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A NEW ESTIMATE FOR LINNIK'S CONSTANT

BY

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1. Introduction

1. Let D be a large positive integer, $(D, k) = 1$, and $p(D, k)$ the least prime in the progression $\{Dn + k\}$. LINNIK has proved that there exists an absolute constant L such that

$$p(D, k) \ll D^L .$$

The best estimate for L to be found in literature is the result $L < 777$ of CHEN JING-RUN [1]. In this paper, we are going to prove that

$$L < 550 . \tag{1}$$

In the proof, the methods and results of [2] and [5] come into application. It may be noted that our proof is somewhat simpler than that in [4], pp. 364—370.

2. Lemmas

2. The first lemma is a combination of theorem 1 and lemmas 3a and 3b of [2].

Lemma 1. *Let* $N(\lambda)$ *stand for the total number of zeros of all L-functions* (mod D) *in the square*

$$1 - \frac{\lambda}{\log D} \leq \sigma \leq 1 , \ |t - t_0| < \frac{\lambda}{2 \log D} \tag{2}$$

with $t_0 \ll D^{\varepsilon}, 0.05 < \lambda \leq \log D$. *Then*

$$N(\lambda) < e^{224\lambda}. \tag{3}$$

The next lemma is contained in theorem 2 of [2].

Lemma 2. *Let us suppose that there exists an »exceptional» zero* $\varrho_1 = 1 - \delta_1$ *of a function* $L(s, \chi_1)$ *with a real character* χ_1 (mod D). *If in the rectangle*

$$1 - \frac{\lambda}{\log D} \leq \sigma \leq 1 , \ |t| \leq D^{\varepsilon} \tag{4}$$

there is any non-exceptional zero of any L-function (mod D), *then*

(5) $$\delta_1 \log D > 322^{-1} e^{-215\lambda} .$$

Note. Actually, in [2], the conditions corresponding to (2) and (4) are a little stronger; it is supposed that these conditions are replaced by $t_0 \ll 1$, $|t| \leq 1$, respectively. But it is obvious that the proofs in [2] allow the modifications needed.

3. For the third lemma, we introduce the following notation: $A = \log D$,

(6) $$K(w) = e^{\frac{L-2}{2} w} \frac{e^{w} - e^{-w}}{2w} , K_1(w) = K(Aw) ,$$

(7) $$R(n) = \frac{1}{2\pi i} \int_{(2)} K_1^2(w) e^{-w \log n} dw ,$$

(8) $$J(\chi) = - \frac{1}{2\pi i} \int_{(2)} K_1^2(w) \frac{L'}{L} (w + 1 , \chi) dw .$$

By the arguments of [5],

(9) $$R(n) = 0 \quad if \quad n < D^{L-4} , \quad or \quad n > D^{L} ,$$

(10) $$R(n) \ll A^{-1} \quad for \quad D^{L-4} \leq n \leq D^{L} .$$

From (7) — (9), we infer that

(11) $$J(\chi) = \sum_{D^{L-4} \leq n \leq D^{L}} \frac{\Lambda(n) R(n) \chi(n)}{n} .$$

On the other hand, removing the integration in (8) to the line $\delta = -\frac{3}{2}$, and writing $E_0 = 0$ or 1 according to whether $\chi = \chi_0$ or $\chi \neq \chi_0$, we obtain by well-known arguments

(12) $$J(\chi) = E_0 - \sum_{\varrho_\chi} K_1^2(\varrho_\chi - 1) + O(D^{-2}) ,$$

where ϱ_χ runs over the nontrivial zeros of $L(s, \chi)$.

