ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

446

ON SPECTRAL DECOMPOSITIONS OF OPERATORS IN *J*-SPACE

BY

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HELSINKI 1969
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1969.446

Communicated 14 March 1969 by R. Nevanlinna and G. Järnefelt

KESKUSKIRJAPAINO
HELSINKI 1969

1. Introduction

1.1. We consider a complex linear space X. Let Q denote a Hermitian inner product on X. We assume it to be non-degenerate ($Q(x, y) = 0$ for all $y \in X$ implies $x = 0$) and indefinite ($Q(x, x)$ has positive and negative values). Further, let X have a decomposition

$$X = X^+ \oplus X^-$$

into direct sum of two Q-orthogonal linear manifolds X^+, X^-:

$$Q(x^+, x^-) = 0 \quad \text{for all} \quad x^+ \in X^+, \quad x^- \in X^-,$$

$$Q(x^+, x^+) > 0 \quad \text{for all} \quad x^+ \in X^+, \quad x^+ \neq 0,$$

$$Q(x^-, x^-) < 0 \quad \text{for all} \quad x^- \in X^-, \quad x^- \neq 0.$$

We assume that in this »Q-canonical» decomposition $(X^+, Q|X^+)$ and $(X^-, -Q|X^-)$ are Hilbert spaces; in this case (X, Q) is called J-space.

Let P^+ and P^- be the projectors onto X^+ and X^-, respectively, satisfying

$$P^+ P^- = P^- P^+ = O, \qquad P^+ + P^- = I.$$

With

$$J = P^+ - P^-$$

the definition

$$H(x, y) = Q(J x, y) \qquad \text{for all} \quad x, y \in X$$

gives a Hilbert inner product H on X. We denote the corresponding Hilbert norm by

$$\|x\| = + \sqrt{H(x, x)} \qquad (x \in X).$$

In the following all topological properties are based on this norm.

1.2. A linear mapping A with the domain $D(A)$ and the range $R(A)$ in X is called an operator. If $D(A)$ is dense in X, the operator A has a uniquely determined H-adjoint A^*, which is a closed operator:

$$H(A\,x\,,y) = H(x\,,A^*\,y) \qquad \text{for all} \quad x \in D(A)\,,\ y \in D(A^*)\,.$$

Take $x \in D(A)$ and $y \in D(A^*\,J)$, then [1])

$$Q(A\,x\,,y) = H(J\,A\,x\,,y) = H(x\,,A^*\,J\,y) = Q(x\,,J\,A^*\,J\,y)\,.$$

The operator

$$A^{c} = J\,A^*\,J \tag{1}$$

is called the Q-adjoint of A. For $x \in D(A)$ and $y \in D(A^{c})$ we have

$$Q(A\,x\,,y) = Q(x\,,A^{c}\,y)\,.$$

In particular, we have

$$J^{c} = J^* = J^{-1} = J\,. \tag{2}$$

Lemma. *We assume that* G *and its inverse* G^{-1} *are continuous operators defined on* X. *Further let* A *be an operator with the domain dense in* X. *Then* $D(A\,G)$ *is dense in* X *and*

$$(A\,G)^* = G^*\,A^*\,.$$

Proof. Let x be an arbitrary element of X. Since $D(A)$ is dense in X there is a sequence $\{x_n\} \subset D(A)$ with $x_n \to G\,x$. Then $G^{-1}\,x_n \in D(A\,G)$ and $G^{-1}\,x_n \to x$ since G^{-1} is continuous. This implies that $D(A\,G)$ is dense in X.

It is wellknown that [2]) $(A\,G)^* \supset G^*\,A^*$. Since $D(G^*) = X$ we have $D(G^*\,A^*) = D(A^*)$. For $x \in D(A)$ and $y \in D((A\,G)^*)$ one derives

$$H(A\,x\,,y) = H(A\,G\,G^{-1}\,x\,,\ y) = H(G^{-1}\,x\,,\ (A\,G)^*\,y)\,.$$

If y is fixed the expression $H(G^{-1}\,x\,,(A\,G^*)\,y)$ is a continuous function of x. This implies by the definition of $D(A^*)$ that $y \in D(A^*) = D(G^*\,A^*)$. Consequently, we have $D((A\,G)^*) \subset D(G^*\,A^*)$ which completes the proof.

