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ON BOUNDARY DERIVATIVES IN CONFORMAL MAPPING

BY

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Introduction

In the following we present an elementary and simplified approach to the study of the differentiability of a conformal map at a boundary point, which at the same time permits us to improve upon existing results. Let Ω denote a simply connected domain, z_0 an accessible boundary point of Ω and f a function which maps the upper half plane conformally onto Ω such that z_0 correspond to the origin $\zeta = 0$. Since the papers by J. Wolff [20] in 1926 and particularly by C. Carathéodory [2] in 1929 appeared, a great deal of interest has centered on obtaining criteria for the existence of the (finite and non-vanishing) »*angular derivate*» of f at $\zeta = 0$. Such criteria were given by Ahlfors [1], the writer [17], Van der Corput [13], Visser [16] and more recently, [4], [5], by J. Ferrand and J. Dufresnoy, and others. A survey of the earlier criteria (until 1948) is presented in [9], Chapter III, and of later ones (until 1955) in [6], Chapter VI.

The derivations of most of these criteria are based on extensions of the Wolff—Carathéodory theorem in conjunction with one or both of Ahlfors' principal inequalities in [1] or on extensions and refinements of these inequalities (cf. [6], Chapter VI). The present writer has also been interested in [17], [18], and [19] in criteria for the existence of the »*unrestricted*» derivative of f at $\zeta = 0$, assuming that the boundary of Ω in a neighborhood of z_0 is a (free) Jordan arc. Another proof of the author's result on the unrestricted derivative was given by M. Tsuji [12] pp. 366—377. A further method for providing the transition from the angular derivative to the unrestricted derivative is Ostrowski's »Faltensatz» [11]; see also the writer's recent new proof of this theorem [19]; cf. also Wolff's earlier result in [21].

In the present paper we consider the inverse function φ of f and show that under proper conditions (which in general cannot be improved) φ has a finite derivative $\varkappa$ at $z = z_0$ for approach in the Ostrowski »*kernel*» of Ω at z_0 (for the definition of the kernel see section 1). This is the content of Theorem 1 (and it, of course, also implies the existence of the angular

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derivative at z_0). However, here $\varkappa$ may be 0 and we give a rather general criterion (in section 5) which insures that $\varkappa \neq 0$.

From Theorem 1 we obtain in a very short manner a criterion for the existence of the unrestricted derivative (Theorem 2).

The method applied here to prove Theorem 1 is different from those previously employed in establishing criteria for the existence of the derivative, and the author believes it of interest even beyond the immediate purpose it serves in this paper.

1. Preliminaries

Let Ω be a simply connected domain in the z-plane, $z = x + iy$, and z_0 a boundary point at $z = 0$ which is accessible along the segment $\{x = 0, 0 < y \leqq r_0\} \subset \Omega$. For $0 < r \leqq r_0$ let k_r denote the subarc of $|z| = r$ which contains $z = ir$ and forms a (circular) cross cut of Ω. Following A. Ostrowski [11] we denote the subdomain $K = \{z \, z \in k_r, 0 < r < r_0\}$ as the kernel of Ω at z_0. (K depends on r_0, but this is not essential for our purpose).

Ω will be, at times, subjected to the following additional conditions at z_0, although not necessarily simultaneously to all three.

(A) The boundary point z_0 is accessible in any »Stolz angle», symmetrical to the imaginary axis of opening less than π, i.e. for every α, $0 < \alpha < \frac{\pi}{2}$, there exists an $r_\alpha > 0$ such that the sector $A_\alpha = \{z \, \alpha \leqq \arg z \leqq \pi - \alpha, 0 < |z| \leqq r_\alpha\} \subset \Omega$.

If the length of k_r is $r\Theta(r)$, and $\pi\varepsilon(r) = \operatorname{Max}\,[\Theta(r) - \pi, 0]$, then the two other conditions are:

$$\text{(B)} \qquad \int_0^{r_0} \frac{\varepsilon(r)}{r}\, dr < \infty$$

and

$$\text{(C)} \qquad \lim_{r \to 0} \varepsilon(r) = 0 \, .$$

Let $\zeta = \varphi(z)$, $\zeta = \xi + i\eta$, map Ω conformally onto the half-plane $\eta > 0$ such that z_0 corresponds to $\zeta = 0$. We may assume that the endpoints of the image $\varphi(k_{r_0})$ of k_{r_0} are finite and on opposite sides of $\zeta = 0$.

If Ω satisfies the hypotheses (A) and (C), then $\varphi(z)$ is »semi-conformal» in the kernel K, i.e. as $z \to z_0, z \in K$

$$\lim_{z \to z_0} \arg \frac{\varphi(z)}{z} = 0 \, . \tag{1.1}$$

This is a direct consequence of a lemma of J. Wolff [22], as stated (with proof) in [19], pp. 84—86.[1]) In [19] it is proved as a »Remark» to Theorem 1 a, p. 89; a direct proof is given in the Appendix of this paper.

2. Auxiliary results

We prove first the following lemmas.

Lemma 1. *Suppose the domain* Ω *satisfies condition* (B). *Then there exists a constant* M *such that for* $z \in K$, $|z| \leqq \frac{r_0}{2}$, $\left|\frac{\varphi(z)}{z}\right| \leqq M$.[2])

Proof. Let l_ϱ denote the length of the image $\gamma_\varrho = \varphi(k_\varrho)$ under the mapping $z \to \varphi(z)$. Then we have

$$l_\varrho^2 = \left(\int_{k_\varrho} |\varphi'(\varrho e^{i\theta})| \varrho d\theta\right)^2 \leqq \int_{k_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \varrho d\theta \cdot \Theta(\varrho)\, \varrho \,. \tag{2.1}$$

Hence, for $0 < r \leqq r_0$, since $\Theta(\varrho) \leqq \pi(1 + \varepsilon(\varrho))$,

$$\int_0^r \frac{l_\varrho^2\, d\varrho}{\varrho(1 + \varepsilon(\varrho))} \leqq \pi \int_0^r d\varrho \int_{k_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \varrho d\theta = \pi A(r) \,. \tag{2.2}$$

