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REMOVABILITY THEOREMS FOR QUASICONFORMAL MAPPINGS

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1. *Introduction.* In this paper we shall study the following removability question: Let D and D' be domains in the euclidean n-space R^n, $n \geq 2$, let $E \subset D$ be closed in D, and let $f: D \to D'$ be a homeomorphism which is locally K-quasiconformal in $D - E$ for some K, which means that for every $x \in D - E$ there is a connected neighborhood U of x such that $f \mid U$ is a K-quasiconformal mapping [8, p. 20]. We ask for conditions on E and on the restriction $f \mid E$ which imply the quasiconformality of f. A special case for $n = 2$ of this situation is considered in [5, Theorem 3] which implies that f is quasiconformal if E is a Jordan curve and if $f \mid E = g \mid E$ for some quasiconformal mapping g of a domain $G \supset E$.

The set E is called an exceptional set if f is always a K-quasiconformal mapping. One of the main results which give conditions for the exceptionality is that the set E is exceptional if E is of σ-finite $(n-1)$-dimensional Hausdorff measure [9, 35.1], [3, Corollary 5]; for the case $n = 2$ see also [7], [1], and [4, Satz V.3.2]. We shall give answers to the given problem in the other direction. It turns out (Theorem 1) that the condition mentioned above, namely the existence of a quasiconformal mapping g of a domain $G \supset E$ such that $f \mid E = g \mid E$, implies the quasiconformality of f even if no further assumptions are made on E. We shall also in Theorem 2 establish another form where the assumption on the restriction $f \mid E$ is weakened but E is assumed to be connected or locally connected. In these results the maximal dilatation of f is in general greater than K.

2. *Notation.* Throughout this paper D and D' are domains in R^n and $n \geq 2$. If $A, B \subset R^n$, $d(A, B)$ denotes the euclidean distance between A and B. For $x \in R^n$ and $r > 0$ we set $B^n(x, r) = \{y \in R^n \mid |y - x| < r\}$ and $S^{n-1}(x, r) = \{y \in R^n \mid |y - x| = r\}$. We also use the abbreviations $B^n(r) = B^n(0, r)$ and $S^{n-1}(r) = S^{n-1}(0, r)$. If $f: D \to D'$ is a homeomorphism, if $x \in D$, and if $0 < r < d(x, \partial D)$, we set

$$L(x, f, r) = \sup_{|y-x|=r} |f(y) - f(x)| ,$$

$$l(x, f, r) = \inf_{|y-x|=r} |f(y) - f(x)| .$$

The linear dilatation of f at x is

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)} .$$

The k-dimensional Lebesgue measure is denoted by m_k. The $(n-1)$-dimensional measure of the unit sphere $S^{n-1}(1)$ is ω_{n-1}.

3. We start with a simple distortion result for quasiconformal mappings. Let v be an increasing function of the interval $[1, \infty)$ into itself and let $\varphi : A \to R^n$, $A \subset R^n$, be an injective mapping. We say that φ has *local v-bounded distortion* if for every $x \in A$ there is an $s > 0$ such that $x_1, x_2 \in A$ and $0 < |x - x_2| \leq |x - x_1| < s$ imply

$$\frac{|\varphi(x_1) - \varphi(x)|}{|\varphi(x_2) - \varphi(x)|} \leq v\left(\frac{|x_1 - x|}{|x_2 - x|}\right) .$$

Lemma 1. *Let $f : D \to D'$ be a K-quasiconformal mapping. Then there exists an increasing function $v : [1, \infty) \to [1, \infty)$ depending only on n and K such that f has local v-bounded distortion.*

