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ON A SUBCLASS OF THREE-DIMENSIONAL HARMONIC FUNCTIONS DEFINED BY THE BERGMAN-WHITTAKER OPERATOR

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Preliminary Remarks

Contour integration can be used successfully to obtain integral operators which transform an analytic function of several complex variables regular in the neighborhood of the origin into a harmonic function. Because the structure of analytic functions is richer than the structure of harmonic functions, the study of such operators may serve as a tool for the investigation of harmonic functions. In the following we shall be concerned with the Bergman-Whittaker operator defined below (see [2], [3]).

F and H Spaces

Let us consider a neighborhood G of the curve $|\zeta| = 1$ in the ζ-plane,

$$G = \{\zeta : 1 - \varepsilon < |\zeta| < 1 + \varepsilon, \varepsilon > 0\}$$

along with the neighborhood U of the origin in the u-plane

$$U = \{u\colon |u| < \delta, \delta > 0\}\,.$$

Let M denote the set of functions $f(u\,,\zeta)$ of two complex variables u and ζ, analytic in the Cartesian product (which depends on f) of the two neighborhoods G_f and U_f and possessing a development of the form

$$f(u\,,\zeta) = \sum_{0 \leq m < \infty} \sum_{l=-m}^{m} a_{ml}\, u^m\, \zeta^l \tag{1}$$

valid in $U_f \times G_f$.

We say that two functions f defined in $U_f \times G_f$ and g defined in $U_g \times G_g$ are equivalent if $f(u\,,\zeta) = g(u\,,\zeta)$ for $(u\,,\zeta) \in (U_f \cap U_g) \times (G_f \cap G_g)$. This is an equivalence relation and the classes determined by it form an algebra with respect to the usual addition, multiplication and multiplication by scalars. We shall denote this algebra by F. In the following, however, in order not to complicate the notation, we shall not distinguish between a function f and an element of F determined by it.

In a similar way as above we define the linear variety H of classes of complex-valued harmonic functions of three real variables x, y, z compatible with each other in a neighborhood of the origin. Both F and H

can be endowed with a topology in a natural way. For instance, we can introduce the notion of convergence in H by saying that $h = \lim h_n$ if there exists a fixed neighborhood of the origin on which a certain sequence of functions equivalent to h_n converges uniformly to the function equivalent to h.

Bergman-Whittaker Operator

Let us consider the linear mapping $B_3 \colon F \to H$ given by the contour integration formula

$$(2)\quad \begin{aligned} &B_3 f = \frac{1}{2\pi i} \int_{|\zeta|=1} f(u\,,\,\zeta)\, \frac{d\zeta}{\zeta} = h \in H\,, \qquad h \equiv h(x, y, z) \text{ for } (x, y, z) \in U_f \\ &u = Z\zeta + X + Z^* \zeta^{-1},\ X = x,\ Z = \frac{iy + z}{2}\,,\ Z^* = \frac{iy - z}{2}\,. \end{aligned}$$

The mapping B_3 (continuous in the natural topology) is called the Bergman-Whittaker operator.

Since the function $f(u\,,\,\zeta) \in F$ has in the neighborhood of $U_f \times G_f$ the development of the form (1), the monomials $u^n \zeta^m$, $|m| \leq n$, form a inearly dense set in the space F. The complex harmonic functions

$$\Gamma_{nm} = B_3(u^n\, \zeta^m) = \frac{1}{2\pi i} \int_{|\zeta|=1} u^n \zeta^{m-1}\, d\zeta, \qquad |m| \leq n$$

are called the *modified spherical harmonics*. They can be represented also in the form

$$\Gamma_{nm} = \frac{n!\, i^{|m|}}{(n + |m|)!}\, R^n\, P_{n,|m|}(\cos \Theta) e^{i\varphi m}\,.$$

Here R, Θ, and φ are the spherical coordinates of the point (x, y, z); $P_{n,m}$ are the Legendre polynomials (see [3]).