Here $A(r)$ denotes the area of image Δ_r of D_r and D_r is that of the two subdomains of Ω formed by k_r which contains the segment $0 < y < r$ of the imaginary axis. For almost all $r, 0 < r \leq r_0, l_r < \infty$ and γ_r is therefore a rectifiable Jordan arc; Δ_r is bounded by γ_r and a segment of the real axis (containing $\zeta = 0$). We reflect γ_r and Δ_r in the real axis, and γ_r and its reflection $\bar{\gamma}_r$ form a closed Jordan curve of length $2l_r$. The area of its interior is $2A(r)$. Hence, we have by the isoperimetric inequality

$$2A(r) \leqq \frac{(2l_r)^2}{4\pi} \quad \text{or} \quad \pi A(r) \leqq \frac{l_r^2}{2} \,.$$

Thus we have for all $r, 0 < r \leq r_0$, (l_r may be $+\infty$ on a set of measure 0)

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[1]) If Ω satisfies (A) and (B), then it is shown in [19], p. 96 that (1.1) holds as $z \to z_0$ in any angle A_α. A necessary and sufficient condition that the map $z \to \varphi(z)$ be »semi-conformal» in any Stolz angle A_α is due to Ostrowski [11]; several proofs of this result were subsequently given by C. Gattegno [7], based on Wolff's lemma, in [22], J. Lelong Ferrand [6], pp. 113—122, by use of Carathéodory's »kernel» theorem, and by the writer [19], pp. 87—89.

[2]) The proof of this lemma may be obtained by use of Ahlfors' First Inequality (Distortion Theorem) [1]. However, we present an independent proof since some of the inequalities we obtain are used later on in this paper.

$$\pi \int_0^r \int_{k_r} |\varphi'(\varrho e^{i\theta})|^2 \varrho d\theta d\varrho \leqq \frac{l_r^2}{2} . \tag{2.3}$$

Since for almost all $r, 0 \leqq r \leqq r_0$

$$A'(r) = \int_{k_r} |\varphi'(re^{i\theta})|^2 \, r d\theta$$

we have from (2.1)

$$l_r^2 \leqq \pi r(1 + \varepsilon(r))A'(r) ,$$

and by use of (2.3)

$$A(r) \leqq \frac{r}{2} (1 + \varepsilon(r))A'(r) , \frac{2}{r(1 + \varepsilon(r))} \leqq \frac{A'(r)}{A(r)} ,$$

or, finally, noting that $1 - \varepsilon(r) \leqq (1 + \varepsilon(r))^{-1}$,

$$\frac{2}{r} - \frac{2\varepsilon(r)}{r} \leqq \frac{A'(r)}{A(r)} .$$

Integration between the limits r and $r_1, r < r_1 \leqq r_0$, yields

$$\frac{A(r)}{r^2} e^{2\int_0^r \frac{\varepsilon(t)}{t} dt} \leqq \frac{A(r_1)}{r_1^2} e^{2\int_0^{r_1} \frac{\varepsilon(t)}{t} dt} . \tag{2.4}$$

Because of condition (B) and the monotone character of the function above,

$$\lim_{r \downarrow 0} \frac{A(r)}{r^2} = \mu \ (\geqq 0) \tag{2.5}$$

exists.

Now by (2.2), since clearly $0 \leqq \varepsilon(r) \leqq 1, \ 0 < r \leqq r_0$,

$$\frac{1}{2} \int_{\frac{r}{2}}^{r} \frac{l_t^2}{t} dt < \int_0^r \frac{l_t^2 \, dt}{t(1 + \varepsilon(t))} \leqq \pi A(r) \leqq M_0 r^2 \tag{2.6}$$

where $M_0 = \pi \dfrac{A(r_0)}{r_0^2} \exp \left[2 \displaystyle\int_0^{r_0} \frac{\varepsilon(t)}{t} dt \right]$. Hence there exists an r_1 with $\frac{r}{2} \leqq r_1 \leqq r$ such that

$$l_{r_1}^2 \log 2 \leqq 2M_0 r^2 \left(\frac{r}{2} \leqq r_1 \leqq r\right). \tag{2.7}$$

Consider a cross cut k_ϱ with $0 < \varrho \leqq \frac{r_0}{2}$. We take $r = 2\varrho$ and apply (2.7), where now $\varrho \leqq r_1 \leqq 2\varrho$. Since k_ϱ is contained in the closure of D_{r_1} it follows that, for $z \in k_\varrho$, $\varphi(z) \in \Lambda_{r_1}$ and therefore $|\varphi(z)| \leqq l_{r_1}$. Hence

$$\left|\frac{\varphi(z)}{z}\right| \leqq \frac{l_{r_1}}{\varrho} \leqq 2\sqrt{\frac{2M_0}{\log 2}} \equiv M \quad (z \in k_\varrho)\,.$$

Lemma 2.[3]) *If the mapping function* φ *in Section* 1 *is »semiconformal» in any Stolz angle* A_α $\left(0 < \alpha < \frac{\pi}{2}\right)$, *i.e. if* $\lim\limits_{z\to z_0} \arg\frac{\varphi(z)}{z} = 0$ *in any* A_α, *then*

$$\lim_{z\to z_0} \frac{z\varphi'(z)}{\varphi(z)} = 1\,, \quad z \in A_\alpha\,. \tag{2.8}$$

Furthermore if Ω *satisfies conditions* (A) *and* (B), *then for* $z \in A_\alpha$

$$\lim_{z\to z_0}\left[\frac{\varphi(z)}{z} - \varphi'(z)\right] = 0 \quad (z \in A_\alpha)\,. \tag{2.9}$$

Proof. Although (2.8) is well known[3]) we present a very short proof here for the sake of completeness. Let $\log\frac{z\varphi'(z)}{\varphi(z)}$ represent the determination of the logarithm for which $\lim\limits_{z\to z_0} \arg\frac{\varphi(z)}{z} = 0$ $(z \in A_\alpha)$. Let $z \in A_\alpha$ and let ϱ denote the radius of the largest circle about z contained in $A_{\alpha/2}$. Then (see e.g. [3] p. 88)

$$i\left[\frac{\varphi'(z)}{\varphi(z)} - \frac{1}{z}\right] = -\frac{1}{\pi\varrho}\int_0^{2\pi} \arg\left[\frac{\varphi(\zeta)}{\zeta}\right] e^{-i\theta}\, d\theta \quad (\zeta = z + \varrho e^{i\theta})\,.$$

If z is sufficiently close to 0, $\frac{\varrho}{|z|} \geqq \sin\frac{\alpha}{2}$ and hence

$$\left|\frac{z\varphi'(z)}{\varphi(z)} - 1\right| \leqq 2\left|\frac{z}{\varrho}\right| \cdot h \leqq 2h \csc\frac{\alpha}{2}\,, \quad h = \operatorname{Max}_{\zeta}\left|\arg\frac{\varphi(\zeta)}{\zeta}\right|$$

for $\zeta \in A_{\alpha/2}$, and $|\zeta| \leqq |z| + \varrho$. As $z \to 0$ in A_α, $h \to 0$.