Proof. Assume $x \in D$. Choose $s > 0$ such that $f\bar{B}^n(x, s) \subset \bar{B}^n(f(x), t) \subset D'$ for some t. Let $0 < |x_2 - x| \leq |x_1 - x| < s$ and set $\alpha_i = |f(x_i) - f(x)|$, $i = 1, 2$. Assume $\alpha_2 < \alpha_1$ and let Γ' be the family of curves which join the boundary components of the ring $A' = \{y \mid \alpha_2 < |y - f(x)| < \alpha_1\}$ in A'. Then the n-modulus $M(\Gamma')$ of Γ' equals $\omega_{n-1}/(\log (\alpha_1/\alpha_2))^{n-1}$ [8, p. 5, 7]. For the n-modulus of the curve family $\Gamma = f^{-1}\Gamma' = \{f^{-1} \circ \gamma' \mid \gamma' \in \Gamma'\}$ we get by [9, 11.7] (see also [2, Theorem 4]) the estimate

$$M(\Gamma) \geq \varkappa_n\left(\frac{|x_1 - x|}{|x_2 - x|}\right)$$

where $\varkappa_n : (0, \infty) \to (0, \infty)$ is a decreasing function which depends only on n. Since f is K-quasiconformal, $M(\Gamma) \leq K\, M(\Gamma')$. Hence

$$\frac{|f(x_1) - f(x)|}{|f(x_2) - f(x)|} \leq \exp\left(\left(\frac{K\, \omega_{n-1}}{\varkappa_n\left(\dfrac{|x_1 - x|}{|x_2 - x|}\right)}\right)^{1/(n-1)}\right) = v\left(\frac{|x_1 - x|}{|x_2 - x|}\right) ,$$

and the lemma is proved.

The main step is the following lemma (cf. [5, Lemma 3]).

Lemma 2. *Let $E \subset D$ be closed in D and let $f : D \to D'$ be a homeomorphism which is locally K-quasiconformal in $D - E$ and such that $f \mid E$ has local v-bounded distortion for some v. Let E_0 be the set of points $x \in E$ such that for every integer j there exists an integer $k \geq j$ such that $(B^n(x, 1/k) - B^n(x, 1/2k)) \cap E = \emptyset$. Then*

(a) $m_n(E_0) = 0$.

(b) *There exists a* $c < \infty$, *depending only on* n, K, *and* v, *such that* $H(x,f) < c$ *if* $x \in D - E_0$.

Proof. Since no point of E_0 is a point of outer density for E_0, $m_n(E_0) = 0$ [6, p. 129].

To prove (b) it suffices by [9, 34.2] to show that a uniform bound exists for $H(x,f)$ for points $x \in E - E_0$. Let $x_0 \in E - E_0$. By performing similarity transformations we may assume that $x_0 = f(x_0) = 0$. There exists an integer k_0 such that $(B^n(1/k) - B^n(1/2k)) \cap E \neq \emptyset$ for $k \geq k_0$. Since $f \mid E$ has local v-bounded distortion, there exists an $s > 0$ such that if $x_1, x_2 \in E$ and if $0 < |x_2| \leq |x_1| < s$, then

$$\frac{|f(x_1)|}{|f(x_2)|} \leq v\left(\frac{|x_1|}{|x_2|}\right).$$

Now let r be such that $0 < r < \min\,(d(0, \partial D), s, 1/k_0)/8$ and such that $\bar{B}^n(L(0,f,r)) \subset D'$. We set

$$L_r = L(0,f,r),\ l_r = l(0,f,r),$$

$$A_1 = \{x \mid r < |x| < 2r\}\,,\ A_2 = \{x \mid r/2 < |x| < r\}\,,$$

$$H_1 = \{x \mid 2r < |x| < 8r\}\,,\ H_2 = \{x \mid r/8 < |x| < r/2\}\,,$$

$$F_i = \bar{A}_i \cup \bar{H}_i\,,\quad i = 1, 2,$$

$$r_1 = \sup_{x \in E \cap F_1} |f(x)|,\quad r_2 = \inf_{x \in E \cap F_2} |f(x)|\,.$$

We shall make use of the fact that the sets $\bar{A}_1 - f^{-1}\bar{B}^n(r_1)$ and $\bar{A}_2 \cap f^{-1} B^n(r_2)$ do not meet E.