The spherical harmonics Γ_{nm} are linearly dense in $B_3(F)$. By a known result of Bergman, B_3 is homeomorphic and onto H. This was used by Bergman to introduce the operation of composition of the elements in H. The composition $h_1 * h_2$ is defined as the unique element $h \in H$, such that

$$\begin{aligned} h &= B_3(f_1 f_2)\,, \\ h &= B_3(f_k), \qquad\qquad k = 1, 2. \end{aligned}$$

With this operation H becomes a topological algebra. One can now study the subalgebras of F which are isomorphic to the algebra of analytic functions of one complex variable and the corresponding subalgebras of H. An example of such a subalgebra is furnished by the set of those functions belonging to F which in a neighborhood of the origin can be developed in the series of monomials of the form

$$(u^K \zeta^P)^n , \qquad n = 0, 1, 2, \ldots , \qquad P \leq K ,$$

K, P are fixed positive integers. This algebra denoted usually by $t_{K,P,O}$ was introduced and investigated by Bergman (see, for instance [3]).

Consider a fixed polynomial W of the variables u, ζ, ζ^{-1}

$$W(u, \zeta, \zeta^{-1}) = u^r \sum_{K=m}^{M} w_K \zeta^K \tag{3}$$

such that $r \geq 1$, $-r \leq m \leq M \leq r$, $w_m \neq 0$, $w_M \neq 0$.

In the following we shall be concerned with the subalgebra F_W of F containing the elements t of the form

$$t(u, \zeta) = t(\eta) = \sum_{k=0}^{\infty} a_k \eta^k ,$$

where $t(\eta)$ is a meromorphic function regular in the neighborhood of the origin and

$$\eta = W(u, \zeta, \zeta^{-1}) .$$

We introduce the following:

Definition. The function $h \in H$ is called W-meromorphic if

$$h = B_3(t), \qquad t \in F_W. \tag{4}$$

The W-meromorphic functions form a subalgebra H_W of H and

$$H_W = B_3(F) .$$

We define the order ϱ of the W-meromorphic function to be the order of the corresponding meromorphic function $t(\eta)$. Let $T(r, t)$ denote the Nevanlinna characteristic function, then

$$\varrho = \overline{\lim_{r\to\infty}} \frac{\log T(r, t)}{\log r} .$$

Let us call the W-meromorphic function, resulting from the entire function $t(\eta)$, the W-entire function. The formula (4) defines the W-entire function $h(x, y, z)$ for all x, y, z , and the modulus of $h(x, y, z)$ of the order ϱ can be estimated by an exponential function of the variable $\sigma = (x^2 + y^2 + z^2)^{1/2}$. Indeed, for each $\varepsilon > 0$, we have $|t(\eta)| \leq \exp(|\eta|^{\varrho+\varepsilon})$ for $|\eta| > N = N(\varepsilon)$. The continuous function $t(\eta)$ is bounded on the disc $|\eta| \leq N$ by a certain positive number M. Denoting $B = (\sum_K |w_K|)^{\varrho+\varepsilon}$, $B > 0$, and taking into account that

$$\max_{|\zeta|=1} |u| = \sigma$$

we obtain

$$|h(x, y, z)| \leq \frac{1}{2\pi} \int_{|\zeta|=1} |t(\eta)| ds \leq \frac{1}{2\pi} \int_{|\zeta|=1} \max\,(e^{|\eta|^{\varrho+\varepsilon}}, M) ds$$
$$\leq \frac{1}{2\pi} \int_{|\zeta|=1} \max\,(e^{B_\sigma{}^{r(\varrho+\varepsilon)}}, M)\, ds \leq \max\,(e^{B_\sigma{}^{r\varrho+\varepsilon*}}, M)$$

where $\varepsilon^* = r\varepsilon$. We can formulate this result as follows. For each $\varepsilon > 0$ the W-entire function of order ϱ satisfies the inequality

$$|h(x, y, z)| \leq e^{\sigma^{r\varrho+\varepsilon}}$$

for sufficiently large σ.