We assume that A is a closed operator with the domain $D(A)$ dense in X. Then $D(A^*)$ is dense in X and by the previous lemma $D(A^{c}) = D(A^*\,J)$ is also dense in X. Further we get

$$(J\,A\,J)^* = (A\,J)^*\,J^* = J^*\,A^*\,J^* = J\,A^*\,J = A^{c}\,.$$

This implies that A^{c} is a closed operator. Since $A^{**} = A$ one obtains

$$A^{cc} = J\,(J\,A^*\,J)^*\,J = J\,(A^*\,J)^*\,J^*\,J = J\,J^*\,A^{**}\,J^*\,J = J^2\,A\,J^2 = A\,.$$

[1]) Obviously $J = J^* = J^{-1}$.

[2]) E.g. [6].

From (1) and (2) one immediately gets the rules (if the operators in question exist):

1) $(A^{-1})^c = (A^c)^{-1}$,

2) $(\alpha A)^c = \bar{\alpha} A^c$,

3) $(A + B)^c \supset A^c + B^c$,

4) $(A B)^c \supset B^c A^c$,

5) $A \subset B$ implies $A^c \supset B^c$.

Especially if A is a continuous operator, the rules 3) and 4) can be replaced by

3′) $(A + B)^c = A^c + B^c$,

4′) $(A B)^c = B^c A^c$.

Let A be an operator (not necessarily continuous) and C a continuous operator. If $C A \subset A C$ we say that A commutes [3]) with C and write $A \smile C$. The notation $A \smile\smile B$ means that A commutes with every continuous operator C commuting with B .

We give the definitions:

(a) The operator A is *Q-self-adjoint* if $A = A^c$.

(b) A continuous operator A with $A^c = A^{-1}$ is called *Q-unitary.*

(c) A closed operator A with the domain dense in X is called *Q-normal* if $A A^c = A^c A$.

1.3. In his theory of linear spaces with indefinite inner products Rolf Nevanlinna expressed the idea [4] that under some restrictive conditions it should be possible to derive, by analogy with the spectral theory of H-self-adjoint operators, a spectral decomposition of Q-self-adjoint operators. Erkki Pesonen [5] studied the question in details in the special case that the self-adjoint operator is continuous and (X , H) is a separable Hilbert space. Applying some results of Heinz Langer [3], Rolf Kühne [2] examined the problem from a different point of view and generalized the results of Pesonen for general Hilbert spaces. Peter Hess recently [1] succeeded in generalizing this for non-continuous Q-self-adjoint operators.

In this paper we shall give such a modification of the results of Hess which is also applicable for Q-unitary and Q-normal operators.

I express my sincerest thanks to Professor I. S. Louhivaara for his kind interest and many valuable advice. I also wish to thank Dr. Peter Hess for his valuable criticism on the first manuscript of this paper.

[3] E.g. [6].

2. Various Hilbert inner products in J-space

2.1. Let Λ be the set of the continuous and Q-self-adjoint operators G for each of which there is a positive number h (depending on G) such that

$$Q(G\,x\,,\,x) \geqq h\,\|x\|^2 \qquad \text{for all} \quad x \in X\,. \tag{3}$$

Theorem 1. *A bilinear form K defined on the space X is a Hilbert inner product topologically equivalent to H if and only if there is an operator $G \in \Lambda$ such that*

$$K(x\,,\,y) = Q(G\,x\,,\,y) \qquad \textit{for all} \quad x\,,\,y \in X\,.$$

Proof. (a) Let K be a Hilbert product equivalent to H. There exists an H-self-adjoint continuous operator C such that

$$K(x\,,\,y) = H(C\,x\,,\,y) = Q(J\,C\,x\,,\,y) \qquad (\,x\,,\,y \in X\,)\,.$$

We write $G = J\,C$. Then we have

$$K(x\,,\,y) = Q(G\,x\,,\,y) \qquad (\,x\,,\,y \in X\,)\,,$$

and G is Q-self-adjoint:

$$G^c = (J\,C)^c = C^c\,J = J\,C^*\,J^2 = J\,C = G\,.$$

Since the Hilbert products H and K give the same topology there is a positive number h such that

$$Q(G\,x\,,\,x) = K(x\,,\,x) \geqq h\,H(x\,,\,x) = h\,\|x\|^2$$

for all $x \in X$. Consequently, we have $G \in \Lambda$.