Combining (11) and (12) we get

Lemma 3. We have

(13) $$\sum_{\substack{D^{L-4} \leq n \leq D^{L} \\ n \equiv k \ (\mathrm{mod}\ D)}} \frac{\Lambda(n) R(n)}{n}$$
$$= \frac{1}{\varphi(D)} \Big\{1 - \sum_{\chi \bmod D} \bar{\chi}(k) \sum_{\varrho_\chi} K_1^2(\varrho_\chi - 1)\Big\} + O(D^{-2}) .$$

4. Finally we state simple estimates for the function $K_1(w)$.

Lemma 4. *Let* $w = (-\lambda + i\tau)/A, \lambda > 0$. *Then*

$$|K_1(w)|^2 < 4.9\, e^{-(L-4)\lambda} \text{ for } \lambda \leq \tfrac{1}{2}, |\tau| \leq \tfrac{1}{2}, \tag{14}$$

and always

$$|K_1(w)|^2 \leq \frac{e^{-(L-4)\lambda}}{\lambda^2 + \tau^2}. \tag{15}$$

Proof. The estimate (15) follows immediately from (6). For (14) note that

$$\begin{aligned} |K_1(w)|^2 &= e^{-(L-4)\lambda} \left| \frac{e^{2Aw} - 1}{2Aw} \right|^2 \\ &= e^{-(L-4)\lambda} \left| \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (2Aw)^{\nu-1} \right|^2 \\ &\leq e^{-(L-4)\lambda} \left| \sum_{\nu=1}^{\infty} \frac{1}{\nu!} (\sqrt{2})^{\nu-1} \right|^2 < 4.9\, e^{-(L-4)\lambda} . \end{aligned}$$

3. Proof of the estimate for L

5. Suppose that the n's in the sum in (13) are all non-primes. Then the sum in consideration is, roughly,

$$\ll D^{L/2} \log D \cdot D^{4-L} + D^{-2} \ll D^{-2} . \tag{16}$$

Hence, for the proof of (1), it is sufficient to verify

Lemma 5. *We have for* $L \geq 550$

$$\sum_{\chi \bmod D} \sum_{\varrho_\chi} |K_1(\varrho_\chi - 1)|^2 < 1 - D^{-1/2} . \tag{17}$$

Proof. We shall carry out a straightforward estimation of the sum in (17). Suppose that there exists an exceptional zero $\varrho_1 = 1 - \delta_1$. We may suppose that

$$\tfrac{1}{2}((L-4)\, \delta_1 \log D)^2 < \delta_1 \log D\,, \ \delta_1 > D^{-\varepsilon} .$$

Then the contribution of ϱ_1 to the sum (17) is

$$< e^{-(L-4)\,\delta_1 \log D} < 1 - (L-5)\, \delta_1 \log D . \tag{18}$$

Now consider the other zeros. Let

$$\lambda_1 = \min_{\varrho} (1 - \operatorname{Re} \varrho) \log D \,, \tag{19}$$

the minimum being taken over all non-exceptional zeros with $|\operatorname{Im} \varrho| \leq D^\varepsilon$. By a result of MIECH [3], we have

$$\lambda_1 > \lambda_0 = 0.05. \tag{20}$$

Next subdivide the zeros $\varrho = 1 - \dfrac{\beta}{A} + \dfrac{\gamma i}{A} \neq \varrho_1$ into four sets H_1, H_2, H_3, H_4 as follows:

H_1: $\lambda_1 \leq \beta \leq \frac{1}{2}\,, \quad |\gamma| \leq \frac{1}{2}\,;$

H_2: $\beta > \frac{1}{2}\,, \quad |\gamma| \leq \frac{1}{2}\,;$

H_3: $\frac{1}{2} < |\gamma| < D^\varepsilon\,;$

H_4: $|\gamma| \geq D^\varepsilon\,.$

The contribution of a zero in H_i is by lemma 4 at most

$$4.9\, e^{-(L-4)\beta} \text{ for } i = 1\,; \tag{21}$$

$$4\, e^{-(L-4)\beta} \text{ for } i = 2\,; \tag{22}$$

$$|\gamma|^{-2} e^{-(L-4)\beta} \text{ for } i = 3,\, 4. \tag{23}$$

Let C_i be the total contribution of the zeros in H_i. By lemma 2 and (18), it is sufficient to show that

