric expressions of all the roots z_ν $(\nu = 1, \ldots, 6)$. Thus, an agreement is made which allows of replacing $\sum_1^N$ in (44) by $\sum_1^6$:

(37) In $\sum_1^6$ take $A_\nu = 0$ if z_ν is a free root.

To arrive at the connections between the coefficients (31) and the symmetric expressions of z_ν mentioned, compare the left and right sides of the identity

$$
\begin{aligned}
(38) \qquad -z^3 g_4(z) &= z^6 + 2\bar{a}_2 z^5 + 3\bar{a}_3 z^4 - 3a_3 z^2 - 2a_2 z - 1 \\
&= (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6) \\
&= z^6 - C_5 z^5 + C_4 z^4 - C_3 z^3 + C_2 z^2 - C_1 z + C_0 \\
&= h_6(z) .
\end{aligned}
$$

Here, coefficients C_ν have the following symmetric expressions:

$$
(39) \qquad \left\{
\begin{aligned}
C_5 &= \sum_{\nu=1}^{6} z_\nu , \\
C_4 &= \sum_{\mu=1}^{5} z_\mu \sum_{\nu=\mu+1}^{6} z_\nu ; \\
C_3 &= \sum_{\lambda=1}^{4} z_\lambda \sum_{\mu=\lambda+1}^{5} z_\mu \sum_{\nu=\mu+1}^{6} z_\nu , \\
C_2 &= z_1 z_2 z_3 z_4 z_5 z_6 \sum_{\mu=1}^{5} z_\mu^{-1} \sum_{\nu=\mu+1}^{6} z_\nu^{-1} , \\
C_1 &= z_1 z_2 z_3 z_4 z_5 z_6 \sum_{\nu=1}^{6} z_\nu^{-1} , \\
C_0 &= z_1 z_2 z_3 z_4 z_5 z_6 .
\end{aligned}
\right.
$$

Comparison of the coefficients in (38) now indicates the connections, which can be regarded as necessary extremum conditions

$$
(40) \qquad \left\{
\begin{aligned}
2\bar{a}_2 &= -C_5 , \\
3\bar{a}_3 &= C_4 , \\
0 &= C_3 , \\
-3a_3 &= C_2 , \\
-2a_2 &= -C_1 , \\
-1 &= C_0 .
\end{aligned}
\right.
$$

It is readily found, that the first and fifth condition (40) are equivalent, and similarly that the second and fourth condition (40) are equivalent to each other. — The first equation (40) gives

$$(41) \qquad 2a_2 = -\bar{C}_5 = -\sum_{\nu=1}^{6} \bar{z}_\nu .$$

Now, if $|z_\nu| = 1$, then $\bar{z}_\nu = \frac{1}{z_\nu}$. If $|z_\nu| \neq 1$ there also exists a free root $z_\mu = \frac{1}{\bar{z}_\nu}$; $\bar{z}_\nu = \frac{1}{z_\mu}$. Thus we can write (41) in the form

$$2a_2 = -\sum_{\nu=1}^{6} \frac{1}{z_\nu} .$$

According to the last equation (40) this is the same as

$$2a_2 = z_1 z_2 z_3 z_4 z_5 z_6 \sum_{\nu=1}^{6} \frac{1}{z_\nu} = C_1 ,$$

and we have arrived at the fifth condition (40). — The equivalence of the second and fourth condition (40) is proved similarly. Hence, four independent necessary conditions are left:

$$(42) \qquad \begin{cases} 2\bar{a}_2 = -C_5 , \\ 3\bar{a}_3 = C_4 , \\ 0 = C_3 , \\ -1 = C_0 , \end{cases}$$

Our final aim is that of utilizing (40) in rewriting the Poisson-Stieltjes presentation for the extremal $f(z)$ given by the extremal $\psi_N(\varphi)$. In the case of a_4, the general form (23) of the presentation concerned, can be written

$$(43) \qquad 1 + z\frac{f''(z)}{f'(z)} = \tfrac{1}{2}\sum_{\nu=1}^{6} \frac{z_\nu + z}{z_\nu - z} \Delta_\nu .$$

Here again, use is made of agreement (37). — Initially, it should be noted that

(44)

$$\sum_{\nu=1}^{6} \frac{z_\nu + z}{z_\nu - z}\Delta_\nu = \frac{-1}{-z^3 g_4(z)} [(z + z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)\Delta_1$$

$$+ \ldots + (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z + z_6)\Delta_6] .$$

For evaluation of the [ ]-expression, we will first rewrite its first term. Let us denote

$$(45) \qquad \tilde{h}_6(z) = (z + z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6) .$$