In the case of a W-entire function which admits a development in spherical harmonics of the form

$$h(x, y, z) = \sum_{n=0}^{\infty} \sum_{K=-n}^{n} \alpha_{n,K}\, \Gamma_{n,K}\,,$$

one can obtain an explicit formula for the order ϱ in terms of the coefficients $\alpha_{n,K}$. Suppose that

$$t(\eta) = \sum_{n=0} a_n\, \eta^n$$

Hence

$$\alpha_{nr,q} = a_n \sum_{\substack{\Sigma\mu_K=n \\ \Sigma K\mu_K=q}} \binom{n}{\mu_m, \ldots, \mu_M} \prod_{K=m}^{M} w_K^{\mu_K}\,.$$

In particular,

$$\alpha_{rn,nm} = a_n (w_m)^n\,.$$

Substituting this into the known formula (see [7])

$$\varrho = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log |\alpha_{nr,nm}|^{-1}}\,.$$

Similarly,

$$\varrho = \overline{\lim_{n\to\infty}} \frac{n \log n}{\log |\alpha_{nr,nM}|^{-1}}\,.$$

If $a \neq t(0) = a_0$, a complex, then the reciprocal $\dfrac{1}{t(\eta) - a} = r_a$ belongs to H_W. By the Mittag-Leffler theorem,

$$\frac{1}{t(\eta) - a} = \sum_{\nu=1}^{\infty} \sum_{s=1}^{q_\nu} \left(\frac{M_{\nu s}}{(\eta - A_\nu(a))^s} - p_\nu(\eta) \right) + l(\eta) \tag{5}$$

where p_ν are the convergence-producing polynomials, $l(\eta)$ is entire, $M_{\nu s}$ are complex constants and $A_\nu(a)$ denotes the a-points of the function $t(\eta)$. By known theorems the order of r_a is ϱ and the series

$$\sum_{\nu=1}^{\infty} \frac{1}{|A_\nu(a)|^{\varrho+\varepsilon}}$$

converges for each $\varepsilon > 0$.

For fixed x^0, y^0, z^0, consider the rectifiable Jordan curve of integration s_1 which does not pass neither through the origin in the ζ-plane, nor through the poles of the integrand

$$g_{\nu s}(\zeta) = \frac{M_{\nu s}}{(\eta - A_\nu)^s}$$

for $\nu = 1, 2, \ldots,$ Then there exists some neighborhood U of the point (x^0, y^0, z^0) such that the curve s^1 still does not pass through the poles of $g_{\nu s}(\zeta)$ for all $(x, y, z) \in U$. Since $A_\nu \to \infty$, the series (5) converges uniformly on $U \times s^1$ and the integration of it over s_1 can be carried out term by term. Consequently, one obtains a harmonic element of r_a defined in U. In the case when s^1 is the unit circle and $x^0 = y^0 = z^0 = 0$ this element belongs to H_W.

In the following we restrict ourselves to the case when $t(\eta)$ has only simple zeros in the η-plane. The general form of the term which contributes to the singularities of r_a is

$$r_a^{(\nu)} = \frac{1}{2\pi i} \int_{|\zeta|=1} g_\nu d\zeta \,, \quad g_\nu = g_{\nu 1} = \frac{M_\nu}{(\eta - A_\nu)\zeta} \,, \quad M_\nu = M_{\nu 1} \,.$$

We shall distinguish three cases depending on the form of the corresponding expression

$$g = \frac{g_\nu}{M_\nu} = \frac{1}{(\eta - \alpha)\zeta} \,, \qquad \alpha \neq 0 \,.$$

i) $m = r$. Then $g = \dfrac{1}{P(\zeta)}$, where

$$P(\zeta) = P(x, y, z; \zeta) = [w_r(Z\zeta^2 + X\zeta + Z^*)^r - \alpha]\, \zeta \,.$$

Since $\alpha \neq 0$, there exists a neighborhood U of the origin 0 in the x, y, z-space such that for $(x, y, z) \in U$, g has only one simple pole $\zeta = 0$ inside the unit disc $|\zeta| \leq 1$.

ii) $m = r - 1$. Then $g = \dfrac{1}{P(\zeta)}$, where

$$P(\zeta) = [w_r(Z\zeta^2 + X\zeta + Z^*)^r - \alpha]\zeta + w_{r-1}(Z\zeta^2 + X\zeta + Z^*)^r .$$

Since at the origin P has only a simple zero $\zeta = 0$, for (x, y, z) belonging to the sufficiently small neighborhood U, P has exactly one zero in the disc $|\zeta| \leq 1$. At this point g has a simple pole.

iii) $m = r - 1 - k$, $0 < k < 2r$. Then $g = \dfrac{\zeta^k}{P(\zeta)}$, where

$$P(\zeta) = \sum_{\mu=0}^{M-m} (Z\zeta^2 + X\zeta + Z^*)\zeta^\mu w_{m+\mu} - \alpha\, \zeta^{k+1} .$$

Since at the origin $P(\zeta)$ has only one root of order $k + 1$, for (x, y, z) belonging to a sufficiently small neighborhood U, P has exactly $k + 1$ zeros in the disc $|\zeta| \leq 1$. Hence g has $k + 1$ poles in this disc if $Z^* \neq 0$ and k poles if $Z^* = 0$, $X \neq 0$.