(b) Suppose

$$K(x\,,\,y) = Q(G\,x\,,\,y) \qquad (\,x\,,\,y \in X\,)$$

where $G \in \Lambda$. Since G is Q-self-adjoint, K is a Hermitian inner product. In accordance with (3) there is a positive number h such that

$$K(x\,,\,x) = Q(G\,x\,,\,x) \geqq h\,H(x\,,\,x)$$

for all $x \in X$. On the other hand

$$K(x\,,\,x) = H(J\,G\,x\,,\,x) \leqq \|J\,G\|\,H(x\,,\,x)$$

for all $x \in X$. Consequently, the forms H and K induce the same topology.

2.2. We shall still consider an operator $G \in \Lambda$ and the corresponding Hilbert product

$$K(x, y) = Q(G x, y).$$

We have for $x, y \in X$

$$K(G x, y) = Q(G^2 x, y) = Q(G x, G y) = K(x, G y),$$

thus G is K-self-adjoint.

Let C be a continuous operator satisfying

$$K(x, y) = H(C x, y) \qquad (x, y \in X).$$

Then we have $G = J C$. The operator C has a continuous inverse C^{-1} defined on X. Since $G^{-1} = C^{-1} J$, the operator G has also a continuous inverse G^{-1} defined on X.

Theorem 2. *Let A and B be two closed operators with the domains dense in X. Then the two following propositions are equivalent.*

(i) *In X there exists a Hilbert product K equivalent to H so that the operators A and B are the K-adjoints of each other.*

(ii) *There exists an operator $G \in \Lambda$ such that $G A = B^c G$.*

Proof. (a) First we assume that there is a Hilbert product K equivalent to H so that B is the K-adjoint of A. We denote for K-adjoint of A by A° that is $B = A^\circ$. According to Theorem 1 there is such an operator $G \in \Lambda$ that

$$K(x, y) = Q(G x, y) \qquad (x, y \in X).$$

For $x \in D(A)$ and $y \in D(B)$ one gets

$$Q(x, G B y) = K(x, B y) = K(A x, y) = Q(G A x, y).$$

This implies $G A \subset (G B)^c = B^c G$.

For $x \in D(B G)$ and $y \in D(B^c G)$ one derives

$$K(B G x, y) = Q(G B G x, y) = Q(G x, B^c G y) = K(x, B^c G y),$$

hence $B^c G \subset (B G)^\circ$.

Since G and G^{-1} are continuous operators defined on X, we obtain $(B G)^\circ = G^\circ B^\circ = G A$ according to the lemma in section 1.2. Thus we have $B^c G \subset G A$.

Consequently, we have $G A = B^c G$.

(b) Let $G \in \Lambda$ be an operator so that $G A = B^c G$.
We define

$$K(x, y) = Q(G x, y) \qquad (x, y \in X).$$

According to Theorem 1 the form K is a Hilbert product equivalent to H.

For $x \in D(A)$ and $y \in D(B)$ one has

$$K(A\,x\,,y) = Q(G\,A\,x\,,y) = Q(B^c\,G\,x\,,y) = Q(G\,x\,,B\,y) = K(x\,,B\,y)\,,$$

therefore $B \subset A^\circ$.

Because of the equation $G\,A = B^c\,G$ we have $x \in D(A)$ if and only if $G\,x \in D(B^c)$. We obtain

$$Q(B^c\,G\,x\,,y) = Q(G\,A\,x\,,y) = K(A\,x\,,y) = K(x\,,A^\circ\,y) = Q(G\,x\,,A^\circ\,y)\,,$$

for $x \in D(A)$ and $y \in D(A^\circ)$. This results in $A^\circ \subset B^{cc} = B$. Consequently A° equals B.

3. Application for the spectral decomposition of Q-self-adjoint, Q-unitary and Q-normal operators

3.1. Let A be a closed operator with the domain dense in X. We assume there is such an operator $G \in A$ that

$$G\,A = A^c\,G\,. \tag{4}$$

According to Theorem 2 there is a Hilbert product K equivalent to H and K is such that A is K-self-adjoint. Consequently, one has a unique K-self-adjoint spectral family $\{\,E_\lambda \mid -\infty < \lambda < \infty\,\}$ having the following properties:

(a) $E_\lambda\,E_\mu = E_\lambda$ for $\lambda \leqq \mu$,

(b) $E_{\lambda+0} = E_\lambda$,

(c) $E_\lambda \to O$ for $\lambda \to -\infty$, $E_\lambda \to I$ for $\lambda \to +\infty$,

(d) $A = \int_{-\infty}^{+\infty} \lambda\,dE_\lambda$,

(e) $E_\lambda \smile\smile A$.