The second part of the lemma, (2.9), follows from the first by use of Lemma 1, since

[3]) C. Visser [14], theorem 7; A. Ostrowski [10], theorem 4.

$$\left|\varphi'(z) - \frac{\varphi(z)}{z}\right| = \left|\left(\frac{z\varphi'(z)}{\varphi(z)} - 1\right)\frac{\varphi(z)}{z}\right| \leqq M\left|\frac{z\varphi'(z)}{\varphi(z)} - 1\right| .$$

Lemma 3. *Suppose* Ω *satisfies conditions* (A), (B) *and* (C). *Then there exist for every* $r \leqq \frac{r_0}{2}$ *and for every fixed* δ, $0 < \delta < 1$ *an* r' *with* $r \leqq r' \leqq r(1+\delta)$ *and an* r'' *with* $(1-\delta)r \leqq r'' \leqq r$ *such that for* $\varrho = r'$ *and* $\varrho = r''$

$$\sigma(\varrho) = \int_{k_\varrho} \left|\varphi'(z) - \frac{\varphi(z)}{z}\right| d\theta \to 0 \quad \text{as} \quad r \to 0 \quad (z = \varrho e^{i\theta}) .$$

Corollary. For $z_1 \in k_\varrho$, $z_2 \in k_\varrho$, for $\varrho = r'$ and $\varrho = r''$ uniformly:

$$\lim_{r\to 0}\left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| = 0 \tag{2.10}$$

The Corollary follows immediately from the Lemma, since

$$\left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| = \left|\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\left[\frac{\varphi(\varrho e^{i\theta})}{\varrho e^{i\theta}}\right] d\theta\right| \leqq \int_{k_\varrho}\left|\varphi'(z) - \frac{\varphi(z)}{z}\right| d\theta .$$

Proof of the Lemma. Let $re^{i\beta_1}$ and $re^{i\beta_2}$ be the endpoints of k_r and let A_α be a sector intersected by k_r $\left(0 < \alpha < \frac{\pi}{2}\right)$. Then

$$\sigma(r) = \int_{\beta_1}^{\beta_2} = \int_{\beta_1}^{\alpha} + \int_{\alpha}^{\pi-\alpha} + \int_{\pi-\alpha}^{\beta_2} .$$

Let $\eta > 0$ be given. Then by Lemma 2, for fixed α,

$$\int_{\alpha}^{\pi-\alpha}\left|\varphi'(z) - \frac{\varphi(z)}{z}\right| d\theta < \eta \quad (z = re^{i\theta})$$

for all sufficiently small r, say $r \leqq r_1(\eta, \alpha)$. Next, by Lemma 1 and the Schwarz inequality (for $z = re^{i\theta}$)

$$\int_{\beta_1}^{\alpha}\left|\varphi'(z) - \frac{\varphi(z)}{z}\right| d\theta \leqq \int_{\beta_1}^{\alpha}\left|\frac{\varphi(z)}{z}\right| d\theta + \int_{\beta_1}^{\alpha}|\varphi'(z)|d\theta$$

$$\leqq (\alpha - \beta_1)M + \left\{(\alpha - \beta_1)\int_{\beta_1}^{\alpha}|\varphi'(z)|^2\, d\theta\right\}^{1/2} ,$$

and an analogous estimate is obtained for $\int\limits_{\pi-\alpha}^{\beta_2} \left| \varphi'(z) - \frac{\varphi(z)}{z} \right| d\theta$.

Noting that

$$\alpha - \beta_1 \leqq \alpha + \pi\varepsilon(r)\,, \quad \beta_2 - (\pi - \alpha) \leqq \alpha + \pi\varepsilon(r)$$

we finally obtain

$$\sigma(r) \leqq \eta + 2(\alpha + \pi\varepsilon(r))M + 2\left\{(\alpha + \pi\varepsilon(r)) \int\limits_{k_r} |\varphi'(re^{i\theta})|^2\, d\theta\right\}^{1/2}. \tag{2.11}$$

Now we determine a bound for the last integral. Since for $r \leqq r_0$,

$$\frac{A(r)}{r^2} < \frac{A(r_0)}{r_0^2} \exp\left[2 \int\limits_0^{r_0} \frac{\varepsilon(t)}{t}\, dt\right] \equiv M_1$$

we have for $r \leqq \frac{r_0}{2}$

$$\frac{A(r(1+\delta)) - A(r)}{r^2} = \frac{A(r(1+\delta))}{r^2(1+\delta)^2}(1+\delta)^2 - \frac{A(r)}{r^2} < 4M_1$$

or

$$\frac{1}{r^2} \int\limits_r^{r(1+\delta)} \int\limits_{k_t} |\varphi'(te^{i\theta})|^2\, t d\theta dt < 4M_1\,.$$

Hence, there exists an r' with $r \leqq r' \leqq r(1+\delta)$ such that

$$\int\limits_{k_\varrho} |\varphi'(\varrho e^{i\theta})|^2\, d\theta\, \frac{1}{r^2} \int\limits_r^{r(1+\delta)} t\, dt < 4M_1 \quad (\varrho = r') \tag{2.12}$$

or

$$\int\limits_{k_\varrho} |\varphi'(\varrho e^{i\theta}|^2\, d\theta \leqq \frac{8M_1}{2\delta + \delta^2} = K(\delta) \quad (\varrho = r')\,. \tag{2.13}$$

Thus, substituting ϱ in place of r in (2.11) and using (2.13) we obtain

$$\overline{\lim_{r \to 0}}\, \sigma(\varrho) \leqq \eta + 2\alpha M + 2\,\{\alpha K(\delta)\}^{1/2} \quad (\varrho = r')\,.$$

Letting here first $\eta \to 0$ and subsequently (for fixed δ) $\alpha \to 0$ we obtain the conclusion for $\varrho = r'$. The derivation of a bound for the integral in (2.13) for $\varrho = r''$ and the remainder of the proof is completely analogous.