In all three cases the element $r_a^{(\nu)} \in H_w$ can be represented in U as a sum

$$r_a^{(\nu)} = \sum_{l=1}^{L} R_l$$

where R_l is a harmonic element defined in the disc $D \subset U$ by

$$R_l = \frac{1}{2\pi i} \int_{s_l} g_\nu \, d\zeta .$$

Here s_l is a sufficiently small curve about the simple pole b_l such that the pole b_j of g for $j \neq l$ does not lie inside s_l.

The question on the possibility of an extension of the harmonic element R_l is partially answered by the following theorems:

Theorem 1. Consider the set T of ordered pairs $(p; \zeta) = (x, y, z; \zeta)$ such that $\zeta \neq 0$ is a simple root of $P(x, y, z, \omega)$. T is a three-dimensional smooth manifold imbedded in $E_3 \times C$. Let $(x^0, y^0, z^0, b_l) \in T$, and let T_1 be the component of T which contains (x^0, y^0, z^0, b_l). The component T_1 with projection

$$\pi(x, y, z, \zeta) = (x, y, z) \in E^3$$

is an unbranched covering manifold of $\pi(T_1)$.

Proof. Consider the point $(x', y', z', \zeta') \in T_1$. By a theorem of Hurwitz there exists a neighborhood U of (x', y', z') and a neighborhood V of ζ' such that for $(x, y, z) \in U$, $P(x, y, z, \omega)$ has exactly one simple zero in V. Hence the projection π on $T_1 \cap (U \times V)$ being continuous is a homeomorphism and provides a coordinate system for T_1.

Theorem 2. The harmonic element given in the neighborhood of $(x^0, y^0, z^0, b_l) \in T_1$ by

$$R_l(x, y, z, \zeta) = R_l(x, y, z) = \frac{1}{2\pi i} \int_{s_l} g_\nu(\omega) d\omega$$

can be extended to the univalent harmonic function T_1.

Proof. The extension of R_l on T_1 is given by the formula

$$R_l(x, y, z, \zeta) = \frac{1}{2\pi i} \int_{s} g_\nu(\omega) d\omega = Res_\zeta \, g_\nu(\omega) \,, \tag{6}$$

where s is a sufficiently small curve about ζ. Since the right-hand side of (6) depends only on (x, y, z, ζ), the extension is univalent.

Theorem 3. Denote by $\tilde{S}$ the set of $(x, y, z, \zeta) \in E^3 \times C$ for which $P(x, y, z, \omega)$ does not vanish identically and possesses a multiple root $\zeta \neq 0$. In this case it holds

$$P(x, y, z, \zeta) = 0, \qquad \frac{\partial}{\partial \omega} P(x, y, z, \omega)|_{\omega=\zeta} = 0 \,.$$

Suppose that $(x', y', z', \zeta') = (q; \zeta') \in \tilde{S} = S \cap \bar{T}_1$, $q = (x', y', z')$. Then the extension $R_l(x, y, z, \zeta)$ is unbounded in the neighborhood of $(q; \zeta')$.

Proof. Let

$$\lim p_n = q \,, \qquad \lim \zeta_n = \zeta' \,, \qquad (p_n; \zeta_n) \in T_1 \,.$$

Then

$$R_l(p_n; \zeta_n) = Res_{\zeta_n} (P(p_n; \omega)) = \frac{M_\nu(\zeta_n)^k}{B_n(\zeta_n - \zeta_n^{(1)})(\zeta_n - \zeta_n^{(2)}) \ldots (\zeta_n - \zeta_n^{(\beta)})} \,, \tag{7}$$

where $\zeta_n = \zeta_n^{(0)}$, $\zeta_n^{(i)}$ denote the roots of $P(p_n; \omega)$ and B_n denotes the leading coefficient of P. One infers from the Hurwitz theorem that at least one factor in the denominator of (7) tends to zero, while all the others are bounded. Since $\zeta' \neq 0$, our statement follows.