This polynomial is closely connected to the polynomial $h_6(z)$ defined by (12). We write

$$(46) \qquad \begin{cases} h_6(z) = \sum_{\nu=6}^{0} (-1)^\nu C_\nu z^\nu & (C_6 = 1) , \\ \tilde{h}_6(z) = \sum_{\nu=6}^{0} (-1)^\nu \tilde{C}_\nu z^\nu & (\tilde{C}_6 = 1) , \end{cases}$$

and will express coefficients $\tilde{C}_\nu$ by the aid of coefficients C_ν. The only alteration needed to get $\tilde{C}_\nu$ from C_ν is to change the sign of z_1 in C_ν. — In simplification of the expression of C_2 we need the connection

$$(47) \qquad \bar{C}_5 = \sum_{\nu=1}^{6} \frac{1}{z_\nu} ,$$

which follows from the considerations connected with (41). Consequently:

$$\begin{aligned}
\tilde{C}_5 &= - 2 z_1 + C_5 , \\
\tilde{C}_4 &= - 2z_1 \sum_{\nu=2}^{6} z_\nu + C_4 = - 2z_1(-z_1 + C_5) + C_4 , \\
\tilde{C}_3 &= - 2z_1 \sum_{\mu=2}^{5} z_\mu \sum_{\nu=\mu+1}^{6} z_\nu + C_3 \\
&= - 2z_1[- z_1 \sum_{\nu=2}^{6} z_\nu + \sum_{\mu=1}^{5} z_\mu \sum_{\nu=\mu+1}^{6} z_\nu] + C_3 \\
&= - 2z_1 [- z_1(- z_1 + C_5) + C_4] + C_3 , \\
\tilde{C}_2 &= - z_1z_2z_3z_4z_5z_6(- 2 z_1^{-1} \sum_{\nu=2}^{6} z_\nu^{-1} + \sum_{\mu=1}^{5} z_\mu^{-1} \sum_{\nu=\mu+1}^{6} z_\nu^{-1}) \\
&= - z_1z_2z_3z_4z_5z_6 [- 2 z_1^{-1}(- z_1^{-1} + \sum_{\nu=1}^{6} z_\nu^{-1}) + \sum_{\mu=1}^{5} z_\mu^{-1} \sum_{\nu=\mu+1}^{6} z_\nu^{-1}] \\
&= - C_0 [- 2z_1^{-1} (- z_1^{-1} + \bar{C}_5)] - C_2 , \\
\tilde{C}_1 &= - z_1z_2z_3z_4z_5z_6 (- 2 z_1^{-1} + \sum_{\nu=1}^{6} z_\nu^{-1}) \\
&= C_0 \cdot 2z_1^{-1} - C_1 , \\
\tilde{C}_0 &= - C_0 .
\end{aligned}$$

Hence quite a simple expression for $\tilde{h}_6(z)$ results, when still applying the necessary conditions (40):

$$\begin{aligned}\tilde{h}_6(z) &= z^6 - (-2z_1 + C_5)z^5 + [-2z_1(-z_1 + C_5) + C_4]\, z^4 \\ &\quad + \{2z_1[-z_1(-z_1 + C_5) + C_4] + C_3\}\, z^3 \\ &\quad + \{-C_0[-2z_1^{-1}(-z_1^{-1} + \bar{C}_5)] - C_2\}\, z^2 \\ &\quad - (C_0 \cdot 2z_1^{-1} - C_1)\, z - C_0 \\ &= z^6 + (2z_1 + 2\bar{a}_2)\, z^5 \\ &\quad + (2z_1^2 + 4\bar{a}_2 z_1 + 3\bar{a}_3)\, z^4 + (2z_1^3 + 4\bar{a}_2 z_1^2 + 6\bar{a}_3 z_1)\, z^3 \\ &\quad + (2z_1^{-2} + 4a_2 z_1^{-1} + 3a_3)z^2 + (2z_1^{-1} + 2a_2)\, z + 1\,.\end{aligned}$$

The first term in the [ ]-expression of (44) is thus

$$\begin{aligned}h_6(z)\, \Delta_1 &= \Delta_1 z^6 + 2(z_1\Delta_1 + \bar{a}_2\Delta_1)z_5 \\ &+ (2z_1^2\Delta_1 + 4\bar{a}_2 z_1\Delta_1 + 3\bar{a}_3\Delta_1)z^4 + 2(z_1^3\Delta_1 + 2\bar{a}_2 z_1^2\Delta_1 + 3\bar{a}_3 z_1\Delta_1)\, z^3 \\ &+ \left(2\,\frac{\Delta_1}{z_1^2} + 4a_2\,\frac{\Delta_1}{z_1} + 3a_3\Delta_1\right) z^2 + 2\left(\frac{\Delta_1}{z_1} + a_2\Delta_1\right) z + \Delta_1\,.\end{aligned}$$