Lemma 4. *Suppose the domain Ω and k_r, $\Theta(r)$, $\varphi(z)$ are defined as in section* 1. *Let l_r be the length of $\varphi(k_r)$. Then for $0 < r < r_0$*

$$r\Theta(r) \operatorname*{Inf}_{z \in k_r} \left| \frac{\varphi(z)}{z} \right| - r\sigma(r) \leqq l_r \leqq r\Theta(r) \operatorname*{Sup}_{z \in k_r} \left| \frac{\varphi(z)}{z} \right| + r\sigma(r) . \tag{2.14}$$

Here l_r and $\sigma(r)$ may be $+\infty$ for some r on a set of measure 0.

Proof. Consider a partition $\theta_0 < \theta_1 < \ldots < \theta_n$ of k_r, $z_\nu = re^{i\theta_\nu}$. Then

$$L_n = \sum_{\nu=0}^{n-1} |\varphi(z_{\nu+1}) - \varphi(z_\nu)| = \sum_{\nu=0}^{n-1} \left| \frac{\varphi(z_{\nu+1})}{z_{\nu+1}} z_{\nu+1} - \frac{\varphi(z_\nu)}{z_\nu} z_\nu \right|$$

$$\leqq \sum_{\nu=0}^{n-1} \left| \frac{\varphi(z_{\nu+1})}{z_{\nu+1}} - \frac{\varphi(z_\nu)}{z_\nu} \right| |z_{\nu+1}| + \sum_{\nu=0}^{n-1} \left| \frac{\varphi(z_\nu)}{z_\nu} \right| |z_{\nu+1} - z_\nu| .$$

Hence

$$L_n \leqq \operatorname*{Sup}_{z \in k_r} \left| \frac{\varphi(z)}{z} \right| \cdot r\,\Theta(r) + r \sum_{\nu=0}^{n-1} \int_{\theta_\nu}^{\theta_{\nu+1}} \left| \varphi'(re^{i\theta}) - \frac{\varphi(re^{i\theta})}{re^{i\theta}} \right| d\theta$$

or

$$l_r \leqq r\Theta(r) \operatorname*{Sup}_{z \in k_r} \left| \frac{\varphi(z)}{z} \right| + r\sigma(r) .$$

Since also

$$L_n \geqq \sum_{\nu=0}^{n-1} \left| \frac{\varphi(z_\nu)}{z_\nu} \right| |z_{\nu+1} - z_\nu| - r\sigma(r)$$

we find easily, by letting the norm of the partition tend to 0, the left hand inequality.

3. The derivative for approach in the kernel

We prove now our first principal theorem.

Theorem 1. *If the domain Ω satisfies hypotheses* (A), (B), *and* (C) *and if $\varphi(z)$ is the mapping function defined in section* 1, *then for $z \in K$,*

$$\lim_{z \to z_0} \frac{\varphi(z)}{z} = \varkappa = \sqrt{\frac{2\mu}{\pi}} .$$

Here $\mu = \lim_{r \to 0} \dfrac{A(r)}{r^2}$ *defined in* (2.5).

Proof. For any r, $0 < r \leqq \frac{r_0}{2}$, and fixed δ, $0 < \delta < 1$, let $\varrho = r'$ or $\varrho = r''$ (Lemma 3). Then by (2.10)

$$\text{(3.1)} \qquad \operatorname*{Sup}_{z \in k_\varrho} \left| \frac{\varphi(z)}{z} - \frac{\varphi(i\varrho)}{i\varrho} \right| = \eta = \eta(\varrho, \delta) \to 0 \quad \text{as} \quad \varrho \to 0 .$$

Hence, by (2.14)

$$\text{(3.2)} \qquad \left(\left| \frac{\varphi(i\varrho)}{i\varrho} \right| - \eta \right) \Theta(\varrho) - \sigma(\varrho) \leqq \frac{l_\varrho}{\varrho} \leqq \left(\left| \frac{\varphi(i\varrho)}{i\varrho} \right| + \eta \right) \Theta(\varrho) + \sigma(\varrho) .$$

As $r \to 0$ we have, therefore, for $\varrho = r'$ and $\varrho = r''$:

$$\text{(3.3)} \qquad \overline{\lim_{r\to 0}} \left| \frac{\varphi(i\varrho)}{i\varrho} \right| \pi \leqq \overline{\lim_{r\to 0}} \frac{l_\varrho}{\varrho} , \quad \underline{\lim_{r\to 0}} \left| \frac{\varphi(i\varrho)}{i\varrho} \right| \pi \geqq \underline{\lim_{r\to 0}} \frac{l_\varrho}{\varrho} .$$

Now, by (2.3), for $0 < r < r_0$

$$2\pi \frac{A(r)}{r^2} \leqq \frac{l_r^2}{r^2}$$

and, therefore, by (2.5)

$$2\pi\, \mu \leqq \underline{\lim_{r\to 0}} \frac{l_r^2}{r^2} .$$

Thus, by the second inequality in (3.3)

$$\text{(3.4)} \qquad \sqrt{\frac{2\mu}{\pi}} \leqq \underline{\lim_{\varrho\to 0}} \left| \frac{\varphi(i\varrho)}{i\varrho} \right| .$$

To estimate the $\overline{\lim\limits_{\varrho\to 0}} \frac{l_\varrho}{\varrho}$ we note that for $0 < r < \frac{r_0}{2}$

$$\frac{1}{r^2} [A(r(1+\delta)) - A(r)] = \frac{A(r(1+\delta))}{r^2(1+\delta)^2} (1+\delta)^2 - \frac{A(r)}{r^2} \to \mu(2\delta + \delta^2)$$

as $r \to 0$. Hence, given any $\varepsilon > 0$, there exists an $r_1 = r_1(\varepsilon, \delta)$, such that for $r < r_1$

$$\frac{1}{r^2} \int_r^{r(1+\delta)} t \int_{k_t} |\varphi'(te^{i\theta})|^2 \, d\theta \, dt < \mu(2\delta + \delta^2) + \varepsilon .$$

Using $\varrho = r'$ as determined in (2.12), we obtain by the same calculation

$$\int_{k_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \, dt < 2\mu + \frac{2\varepsilon}{2\delta + \delta^2} \, . \tag{3.5}$$

Now, by (2.1) and (3.5)

$$\frac{l_\varrho^2}{\varrho^2} \leqq \Theta(\varrho) \int_{k_\varrho} |\varphi'(\varrho e^{i\theta})|^2 \, d\theta < 2\mu\Theta(\varrho) + \frac{2\varepsilon\Theta(\varrho)}{2\delta + \delta^2}$$

and, therefore, for $\varrho = r'$

$$\overline{\lim_{\varrho \to 0}} \frac{l_\varrho^2}{\varrho^2} \leqq 2\pi\mu + \frac{2\varepsilon\pi}{2\delta + \delta^2} \, . \tag{3.6}$$