Thus we have obtained for the operator A a spectral decomposition defined above. However, the spectral family $\{E_\lambda\}$ is in this case not necessarily Q-self-adjoint.

Now we assume in addition to (4) that A is Q-self-adjoint: $A = A^c$. Then one has

$$G\,A = A\,G \quad \text{and} \quad G^{-1}\,A = A\,G^{-1}\,.$$

From (e) it follows that $G^{-1} \smile E_\lambda$. Hence we derive

$$Q(E_\lambda\,x\,,y) = K(G^{-1}\,E_\lambda\,x\,,y) = K(E_\lambda\,G^{-1}\,x\,,y)$$
$$= K(G^{-1}\,x\,,E_\lambda\,y) = Q(x\,,E_\lambda\,y)$$

for all $x\,,y \in X$. Consequently $E_\lambda^c = E_\lambda$.

The Q-self-adjoint spectral family $\{E_\lambda\}$ having the properties (a)—(e) is uniquely determined (not depending on the special choice of K). In fact, let $\{F_\lambda\}$ be another spectral family with the same properties. Since $F_\lambda \smile\smile A$ we obtain $F_\lambda \smile G$; this results in $\{F_\lambda\}$ being K-self-adjoint. The implication of the last fact is that $\{F_\lambda\} = \{E_\lambda\}$.

Thus we have the following result of Peter Hess:

Corollary 1. *Let A be a Q-self-adjoint operator. We assume the existence of an operator $G \in \Lambda$ satisfying $A\,G = G\,A$. Then there is a unique Q-self-adjoint spectral family $\{ E_\lambda \mid -\infty < \lambda < +\infty \}$ having the properties (a)—(e).*

3.2. Now we assume that A and A^{-1} are continuous. Further, we assume there is an operator $G \in \Lambda$ such that

$$G\,A = (A^{-1})^c\, G\,. \tag{5}$$

According to Theorem 2 there is a Hilbert product K equivalent to H so that A and A^{-1} are the K-adjoints of each other. Therefore A is K-unitary. There exists a unique K-self-adjoint spectral family $\{ E_\varphi \mid 0 \leqq \varphi \leqq 2\pi \}$ having the following properties:

(a) $E_\varphi E_\psi = E_\varphi$ for $\varphi \leqq \psi$,

(b) $E_{\varphi+0} = E_\varphi$,

(c) $E_0 = O$, $E_{2\pi} = I$,

(d) $A = \int_0^{2\pi} e^{i\varphi}\, dE_\varphi$,

(e) $E_\varphi \smile\smile A$.

Let us assume in addition to (5) that A is Q-unitary. Then $A^{-1} = A^c$ and $G\,A = A\,G$. Now we can prove as we did in section 3.1 that E_φ is Q-self-adjoint. Besides, this spectral family possessing the properties (a)—(e) is unique.

Corollary 2. *Let A be a Q-unitary operator. We assume there exists an operator $G \in \Lambda$ with the property $A\,G = G\,A$. Then there is a unique Q-self-adjoint spectral family $\{ E_\varphi \mid 0 \leqq \varphi \leqq 2\pi \}$ having the properties (a)—(e).*

3.3. Let A be a closed operator with the domain dense in X. We assume that there is a closed operator B with the domain dense in X such that $A\,B = B\,A$. Moreover, we assume the existence of an operator $G \in \Lambda$ with the property

$$G\,A = B^c\, G\,. \tag{6}$$

In agreement with Theorem 2 there is a Hilbert product K equivalent to H so that A and B are the K-adjoints of each other. Since $A B = B A$ the operator A is K-normal. There exists a unique K-self-adjoint spectral measure E defined for the Borel sets of complex numbers so that the following properties are valid [4]):

(a) $E(\boldsymbol{C}) = I$,

(b) $A = \int_{\boldsymbol{C}} \lambda \, dE$,

(c) $E(M) \smile A$ for each Borel set M of $\boldsymbol{C}$.

We assume especially that $B = A^c$. Then the operator A is Q-normal: $A A^c = A^c A$. Since, according to (6), $G A = A G$ it follows from (c) that $E(M) \smile G$. This implies that the spectral measure E is Q-self-adjoint.

Corollary 3. *Let A be a Q-normal operator. We suppose there exists an operator $G \in A$ satisfying $A G = G A$. Then there is a unique Q-self-adjoint spectral measure E possessing the properties (a)—(c).*

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4) The set of all the complex numbers is denoted by $\boldsymbol{C}$.

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Printed June 1969