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A REMARK ON BLOCH'S CONSTANT FOR SCHLICHT FUNCTIONS

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A remark on Bloch's constant for schlicht functions

We denote by S the class functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular and univalent in $|z| < 1$. Let r be such a constant that any $f(z) \in S$ assumes all values in some circle of radius r. The supremum of all such r is called Bloch's constant for schlicht functions, and we denote it by A.

Reich [4] has proved that $A > .569$. Jenkins [2] has shown that $A > .5705$. The object of the present paper is to show that by modifying the method given by Reich [4] a still better bound can be obtained. The improvement is very small, but our method is perhaps a little simpler than that of Reich.

Theorem. $A \geq .5708$.

Proof. As pointed out by Landau [3], it is sufficient to consider functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S$ satisfying

$$(1) \qquad (1 - |z|^2)|f'(z)| \leq 1 .$$

Inequality (1) implies (Landau [3])

$$(2) \qquad a_2 = 0 , \quad |a_3| \leq 1/3 ,$$

and

$$(3) \qquad |f(z)| \leq \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} = |z| M(|z|) .$$

For $.8 < t < 1$, we set $f(z , t) = f(tz)/t$. Thus $|f(z , t)| \leq M(t)$ in $|z| < 1$. Let

$$g(z , t) = M(t)\Phi(f(z , t)/M(t)) ,$$

where $\Phi(z) = z/(1 - z)^2$. If $f(z)$ omits γ, $0 < \gamma < 1$, then $g(z , t)$ omits

$$(4) \qquad \gamma(t) = \gamma/t(1 - \gamma/tM(t))^2 .$$

Let

$$w(z\,,\,t) = \frac{g(z\,,\,t)}{1 - g(z\,,\,t)/\gamma(t)} = z + b_2 z^2 + b_3 z^3 + \ldots .$$

We have

$$b_2 = 1/\gamma(t) + 2/M(t)\,, \tag{5}$$

$$b_3 = a_3 t^2 + 3/M^2(t) + 4/M(t)\gamma(t) + 1/\gamma^2(t)\,.$$

Since $w(z\,,\,t) \in S$, we get by Goluzin [1, p. 46] that

$$|b_3 - \alpha b_2^2| \leq 1 + 2\exp\{-2\alpha/(1-\alpha)\}$$

for all $0 \leq \alpha \leq 1$. This gives by (4) and (5)

$$\gamma \geq t\left[\frac{1-\alpha}{1 + 2\exp\left\{\dfrac{-2\alpha}{1-\alpha}\right\} + |a_3|t^2 - \dfrac{1-2\alpha}{M^2(t)} - \dfrac{(1-\alpha)\gamma^2}{t^2 M^4(t)}}\right]^{1/2}$$

for $0 \leq \alpha \leq 1$, $.8 < t < 1$. Setting $\alpha = .34$, $t = (e^{6.5} - 1)/(e^{6.5} + 1)$, and $|a_3| \leq 1/3$, we get by a numerical computation that $\gamma > .5708$. Hence we deduce that $f(z)$ assumes all values in $|w| \leq .5708$.

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References

[1] Goluzin, G. M.: Some questions of the theory of univalent functions. - Trudy Mat. Inst. Steklov. 27 (1949).

[2] Jenkins, J. A.: On the schlicht Bloch constant. - J. Math. Mech. 10 (1961), 729–734.

[3] Landau, E.: Über die Blochsche Konstante und zwei verwandte Weltkonstanten. - Math. Z. 30 (1929), 608–634.

[4] Reich, E.: On a Bloch–Landau constant. - Proc. Amer. Math. Soc. 7 (1956), 75–76.

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