A completely analogous argument shows that for $\varrho = r''$ (Lemma 3)

$$\overline{\lim_{\varrho \to 0}} \frac{l_\varrho^2}{\varrho^2} \leqq 2\pi\mu + \frac{2\varepsilon\pi}{2\delta - \delta^2} \, .$$

Thus we obtain for $\varrho = r'$ and $\varrho = r''$, using the inequality on the left in (3.3),

$$\overline{\lim_{\varrho \to 0}} \left| \frac{\varphi(i\varrho)}{i\varrho} \right| \leqq \left\{ \frac{2\mu}{\pi} + \frac{2\varepsilon}{\pi(2\delta - \delta^2)} \right\}^{1/2} .$$

Since this holds for every $\varepsilon > 0$ we have for $\varrho = r'$ and $\varrho = r''$

$$\overline{\lim_{\varrho \to 0}} \left| \frac{\varphi(i\varrho)}{i\varrho} \right| \leqq \sqrt{\frac{2\mu}{\pi}} \, . \tag{3.7}$$

Combining (3.4), (3.7), and (3.1) we find for $z \in k_\varrho$

$$\lim_{z \to 0} \left| \frac{\varphi(z)}{z} \right| = \sqrt{\frac{2\mu}{\pi}} \, . \tag{3.8}$$

Finally, to show that (3.8) holds when $z \to z_0$ in the kernel K, we keep δ, $0 < \delta < 1$, fixed and determine for a given r the radii r' and r''. For any $z \in k_r$ there exist points $z' \in k_{r'}$ and $z'' \in k_{r''}$ such that

$$(1 - \delta) \left| \frac{\varphi(z'')}{z''} \right| < \left| \frac{\varphi(z)}{z} \right| < \left| \frac{\varphi(z')}{z'} \right| (1 + \delta)$$

(To find z' and z'' we merely determine *proper* points in which $\arg \zeta = \arg \varphi(z)$ intersects the arcs $\varphi(k_{r'})$ and $\varphi(k_{r''})$, respectively. The inverse images of these points are z' and z''). Now, as $r \to 0$, $z \in k_r$

$$(1 - \delta) \sqrt{\frac{2\mu}{\pi}} \leqq \underline{\lim_{r \to 0}} \left| \frac{\varphi(z)}{z} \right| \leqq \overline{\lim_{r \to 0}} \left| \frac{\varphi(z)}{z} \right| \leqq \sqrt{\frac{2\mu}{\pi}} (1 + \delta) \, . \tag{3.9}$$

Since δ may be chosen arbitrarily small, (3.8) is valid as $z \to z_0$ in K. To complete the proof of our theorem we note that, because of hypothesis (C), for $z \in K$, $\lim\limits_{z \to 0} \arg \dfrac{\varphi(z)}{z} = 0$.

Remark. *Under the hypotheses of Theorem* 1, *for* $z \in A_\alpha$ *for any* α, $0 < \alpha < \frac{\pi}{2}$, $\lim\limits_{z \to z_0} \varphi'(z) = \varkappa$. (see (2.9)).

4. Existence of the derivative for unrestricted approach

Suppose C is a closed Jordan curve which passes through $z = 0$. Let $z(t)$, $-T < t \leqq T$ be a parametrization of C such that $z(0) = 0$. For $0 < r < |z(T)|$ draw the circle $|z| = r$; let

$$t_r = \max_{|z(t)| = r} |t| , \quad \text{where} \quad |t| < T ,$$

and

$$\Delta(r) = \max_{|t| \leqq t_r} |z(t)| .$$

Thus the arc $z = z(t)$ for $|t| \leqq t_r$ is entirely contained in $|z| \leqq \Delta(r)$. We shall say that C *satisfies condition* (U) *at* $z = 0$ *if*

$$\lim_{r \to 0} \frac{\Delta(r)}{r} = 1^{4)}.$$

This condition is independent of the parametrization of C.

Remark. Let $t_r^* = \min\limits_{|z(t)| = r} |t|$ and $\delta(r) = \min\limits_{t_r^* \leqq |t| \leqq t_r} |z(t)|$.

Then it is easily seen from the definition of $\Delta(r)$ that

$$\Delta(\delta(r)) \geqq r .$$

Theorem 2. *Suppose that* C *is a closed Jordan curve passing through* $z = 0$ *which has the following properties:*

(a) *The x-axis is a tangent to* C *at* $z = 0$ *and the positive y-axis points in the direction of the interior normal.*
(b) C *satisfies condition* (U) *at* $z = 0$.
(c) *The interior domain* Ω *bounded by* C *satisfies condition* (B) *of section* 1.[5)]

[4)] In [17] this condition is called »reguläre Unbewalltheit«.

[5)] Conditions (A) and (C) are fulfilled since C has a tangent at $z = 0$.

If $\zeta = \varphi(z)$ *is the mapping function defined in section* 1, *then*

$$\lim_{z\to 0} \frac{\varphi(z)}{z} = \varkappa \ (0 \leqq \varkappa < \infty$$

exists for unrestricted approach in $\Omega \cup C$.

Proof. Let ε be given, $0 < \varepsilon < 1$. Because of hypothesis (b) there exists an $R(\varepsilon) > 0$ such that for $0 < r \leqq R(\varepsilon)$

$$\Delta(r) < (1 + \varepsilon)r .$$

Let $\delta(r)$ be defined as above (see Remark). Then $\Delta(\delta(r)) < (1 + \varepsilon)\delta(r)$, and since $\Delta(\delta(r)) \geqq r$ we have

$$r < (1 + \varepsilon)\delta(r) .$$

Thus we have the inequalities

$$r(1 - \varepsilon) < \frac{r}{1 + \varepsilon} < \delta(r) \leqq r \leqq \Delta(r) < (1 + \varepsilon)r . \tag{4.1}$$

