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THE NON-PLESSNER POINTS FOR THE SCHWARZ TRIANGLE FUNCTIONS

BY

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1. Introduction

Let $\mu(z)$ be the elliptic modular function defined in the unit disc $D : |z| < 1$. It has been shown by Bagemihl [1, theorem 1] that the set of Plessner points [6, p. 59] for $\mu(z)$ is of measure 2π and residual in the unit circle $C : |z| = 1$. On the other hand $\mu(z)$ has angular limits at each of countably many parabolic vertices for $\mu(z)$, and so these points are certainly not Plessner points. It is natural to ask whether these are the only non-Plessner points. In this note it is shown that there are other non-Plessner points for $\mu(z)$. Indeed it is shown that the set of non-Plessner points for $\mu(z)$ has the cardinality of the real line.

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2. Preliminary Results

For any two points $z', z \in D$, we denote by $\varrho(z', z)$ the non-Euclidean hyperbolic distance between z' and z; for any two points w and w' of the Riemann sphere, we denote by $\chi(w, w')$ the chordal distance between w and w'. For a set $S \subset D$ and $r > 0$, we define

$$H(S, r) = \{z : \varrho(S, z) \leq r\} .$$

The cluster set $C(f, S)$ of $f(z)$ on S is the set of all values w of the Riemann sphere such that there is a sequence $\{z_n\}$, $z_n \in S$, such that $|z_n| \to 1$ and $f(z_n) \to w$. For S a Stolz angle or a segment in D with only one end point on C, Rung [7] has introduced the notation

$$\hat{C}(f, S) = \bigcap_{\Delta} C(f, \Delta) ,$$

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where Δ varies over all Stolz angles which properly contain S. We note that

$$\hat{C}(f, S) = \bigcap_{r>0} C(f, H(S, r)) . \tag{1}$$

For this reason we now define $\hat{C}(f, S)$ by (1) for any set $S \subset D$. A subset $S \subset D$ is called a ϱ-set if there is a sequence $\{z_n\}$ of ϱ-points [3] for $f(z)$ with $z_n \in S$, $n = 1, 2, \ldots$

Theorem 1. Let $w = f(z)$ be a function meromorphic in D and let $S \subset D$. Then either S is a ϱ-set or

$$C(f, S) = \hat{C}(f, S) . \tag{2}$$

Proof. If the closure of S does not intersect C, then

$$C(f, S) = \hat{C}(f, S) = \emptyset ,$$

and so there is nothing to prove.

Suppose on the other hand that the closure of S meets C and that

$$C(f, S) \neq \hat{C}(f, S) .$$

Then since

$$C(f, S) \subset \hat{C}(f, S) ,$$

there must be a point $w_0 \in \hat{C}(f, S)$ for which $w_0 \notin C(f, S)$. For each positive integer n, $w_0 \in C(f, H(S, 1/n))$, and so one can find a point z_n in $H(S, 1/n)$ such that $|z_n| > 1 - 1/n$, and

$$\chi(f(z_n), w_0) < 1/n . \tag{3}$$

Since z_n is in $H(S, 1/n)$, there is a point z_n' in S such that $\varrho(z_n, z_n') \leq 1/n$. For large n, the chordal distance between $f(z_n')$ and w_0 must be bounded away from zero since $w_0 \notin C(f, S)$. Hence by (3) we have that

$$\varrho(z_n, z_n') \to 0 , \quad \text{and} \quad \chi(f(z_n), f(z_n')) > R , \quad \text{for} \quad n = 1, 2, \ldots , \tag{4}$$

where R is some fixed positive number. It follows [3] from (4) that $\{z_n'\}$ is a sequence of ϱ-points, and so S is a ϱ-set. This concludes the proof.

Corollary 1. Let $w = f(z)$ be a normal meromorphic [6, p. 86] function in D, and let $S \subset D$. Then

$$\hat{C}(f, S) = C(f, S) . \tag{5}$$

This generalizes a result due to Rung [7].

Proof. By modifying a result of Gavrilov [4], we have shown [3] that a normal meromorphic function cannot possess a sequence of ϱ-points. Hence (5) follows from theorem 1.