$$\sum_{i=1}^{4} C_i < \frac{L-6}{322}\, e^{-215\,\lambda_1}\,. \tag{24}$$

6. *Estimation of* C_1. We may suppose that $\lambda_1 \leq \frac{1}{2}$ for otherwise $C_1 = 0$. By lemma 1, the number of zeros such that $\lambda_1 \leq \beta \leq \lambda\,, \ |\gamma| \leq \frac{1}{2}$ is at most

$$e^{224\,\lambda}(1 + \lambda^{-1})\,.$$

By this, (21), and the usual rules of Stieltjes-integration, we have

$$C_1 < 4.9\, e^{-(L-4)\,\lambda_1}\, e^{224\,\lambda_1}\, (1 + \lambda_1^{-1})$$
$$+ \int_{\lambda_1}^{\infty} 4.9\, e^{-(L-4)\,\lambda}\, e^{224\,\lambda} (224 + 224\,\lambda^{-1} - \lambda^{-2})\, d\lambda\,.$$

By a simple calculation, we obtain

$$C_1 < e^{-(L-228)\,\lambda_1} \left(105 + \frac{21300}{L - 228}\right). \tag{25}$$

7. *Estimation of* C_2. The number of zeros such that $\frac{1}{2} \leq \beta \leq \lambda$, $|\gamma| \leq \frac{1}{2}$ is at most $2e^{224\lambda}$. Let $\lambda_2 = \max(\frac{1}{2}, \lambda_1)$. Then, similarly as above,

$$
\begin{aligned}
C_2 &< 8e^{-(L-228)\lambda_2} + 8 \cdot 224 \int_{\lambda_2}^{\infty} e^{-(L-228)\lambda} \, d\lambda \\
&\leq 8e^{-(L-228)\lambda_1}\left(1 + \frac{224}{L-228}\right) .
\end{aligned}
\tag{26}
$$

8. *Estimation of* C_3. The number of zeros such that $\beta \leq \lambda$, $\nu\lambda_0 \leq |\gamma| < (\nu + 1)\lambda_0$ is at most $2e^{224\lambda}$, for any ν from the interval $10 \leq \nu \leq \lambda_0^{-1}D^{\varepsilon} = \nu_0$. These zeros give a contribution

$$
C_3^{(\nu)} \leq 2e^{-(L-228)\lambda_1}\left\{1 + \frac{224}{L-228}\right\} \lambda_0^{-2}\nu^{-2} .
$$

Hence, on summation over ν,

$$
C_3 = \sum C_3^{(\nu)} < e^{-(L-228)\lambda_1}\left\{89 + \frac{20000}{L-228}\right\} . \tag{27}
$$

9. *Estimation of* C_4. Let $\nu \geq \nu_0$. For the number of the zeros such that $\beta \leq \lambda$, $\nu\lambda_0 \leq |\gamma| \leq (\nu + 1)\lambda_0$, we have the upper estimates ((29) is classical, and for (28) see [4], p. 299)

$$
\begin{aligned}
&(D^4(\nu\lambda_0)^{2/3}(\nu\lambda_0 + D)^2)^{\frac{\lambda}{\log D}} \log^8(D\nu) \\
&\ll e^{6\lambda}\nu^{\frac{8\lambda}{3\log D}} \log^8(D\nu) ,
\end{aligned}
\tag{28}
$$

and

$$
\ll D \log (D\nu) . \tag{29}
$$

Hence, by obvious arguments, C_4 is of a lower order of magnitude than our estimates for C_1, C_2, and C_3.

10. *Completion of the proof.* We are looking for a number L such that (24) holds. Now, using (25) — (27), we get for L the condition

$$
\frac{L-6}{322} e^{(L-443)\lambda_0} > 202 + \frac{43100}{L-228} .
$$

This holds if $L = 550$, and the proof is complete.

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