From the definitions of $\Delta(r)$ and $\delta(r)$ we note: if $z = re^{i\theta} \in C$, then for any other point $z' \in C$ with $|z'| > \Delta(r)$ on the »same side» of $z = 0$ as z

$$|\varphi(z')| > |\varphi(z)|$$

and for any $z'' \in C$ with $|z''| < \delta(r)$ on the »same side» of $z = 0$ as z

$$|\varphi(z)| > |\varphi(z'')| .$$

Consider two circular cross cuts $k_{r''}$ and $k_{r'}$ with $(1 - \varepsilon)\delta(r) \leqq r'' < \delta(r)$ and $\Delta(r) < r' \leqq (1 + \varepsilon)\Delta(r)$. Let z'' be an endpoint of $k_{r''}$ and z' and endpoint of $k_{r'}$ (z' and z'' are points of C). Then for z, z', z'' on the »same side» of $z = 0$

$$\frac{r''}{r} \left| \frac{\varphi(z'')}{z''} \right| \leqq \left| \frac{\varphi(z)}{z} \right| \leqq \left| \frac{\varphi(z')}{z'} \right| \frac{r'}{r} .$$

We let now $r \to 0$ and obtain from Theorem 1 and (4.1)

$$\varkappa(1 - \varepsilon)^2 \leqq \varliminf_{r\to 0} \left| \frac{\varphi(z)}{z} \right| \leqq \varlimsup_{r\to 0} \left| \frac{\varphi(z)}{z} \right| = \varkappa(1 + \varepsilon)^2 .$$

Since ε is arbitrary, we have for approach along C from either side of $z = 0$,

$$\lim_{z\to 0} \left| \frac{\varphi(z)}{z} \right| = \varkappa .$$

By a well known theorem of Lindelöf

$$\lim_{z\to 0} \arg (\varphi(z)/z) = 0\,, \quad (z \in \Omega \cup C)$$

so that we have as $z \to 0$ *along* C

$$\lim_{z\to 0} \frac{\varphi(z)}{z} = \varkappa\,(\geqq 0)\,. \tag{4.2}$$

To prove that (4.2) holds also for unrestricted approach in $\Omega \cup C$ it is sufficient to show that $\left|\frac{\varphi(z)}{z}\right|$ is bounded in Ω. Let $\{k_{\varrho_n}\}$ be a sequence of circular cross cuts with $\lim_{n\to\infty} \varrho_n = 0$. Then Ω may be exhausted by a sequence of Jordan domains $\{\Omega_n\}$ bounded by k_{ϱ_n} and an arc γ_n of C with does *not* contain $z = 0$. By Lemma 1, $\left|\frac{\varphi(z)}{z}\right| \leqq M$ on all k_{ϱ_n} and by (4.2), $\left|\frac{\varphi(z)}{z}\right| \leqq M'$, for a suitable M' on $C(z \neq 0)$. Since $\left|\frac{\varphi(z)}{z}\right|$ is continuous in the closure of each Ω_n it follows that it is uniformly bounded in all Ω_n and hence in Ω. This completes the proof.

5. Conditions for $\varkappa$ to be positive

To show that $\varkappa > 0$ it is sufficient to find a domain $\Omega_1 \subset \Omega$ which has $z_0 = 0$ as an accessible boundary point along the y-axis, such that for its mapping function $\varphi_1(z)$ onto $\operatorname{Im}(\zeta) > 0$, with $\varphi_1(z_0) = 0$, $\varliminf_{y\downarrow 0} \left|\frac{\varphi_1(iy)}{y}\right| > 0$. In particular, any $\Omega_1 \subset \Omega$ for which φ_1 has a non-vanishing angular derivative may be used. We give the following simple and quite general criterion. Ω, φ are defined in § 1, $\varkappa = \lim_{y\to 0} \frac{|\varphi(iy)|}{y}$.

Theorem 3. *Suppose that, for all real* x, $h(x)$ *is a non-negative, continuous, even function, such that*

$$\text{(a)} \quad \int_0^\delta \frac{h(x)}{x^2}\, dx < \infty \; (\delta > 0) \quad \text{and} \quad \text{(b)} \int_{x-h(x)}^{x+h(x)} h(t)dt \geqq ch^2(x)$$

for some constant $c > 0$. *If for some* $a > 0$, $y_0 > 0$ *the domain* $\{z = x + iy \mid -a < x < a,\; h(x) < y < y_0\} \subset \Omega$, *then* $\varkappa > 0$.[6)]

[6)] Our Theorem 3 combined with Theorem 1 is sharper than the earlier criteria for existence of the nonvanishing angular derivative cited in [9] pp. 22—28 and some of the more recent ones in [6], Chapter VI. However it is restricted by the fact that the curve $y = h(x)$ and hence the domain Ω_1 which we construct are contained in $y \geqq 0$. For the case that Ω is an infinite domain with z_0 at ∞ and is mapped onto a parallel strip we proved a similar result in [19], Theorem 6; cf. also the Remark in [19] p. 100 where comparisons with earlier criteria are given.

Remarks 1. The continuity of h combined with (a) implies $h(0) = 0$, and (b) combined with (a) imply $h(x) = o(x)$ as $x \to 0$.

2. Condition (b) is clearly satisfied if $h(x)$ is monotonically increasing for $x > 0$; in that case the theorem yields a well known result. Other sufficient conditions that (b) hold are that, for some constant $k > 0$, either

$$\frac{h(t) - h(x)}{t - x} \geqq -k \quad \text{or} \quad \frac{h(t) - h(x)}{t - x} \leqq k \ (t > x) .$$

These inequalities facilitate the application of our theorem in many cases.

Proof.[7] We may assume $2a < y_0$. Traverse the circle $|z - ia| = a$ from the point $z = 2ai$ in both directions to the *first* points of intersection, $\pm b + ih(b)$, with the curve $y = h(x)$ and denote the (circular) arc so described by Γ. Clearly $\Gamma \subset \Omega$. Then Γ and the arc $\gamma = \{|x| \leqq b, y = h(x)\}$ form a closed Jordan curve contained in $\overline{\Omega}$. Denote its interior by Ω_1; $\Omega_1 \subset \Omega$. We may assume $b > 0$, for $b = 0$ implies that Ω_1 is the disk $|z - ia| = a$, and in this case the conclusion of our theorem is well known. We may also suppose $h(x) \leqq \frac{1}{2}|x| \leqq \frac{1}{2}a$ for $|x| \leqq b$ (by taking a sufficiently small).

Let $G_1(z, ia) = \log \frac{a}{r} - v(z)$ denote the Green's function of Ω_1 with the pole at $z = ia$, where $r = |z - ia|$ and $v(z)$ is harmonic in Ω_1. For $z \in \Gamma \cup \gamma$ we have $v(z) = \log \frac{a}{r}$.