By cyclic permutation, and addition the [ ]-expression of (44) assumes the form

$$\begin{aligned}[\ \ ] &= \sum_1^6 \Delta_\nu \cdot z^6 + 2\Big(\sum_1^6 z_\nu\, \Delta_\nu + \bar{a}_2 \sum_1^6 \Delta_\nu\Big)\, z^5 \\ &\quad + \Big(2\sum_1^6 z_\nu^2\, \Delta_\nu + 4\bar{a}_2 \sum_1^6 z_\nu\, \Delta_\nu + 3\bar{a}_3 \sum_1^6 \Delta_\nu\Big)\, z^4 \\ &\quad + 2\Big(\sum_1^6 z_\nu^3\, \Delta_\nu + 2\bar{a}_2 \sum_1^6 z_\nu^2\, \Delta_\nu + 3\bar{a}_3 \sum_1^6 z_\nu\, \Delta_\nu\Big)\, z^3 \\ &\quad + \left(2\sum_1^6 \frac{\Delta_\nu}{z_\nu^2} + 4a_2 \sum_1^6 \frac{\Delta_\nu}{z_\nu} + 3a_3 \sum_1^6 \Delta_\nu\right) z^2 \\ &\quad + 2\left(\sum_1^6 \frac{\Delta_\nu}{z_\nu} + a_2 \sum_1^6 \Delta_\nu\right) z + \sum_1^6 \Delta_\nu\,.\end{aligned}$$

All the sums $\sum\limits_1^6$ can now be expressed in the coefficients a_ν by using expressions (32) and agreement (37). This finally gives

$$
\begin{aligned}
[\ ] &= 2z^2 + 2(2\bar{a}_2 + \bar{a}_2 \cdot 2)\, z^5 \\
&+ [2(6\bar{a}_3 - 4\bar{a}_2^2) + 4\bar{a}_2 \cdot 2\bar{a}_2 + 3\bar{a}_3 \cdot 2]\, z^4 \\
&+ 2\,[12a_4 - 18\bar{a}_2\bar{a}_3 + 8\bar{a}_2^3 + 2\bar{a}_2(6\bar{a}_3 - 4\bar{a}_2^2) + 3\bar{a}_3 \cdot 2\bar{a}_2]\, z^3 \\
&+ [2(6a_3 - 4a_2^2) + 4a_2 \cdot 2a_2 + 3a_3 \cdot 2]\, z^2 \\
&+ 2(2a_2 + a_2 \cdot 2)z + 2 \\
&= 2z^6 + 8\bar{a}_2 z^5 + 18\bar{a}_3 z^3 + 24a_4 z^3 + 18a_3 z^2 + 8a_2 z + 2\,.
\end{aligned}
$$

The right side of (43) has thus assumed the form

$$
\begin{aligned}
\tfrac{1}{2}\sum_{\nu=1}^{6} \frac{z_\nu + z}{z_\nu - z}\, \Delta_\nu &= \tfrac{1}{2}\, \frac{-1}{-z^3 g_4(z)}\, [\ ] \\
&= \frac{z^3 + 4\bar{a}_2 z^2 + 9\bar{a}_3 z + 12a_4 + \dfrac{9a_3}{z} + \dfrac{4a_2}{z^2} + \dfrac{1}{z^3}}{g_4(z)}\,.
\end{aligned}
$$

Theorem. *Let $f(z) \in S_k$ be an extremal function for the coefficient $a_4 > 0$, which is generated by a step-function $\psi_N(\varphi)$ with the points of discontinuity φ_μ ($\mu = 1, \ldots, N$). The pre-images $t_\mu = e^{i\varphi_\mu}$ of the corner points of the extremal polygon satisfy the necessary condition*

$$
g_4(z) = \frac{1}{z^3} + \frac{2a_2}{z^2} + \frac{3a_3}{z} - 3\bar{a}_3 z - 2\bar{a}_2 z^2 - z^3 = 0\,. \tag{48}
$$

The Poisson-Stieltjes presentation for $f(z)$ is

$$
1 + z\frac{f''(z)}{f'(z)} = \tfrac{1}{2}\sum_{\nu=1}^{6} \frac{z_\nu + z}{z_\nu - z}\, \Delta_\nu\,. \tag{49}
$$

Here z_ν are the roots of equation (48) *and for the corresponding Δ_ν, agreement* (37) *holds. Equation* (49) *can be written in the form*

$$
1 + z\frac{f''(z)}{f'(z)} = \frac{G_4(z)}{g_4(z)}\,, \tag{50}
$$

where

$$
G_4(z) = \frac{1}{z^3} + \frac{4a_2}{z^2} + \frac{9a_3}{z} + 12a_4 + 9\bar{a}_3 z + 4\bar{a}_2 z^2 + z^3\,. \tag{51}
$$

2. *On determination of max $|a_4|$*

Result (50), which was proved to hold for every extremal function $f(z)$ generated by step-functions is the same as that obtained from formulae well known in the theory of variation of univalent functions [4]. We consider it important to have arrived at the result (50) without any application of the variational theory mentioned.

In [5] SHIFFER and TAMMI, were able to determine a sharp upper bound for $|a_4|$. This was done by starting from the Poisson-Stieltjes presentation (49), and the necessary condition (50). Comparison of the right sides of these equations provides the necessary conditions (15) and (16) of [5]. From these, maximalization follows laboriously by a proper use of Schwarz's inequality. — It should be remarked, that on the ground of the present paper, the conditions (15) and (16) of [5] appear to be direct consequences of the necessary conditions (42). Thus our conclusion is:

The sharp upper bound of $|a_4|$ *in the class* S_k *follows already from the necessary condition* (48).

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