Assume $L_r > r_1$ and let $z \in fS^{n-1}(r)$ be such that $|z| = L_r$. There exists $\tau_1 > 1$ such that the line segment $J = \{\tau z \mid 1 \leq \tau \leq \tau_1\}$ is contained in $f\bar{A}_1$ and such that $\tau_1 z \in fS^{n-1}(2r)$. Assume $r_1 < \sigma < L_r$ and let Γ be the family of curves which join $f^{-1}J$ and $f^{-1}S^{n-1}(\sigma)$ in $\bar{A}_1 - f^{-1}B^n(\sigma)$.

Next we derive a positive lower bound for the n-modulus $M(\Gamma)$ of Γ. Let $r < t < 2r$ and set $S = S^{n-1}(t)$. Then $S \cap f^{-1}J \neq \emptyset$. We show that also $S \cap f^{-1} S^{n-1}(\sigma) \neq \emptyset$ holds. To prove this we first note that $\bar{H}_1 \cap E \neq \emptyset$. There is therefore a point $u \in \bar{B}^n(r_1) \cap f\bar{H}_1$. The line segment $\{\lambda u \mid 0 < \lambda < 1\}$ meets fS. The assertion then follows from the fact that fS has points in both components of the complement of $S^{n-1}(\sigma)$. We now choose a point $y \in S \cap f^{-1}J$. Since y does not belong to the non-empty closed set $S \cap f^{-1} S^{n-1}(\sigma)$, there exists an open half space M such that $y \in M$, $M \cap S \subset S - f^{-1} S^{n-1}(\sigma)$, and $\bar{M} \cap S \cap f^{-1} S^{n-1}(\sigma) \neq \emptyset$. Denote by Γ_t the family of curves $\gamma \in \Gamma$ which lie in $\bar{M} \cap S$.

For the n-modulus $M_n^S(\Gamma_t)$ of Γ_t with respect to S the estimate $M_n^S(\Gamma_t) \geq b_n/t$ holds where $b_n > 0$ is a constant which depends only on n [9, 10.2]. If $\varrho : R^n \to [0, \infty]$ is a Borel function such that

$$\int_\gamma \varrho \, ds \geq 1$$

for every rectifiable $\gamma \in \Gamma$, we have

$$\int_S \varrho^n \, dm_{n-1} \geq M_n^S(\Gamma_t)$$

by definition. Hence

$$\int \varrho^n \, dm_n \geq \int_{A_1} \varrho^n \, dm_n = \int_r^{2r} \left(\int_{S^{n-1}(t)} \varrho^n \, dm_{n-1} \right) dt$$

$$\geq \int_r^{2r} \frac{b_n}{t} \, dt = b_n \log 2 \, .$$

This gives $M(\Gamma) \geq b_n \log 2 > 0$.

On the other hand, the ring $B^n(L_r) - \bar{B}^n(\sigma)$ separates J and $S^{n-1}(\sigma)$. Consequently, $M(f\Gamma) \leq \omega_{n-1}/(\log(L_r/\sigma))^{n-1}$ [8, p. 7] where $f\Gamma = \{f \circ \gamma \mid \gamma \in \Gamma\}$. Let D_1 be the component of $D - E$ which contains $f^{-1}J$. Then every curve of Γ lies in D_1. Since $f \mid D - E$ is locally K-quasiconformal, $f \mid D_1$ is K-quasiconformal, and we have $M(\Gamma) \leq K \, M(f\Gamma)$. This gives $b_n \log 2 \leq K \, \omega_{n-1}/(\log(L_r/\sigma))^{n-1}$. Hence

$$\frac{L_r}{r_1} \leq \exp\left(\left(\frac{K \, \omega_{n-1}}{b_n \log 2}\right)^{1/(n-1)}\right) = a_n \, .$$

Similarly one proves $l_r/r_2 \geq a_n^{-1}$.

Let $x_i \in E \cap F_i$ be such that $|f(x_i)| = r_i$, $i = 1, 2$. Then $|x_1|/|x_2| \leq 64$. Finally we obtain the estimate

$$\frac{L_r}{l_r} \leq a_n^2 \, v(64) \, ,$$

which proves (b).