Now for $z = x + iy \in \gamma$ we have, if $r < a$,

$$\log \frac{a}{r} = \log \left(1 + \frac{a - r}{r}\right) < \frac{a - r}{r} \leqq \frac{2}{a}(a - r) \leqq \frac{2}{a} h(x)$$

and the inequality between the first and last term still holds if $r \geqq a$, since then $\log \frac{a}{r} \leqq 0$. Thus for $z \in \gamma$

$$v(z) \leqq \frac{2}{a} h(x) \qquad (z = x + ih(x)) . \tag{5.1}$$

Let

$$u(z) = \frac{1}{\pi} \int_{-2a}^{2a} h(t) \frac{y\, dt}{(t - x)^2 + y^2}$$

[7]) The proof generalizes a method of M. Tsuji [12] p. 368. To make this paper self-contained we include the proof, although it is similar to that of our Theorem 6 of [19].

which is harmonic in $y > 0$. For $z \in \gamma$ (if $y = 0$, $u(z) = h(x) = 0$):

$$u(z) \geqq \frac{1}{\pi} \int_{x-y}^{x+y} h(t) \frac{y\, dt}{(t-x)^2 + y^2} \geqq \frac{1}{2\pi y} \int_{x-y}^{x+y} h(t) dt \geqq \frac{ch(x)}{2\pi}$$

since $|t - x| \leqq y$ and by property (b). Hence, by (5.1) for $z \in \gamma$

$$v(z) \leqq \frac{4\pi}{ac} u(z) . \tag{5.2}$$

Since on Γ, $v(z) = 0$ and $u(z) > 0$ for $y > 0$, (5.2) holds in Ω_1. Now, for $z = iy$, $0 < y < a$,

$$u(iy) = \frac{1}{\pi} \int_{-2a}^{2a} h(t) \frac{y\, dt}{t^2 + y^2} \leqq \frac{y}{\pi} \int_{-2a}^{2a} \frac{h(t)}{t^2} dt$$

and thus

$$v(iy) \leqq \frac{4y}{ac} \int_{-2a}^{2a} \frac{h(t)}{t^2} dt . \tag{5.3}$$

Hence we have for $0 < y < \frac{a}{2}$ $(a - r = y)$, by (5.3),

$$G_1(iy, ia) = \log\left(1 + \frac{a-r}{r}\right) - v(iy) \geqq \frac{1}{2} \frac{a-r}{a} - v(iy) = \frac{y}{2a} - v(iy)$$

$$\geqq \frac{y}{a}\left\{\frac{1}{2} - \frac{4}{c} \int_{-2a}^{2a} \frac{h(t)}{t^2} dt\right\} = yA .$$

We can choose a priori a so small that $A > 0$. If $G(z, ia)$ denotes the Green's function of Ω with pole at $z = ai$, we have for $z \in \Omega_1$, $G(z, ia) \geqq G_1(z, ia)$ and consequently

$$\varliminf_{y \downarrow 0} \frac{G(iy, ia)}{y} \geqq \varliminf_{y \downarrow 0} \frac{G_1(iy, ia)}{y} \geqq A > 0 .$$

Suitable linear transformations applied to the mapping function φ show then that $\varkappa > 0$.

APPENDIX

We indicate a direct proof of (1.1) stated in section 1.

Theorem. *If the domain* Ω *satisfies conditions* (A) *and* (C) *of section* 1, *then* $\lim\limits_{z \to z_0} \arg \dfrac{\varphi(z)}{z} = 0$, *uniformly as* $z \to z_0$ *in* K.

The function $w = \log \dfrac{i}{z}$, $w = u + iv$, maps Ω onto a simply connected domain S which contains a part L: $\{u \geqq u_0,\ v = 0\}$ of the real axis, for some u_0. Let w_∞ denote the boundary point at $w = \infty$ of S accessible along L (corresponding to z_0). For $u \geqq u_0$ let θ_u be the largest (open) segment in S which intersects L (corresponding to k_r) and let $\Theta(u)$ be its length. The domain $\Gamma = \{w \in \theta_u,\ u \geqq u_0\}$ is the »kernel» of S at w_∞ (depending on u_0). Similarly $Z = \log \dfrac{i}{\zeta}$, $Z = X + iY$, maps the half-plane $\operatorname{Im}(\zeta) > 0$ onto the infinite parallel strip $\Sigma = \{|Y| < \dfrac{\pi}{2},\ -\infty < X < \infty\}$ and the function $Z(w) = X(w) + iY(w) = -\log[i^{-1}\varphi(ie^{-w})]$ maps S conformally onto Σ such that $\lim\limits_{u \to +\infty} X(u) = +\infty$. For all sufficiently large u, $Z(w)$ maps θ_u onto an arc $\sigma_u \subset \Sigma$ which connects a (finite) point of $Y = \dfrac{\pi}{2}$ to a (finite) point on $Y = -\dfrac{\pi}{2}$. We assume that u_0 is so large that this is the case for $u \geqq u_0$. The cross cut θ_{u_0} divides S into two subdomains; let S_0 denote the one containing the part of L with $u > u_0$.

Let α be an endpoint of a θ_u for $u > u_0$. We describe a circle C_ϱ of radius ϱ about α where $\varrho < \operatorname{Min}(u - u_0,\ |\alpha - u|)$ so that C_ϱ does not intersect θ_{u_0} and the u-axis. C_ϱ crosses θ_u at a point α'. Let k_ϱ denote the largest (open) arc of C_ϱ which contains α' and is contained in S; it is also contained in S_0; k_ϱ divides S_0 into two domains and the one containing the segment $\alpha\alpha'$ of θ_u will be denoted by D_ϱ. Then we have (Wolff [22], Warschawski [19] p. 84):

Lemma. *For every* δ, $0 < \delta < \operatorname{Min}(e^{-32},\ u - u_0,\ |\alpha - u|)$ *there exists a* ϱ, $\delta^2 < \varrho < \delta$, *such that the image of* D_ϱ *under the mapping* $w \to Z(w)$ *is a domain* Δ_ϱ *bounded by the image of* k_ϱ *and a finite segment of the line* $Y = \dfrac{\pi}{2}$ *or* $Y = -\dfrac{\pi}{2}$. *The diameter of* Δ_ϱ *does not exceed* $M\left(\log \dfrac{1}{\delta}\right)^{-1/2}$ *where* $M \leqq 6\pi\sqrt{2}$.