Consider now the element $r_a \in H_W$. We say that the point $p \in E^3$ is a singular point of r_a if the extension of r_a along the curve $\gamma(\Theta)$, $0 \leq \Theta \leq 1$, joining the origin with p, is possible for $0 \leq \Theta \leq 1$, but for $0 \leq \Theta \leq 1$ it is not possible for any curve $\tilde{\gamma}(\Theta)$, $\gamma(0) = 0$, $\tilde{\gamma}(1) = p$. Let S and $S^{(\nu)}$ be the set of singular points for r_a and $r^{(\nu)}$, respectively. Let d_ν be the minimal distance from $S^{(\nu)}$ to the origin. Then $d_\nu > 0$. For each disc Q,

$$S \cap Q = \left(\bigcup_\nu S^{(\nu)} \right) \cap Q \, ,$$

where the sum $\bigcup_\nu$ is finite. Hence

$$S = \bigcup_{\nu=1}^{\infty} S^{(\nu)} \, .$$

We set

$$k_* = \min_{|\alpha|=1} d(\alpha) \, , \qquad k^* = \max_{|\alpha|=1} d(\alpha) \, ,$$

where $d(\alpha)$ is the minimal distance to the origin from the set of the singular points of the element

$$r^{(\nu)}(\alpha) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{M_\nu \, d\zeta}{(\eta - \alpha)\zeta} \, .$$

Since the integrand is homogeneous of degree r with respect to the variables $x, y, z,$ we have the inequalities

$$k_* |A_\nu|^{1/r} \leq d_\nu \leq k^* |A_\nu|^{1/r} \, , \tag{8}$$

where $d_\nu = d(A_\nu)$.

In conclusion we obtain the following theorem characterizing the relation between the order of the W-meromorphic function and the geometric properties of the singularities of its reciprocal:

Theorem 4. The set of singular points of r_a is given by $\bigcup_\nu S^{(\nu)}$. If ϱ is the order of the corresponding W-meromorphic function, then for each $\varepsilon > 0$ the series

$$\sum_\nu \frac{1}{d_\nu(a)^{\varrho r + \varepsilon}} \tag{9}$$

converges for each value of $a \neq a_0$. Also, if the series

$$\sum_\nu \frac{1}{d_\nu(a)^{\varrho r}} \tag{10}$$

converges for three different values of $a \neq a_0$, then it converges for every value of a, and the order of the corresponding W-meromorphic function is not greater than ϱ.

Proof. By the inequality (8) the series (9) and (10) converge simultaneously with the series

$$\sum \frac{1}{|A_\nu|^{\varrho+\delta}}, \qquad \delta > 0$$

and

$$\sum \frac{1}{|A_\nu|^{\varrho}},$$

respectively. The results follows from the Nevanlinna theorem, see [7].

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Bibliography

1. BERGMAN, S., Zur Theorie der algebraischen Potentialfunktionen des dreidimensionalen Raumes - Math. Ann. 99, 1928, pp. 629—659 and 101, 1929, pp. 534—558.
2. —»— Integral Operators in the Theory of Linear Partial Differential Equations - Ergebnisse der Mathematik, N.F. 23, Berlin, 1961.
3. —»— Integral operators in the study of an algebra and of a coefficient problem in the theory of three-dimensional harmonic functions - Duke Math. J. 30, 1963, pp. 447—460.
4. GILBERT, R., On generalized axially symmetric potentials - J. Reine Angew. Math. 212, 1963, pp. 158—168.
5. KREYSZIG, E., On a class of integral operators of Bergman-Whittaker type - Arch. Rational Mech. Anal. 10, 1962, pp. 242—248.
6. MITCHELL, J., Properties of harmonic functions of three real variables - Canad. J. Math. 15, 1963, pp. 157—168.
7. NEVANLINNA, R., Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes - Paris, 1929.
8. WHITTAKER, E., On the partial differential equations of mathematical physics - Math. Ann. 57, 1903, pp. 333—355.

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