We shall call two sets, $S_1 \subset D$ and $S_2 \subset D$, equivalent if for each $r > 0$, there is a $\delta > 0$ such that

$$S_2 \cap \{z : |z| > 1 - \delta\} \subset H(S_1, r) \quad \text{and}$$
$$S_1 \cap \{z : |z| > 1 - \delta\} \subset H(S_2, r) . \tag{6}$$

This definition obviously has the following corollary.

Corollary 2. Let $w = f(z)$ be a meromorphic function in D, and let S_1 and S_2 be equivalent subsets of D. Then

$$\hat{C}(f, S_1) = \hat{C}(f, S_2) .$$

Theorem 2. Let $w = f(z)$ be a normal meromorphic function in D, and let S_1 and S_2 be equivalent subsets of D. Then

$$C(f, S_1) = C(f, S_2) . \tag{7}$$

Proof. This theorem follows directly from corollary 1 and corollary 2.

3. Triangle functions

Lemma 1. Any Schwarz triangle function in D is normal.

Proof. Suppose $f(z)$ is a Schwarz triangle function (see [2]) defined in D, and suppose that $\{z_n\}$ is a sequence of ϱ-points for $f(z)$. For each z_n, let ζ_n be a point of the fundamental polygon for $f(z)$ such that z_n is equivalent to ζ_n. Then $\{\zeta_n\}$ is also a sequence of ϱ-points. Let ζ_0 be a limit point of $\{\zeta_n\}$, and let $\{\zeta_{n(k)}\}$ be a subsequence of $\{\zeta_n\}$ which converges to ζ_0. $\{\zeta_{n(k)}\}$ is also a sequence of ϱ-points, and so $f(z)$ assumes each value of the Riemann sphere, with at most two exceptions, infinitely often in every neighborhood of ζ_0. Hence ζ_0 lies on the unit circle. But ζ_0 is a boundary point of the fundamental polygon, and so ζ_0 is a parabolic vertex. It follows that $f(z)$ has an angular limit at ζ_0. The sequence $\{\zeta_{n(k)}\}$ lies in the fundamental polygon and so it must approach ζ_0 non-tangentially. Hence $f(z)$ assumes every value of the Riemann sphere, with at most two exceptions, infinitely often in some Stolz angle having vertex at ζ_0. This contradicts the fact that $f(z)$ has an angular limit at ζ_0. Thus the supposition that $f(z)$ has a sequence of ϱ-points was false. By a theorem of Gavrilov (see [3]), then, $f(z)$ must be normal.

Theorem 3. The set of non-Plessner points for any Schwarz triangle function $f(z)$ has the cardinality of the real line.

Proof. Let $\{T_n\}$ be the group of linear transformations with respect to which $f(z)$ is an automorphic function. For any $S \subset D$ we write S_n for $T_n(S)$. We now define a subset $\bar{M}_0$ of the unit circle. We shall say that $\alpha \in \bar{M}_0$ if there is some non-Euclidean straight line L which intersects the unit circle at α and for which the set $\cup \{L_n : n = 1, 2, \ldots\}$ is not dense in D.

Let $\alpha \in \bar{M}_0$ and let L be as in the above paragraph. Since $\cup \{L_n : n = 1, 2, \ldots\}$ is not dense in D, there must be an open set G contained in a fundamental region for $f(z)$ such that $\cup \{L_n : n = 1, 2, \ldots\}$ does not meet G. Hence $f(G)$ does not meet $f(L)$. Since $f(G)$ is open, $f(L)$ cannot be dense in the Riemann sphere, and so $C(f, L)$ is not total. Let β be the other (besides α) point at which L intersects the unit circle, and let $S = L \cap \{z : |z - \alpha| < |z - \beta|\}$. Let R be the radius $\arg z = \arg \alpha$. Then clearly R is equivalent to S and so by theorem 2,

$$C(f, R) = C(f, S) . \tag{8}$$

Since $S \subset L$ and $C(f, L)$ is not total, it follows from (8) that $C(f, R)$ is not total. According to a theorem of Bagemihl [1, lemma], it follows that the point α is not a Plessner point for $f(z)$. Hence no point of $\bar{M}_0$ is a Plessner point. Now Myrberg [5, p. 408] has shown that the set $\bar{M}_0$ has the cardinality of the real line, and so the proof is complete.

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