In particular, we have for $w_1,\ w_2 \in D_\varrho$,

$$|Y(w_1) - Y(w_2)| \leqq M \left(\log \frac{1}{\delta}\right)^{-1/2},$$

and if we let $w_1 \to \alpha$ *along* θ_u, *for any* $w \in D_\varrho$,

$$\left|\frac{\pi}{2} - |Y(w)|\right| \leqq M\left(\log\frac{1}{\delta}\right)^{-1/2}. \tag{A.1}$$

Now, to prove the theorem, assume $u > u_0$ and choose any $\varepsilon > 0$ such that $\sqrt{\varepsilon} < \mathrm{Min}\,(e^{-32},\ u - u_0,\ |\alpha - u|)$ where α (as above) is an endpoint of θ_u. Then there exists an R_ε such that (because of (A))

(i) $$S_{\frac{\pi}{2}-\varepsilon} = \left\{w = u + iv \mid u \geqq R_\varepsilon\,,\ |v| \leqq \frac{\pi}{2} - \varepsilon\right\} \subset S\,,$$

and (because of (C)) for $u > R_\varepsilon$, $w = u + iv \in \theta_u$

(ii) for $v > 0$: $v < \dfrac{\pi}{2} + \varepsilon$ and for $v < 0$: $v > -\dfrac{\pi}{2} - \varepsilon$.

Applying (A.1) with $\delta^2 = \varepsilon$ we see that for every $w \in \theta_u$, $u > R_\varepsilon$, and $|v| \geqq \dfrac{\pi}{2} - \varepsilon$

$$\left|\frac{\pi}{2} - |Y(w)|\right| \leqq 2M\left(\log\frac{1}{\varepsilon}\right)^{-1/2}.$$

Hence, for every such w with $v \geqq \dfrac{\pi}{2} - \varepsilon$, by (ii)

$$|Y(w) - v| \leqq \left|Y(w) - \frac{\pi}{2}\right| + \left|v - \frac{\pi}{2}\right| \leqq 2M\left(\log\frac{1}{\varepsilon}\right)^{-1/2} + \varepsilon \tag{A.2}$$

and for w with $v \leqq -\dfrac{\pi}{2} + \varepsilon$

$$|Y(w) - v| \leqq \left|\frac{\pi}{2} + Y(w)\right| + \left|\frac{\pi}{2} + v\right| \leqq 2M\left(\log\frac{1}{\varepsilon}\right)^{-1/2} + \varepsilon\,. \tag{A.3}$$

Finally since $Y(w) - v$ is harmonic and bounded in $S_{\frac{\pi}{2}-\varepsilon}$ and (A.2) and (A.3) are satisfied on the horizontal boundaries of this strip, it follows that for $w \in S_{\frac{\pi}{2}-\varepsilon}$, uniformly,

$$\overline{\lim_{u\to+\infty}}\, |Y(w) - v)| \leqq 2M\left(\log\frac{1}{\varepsilon}\right)^{-1/2} + \varepsilon\,. \tag{A.4}$$

But by (A.2) and (A.3) it is also true uniformly for $w \in \Gamma$ outside of $S_{\frac{\pi}{2}-\varepsilon}$ as $u \to +\infty$. Hence $\lim_{u \to +\infty} (Y(w) - v) = 0$, uniformly for $w \in \Gamma$. This is equivalent to the conclusion of the theorem.

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Bibliography

1. L. AHLFORS, Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen, Acta Societatis Scientiarum Fennicæ (n.s.) A, 1 (1930): 1—40.
2. C. CARATHÉODORY, Über die Winkelderivierten von beschränkten analytischen Funktionen, Sitz. Berichte, Berliner Akad. der Wissenschaften, 32 (1929): 39—54.
3. E. T. COPSON, Introduction to the theory of functions of a complex variable, Oxford Univ. Press, 1935.
4. J. DUFRESNOY et J. FERRAND, Extension d'une inégalité de M. Ahlfors et application au problème de la dérivee angulaire, Bulletin des Sciences Mathématiques 69 (1945): 165—174.
5. J. FERRAND, Sur l'inégalité d'Ahlfors et son application au problème de la dérivée angulaire, Bulletin de la Société Math. de France, 72 (1944): 178—192.
6. J. LELONG-FERRAND, Représentation conforme et transformations a intégrale de Dirichlet bornée, Gauthier-Villars, Paris, 1955.
7. C. GATTEGNO, Nouvelle démonstration d'un théorème de M. Ostrowski sur la représentation conforme, Bulletin des Sciences Math. 62 (1938) 12—21.
8. C. GATTEGNO et A. OSTROWSKI, Représentation conforme à la frontière: domains generaux, Mémorial des Sciences Math. Fasc. 109 (1949), Gauthier-Villars.
9. —»— Représentation conforme à la frontière: domains particuliers, ibid. Fasc. 110 (1949).
10. A. OSTROWSKI, Über den Habitus der konformen Abbildung on Rande des Abbildungsbereiches, Acta Math. 64 (1934): 81—185.
11. —»— Zur Randverzerrung bei konformer Abbildung. Prace Matematiczno Fiziczne, 44 (1937): 371—471.
12. M. TSUJI, Potential theory in modern function theory, Maruzen Co., Ltd., Tokyo, 1959.
13. J. G. VAN DER CORPUT, Über die Winkelableitung bei konformer Abbildung, Proc. Ned. Akademie van Wetenschappen, 35 (1932): 330—338.
14. C. VISSER, Über beschränkte analytische Funktionen und die Randverhältnisse bei konformen Abbildungen, Math. Annalen 107 (1932): 28—39.
15. —»— Über die Ränderzuordnung bei konformen Abbildungen, Proc. Ned. Akad. van Wetenschappen 38 (1935): 411—414.
16. —»— Sur la dérivée angulaire des fonctions univalentes: I. Proc. Ned. Akad. van Wetenschappen 38 (1935); 402—411; II. 40 (1937): 223—226.
17. S. E. WARSCHAWSKI, Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung, Math. Zeitschrift 35 (1932): 322—456.
18. —»— Zur Randverzerrung bei konformer Abbildung, Compositio Mathematica 1 (1935): 314—343.
19. —»— On the boundary behavior of conformal maps, Nagoya Mathematical Journal 30 (1967): 83—101.

20. J. Wolff, Sur une géneralization d'un lemme de Schwarz, C. R. Acad. Sc. Paris 183 (1926): 500—502.

21. —»— Sur la représentation conforme des bandes, Compositio Mathematica 1 (1934): 217—222.

22. —»— Démonstration d'un théorème sur la conservation des angles dans la représentation conforme on voisinage d'un point frontière, Proc. Ned. Akad. van Wetenschappen, 38 (1935): 46—50.

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