Theorem 1. *Let $E \subset D$ be closed in D and let $f : D \to D'$ be a homeomorphism which is locally K-quasiconformal in $D - E$ for some K. Suppose that there exists a quasiconformal mapping g of a domain G, $E \subset G \subset D$, such that $g \mid E = f \mid E$. Then f is quasiconformal.*

Proof. By Lemma 1, $f \mid E$ has local v-bounded distortion where v depends only on n and on the maximal dilatation $K(g)$ of g. By Lemma 2, there exists a set $E_0 \subset E$ of measure zero such that $H(x,f) < c < \infty$ for $x \in D - E_0$ where c depends only on n, K, and $K(g)$. By an n-dimensional version of [8, Lemma 6.3], f is differentiable almost everywhere. If f is differentiable at $x \in D - E_0$, $|f'(x)|^n \leq c^{n-1}|J(x,f)|$ where $f'(x)$ is the derivative and $J(x,f)$ the Jacobian of f at x. We shall show that f is ACL [8, p. 15]. The quasiconformality of f then follows from an n-dimensional version of [8, Theorem 6.11].

It suffices to prove that f is ACL in G. To show this, let Q be an open n-interval such that $\bar{Q} \subset G$. Let $P : R^n \to R^{n-1}$ be the orthogonal projection. For each Borel set $A \subset PQ$ we set $Z_A = Q \cap P^{-1}A$ and $\psi(A) = m_n(fZ_A)$. By Lebesgue's theorem, the set function ψ has a finite derivative $\psi'(y)$ for almost every $y \in PQ$. Furthermore, g is absolutely continuous on Z_y and $m_1(E_0 \cap Z_y) = 0$ for almost every $y \in PQ$. Fix $y \in PQ$ such that all these three conditions are satisfied. By symmetry, it is sufficient to prove that f is absolutely continuous on $\bar{Z}_y$.

Let F be a compact subset of Z_y. Since g is absolutely continuous on Z_y, since $g \mid E_0 = f \mid E_0$, and since $m_1(E_0 \cap F) = 0$, we have $\Lambda_1(f(E_0 \cap F)) = 0$ where Λ_1 is the 1-dimensional Hausdorff measure. Hence $\Lambda_1(fF) = \Lambda_1(f(F - E_0))$. Let k_0 be an integer such that $0 < 1/k_0 < d(F, \partial Q)$. For each integer $k \geq k_0$ we define the set F_k of points $x \in F$ such that $0 < r < 1/k$ implies $L(x, f, r) \leq c\, l(x, f, r)$. For every $k \geq k_0$ F_k is compact and $F_k \subset F_{k+1}$. Since $H(x,f) < c$ for $x \in F - E_0$, we have

$$F - E_0 \subset \bigcup_{k=k_0}^{\infty} F_k = \hat{F}$$

and one can prove the inequality (see [9, (31.3)] and [1, p. 10])

$$\Lambda_1(f\hat{F})^n \leq \alpha\, c^n\, \psi'(y)\, m_1(F)^{n-1} \tag{1}$$

where $\alpha < \infty$ is a constant which depends only on n. Consequently, also $\Lambda_1(fF)^n$ has the right hand side of (1) as an upper bound. After this a simple limiting process shows that f is absolutely continuous on $\bar{Z}_y$. The theorem is proved.

Theorem 2. *Let $E \subset D$ be connected or locally connected and closed in D. Let $f : D \to D'$ be a homeomorphism which is locally K-quasiconformal in $D - E$ for some K. Suppose further that $f \mid E$ has local v-bounded distortion for some v. Then f is quasiconformal.*

Proof. The set $E_0 \subset E$ defined in Lemma 2 consists in this case of isolated points only, and $D - E_0$ is a domain. By (b) in Lemma 2 and by [9, 34.1] $f \mid D - E_0$ is quasiconformal. But E_0 is removable [9, 17.3], and the theorem is proved.

Remark. If φ has local v-bounded distortion for some v, it does not necessarily follow that φ is a restriction of a quasiconformal mapping. This is shown by an n-dimensional version of the example presented in [5, p. 388]. Hence the condition on $f \mid E$ is in this sense weaker in Theorem 2 than in Theorem 1.

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