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AN ESTIMATE FOR THE CLASS NUMBER OF THE ABELIAN FIELD

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Introduction. Let K be an Abelian field of degree n over the field P of rational numbers. It is a class field with the rational group H of residue classes, which has the index n and the conductor f. H is thus a subgroup of the group G_f, which consists of all prime residue classes modulo f. K is a subfield of the cyclotomic field $P_f = P(\zeta)$, where ζ is a primitive fth root of unity. Let $\bar{a}$ denote the residue class which contains a. K is then left invariant under the automorphisms $\zeta \to \zeta^a$, where $\bar{a} \in H$, and the Galois group of K is isomorphic to the Abelian group G_f/H. In this paper we consider the class number of this Abelian field.

Let Y be the group of the characters of G_f. Consider the group X consisting of all the characters $\chi \in Y$ such that $\chi(\bar{a}) = 1$ for all $\bar{a} \in H$. The characters belonging to the group X are the characters of the Abelian group G_f/H. ($\chi(\bar{a}H)$ is then understood to be equal to $\chi(\bar{a})$). The characters in Y are also numerical characters $(\operatorname{mod} f)$ $(\chi(a) = 0$ if $(a\,,f) > 1)$. We denote by $f(\chi)$ the conductor of χ (the fundamental modulus of the character χ). It then follows that f is the least common multiple of the conductors $f(\chi)$. For the absolute value d of the discriminant of the field K we have an important equation

$$d = \prod_{\chi \in X} f(\chi)\,. \tag{1}$$

(cf. e.g. [2], p. 4—8).

In the following we must distinguish two cases, where K is real or imaginary respectively. If K is imaginary, then $n = 2n_0$ is even and K is of degree 2 over the field K_0, which is its real maximal subfield. The group X of the n characters χ is then divided into two parts. Those n_0 characters of X, for which $\chi(-1) = 1$, form a subgroup X_0 of X, and the other characters, for which $\chi(-1) = -1$, form a coset X_1 of X_0 in X (cf. [2], p. 5). If K is real then $K = K_0$ and $X = X_0$.

Some preliminary results. Our main object is to study the product hR, where h denotes the class number of the field K and R the regulator of this field. We first want to mention the following result of BRAUER (cf. [1], p. 745).

Theorem 1. *If K ranges over a sequence of fields normal over P, for which $n/\log d \to 0$, then*

$$\log (hR) \sim \log \sqrt{d}\,.$$

In the case of the cyclotomic field P_f we have got in [6] a sharper result:

Theorem 2. *If c and $c(\varepsilon)$ denote respectively an absolute positive constant and a positive constant depending on parameter ε alone and $\omega(f)$ denotes the number of different prime factors of f, then*

$$c(\varepsilon) f^{-\varepsilon} < Rh/G < \exp\,(c(\log\log f + \omega(f)))\,,$$

where

$$G = (2\pi)^{-\frac{1}{2}\varphi(f)}\, w \sqrt{d}$$

and

$$w = \begin{cases} 2f & \text{if } 2\nmid f\,, \\ f & \text{if } 2\mid f \end{cases}$$

is the number of roots of 1 contained in P_f.

The special case of this result (f is an odd prime) is proved by TATUZAWA [8] (see p. 111).

If we in the set of all Abelian fields pick up a sequence of fields in such a way that $n \to \infty$, then the conditions of theorem 1 are satisfied. Let us consider this question closely. We first write, by (1),

$$\log d = \sum_{\chi \in X} \log f(\chi)\,, \tag{2}$$

where the sum in the right-hand side contains n members, because the order of X equals the degree of the field K. Let x be the least positive integer such that

$$n < \sum_{k=1}^{x-1} \varphi(k) \qquad (n \geqq 4)\,. \tag{3}$$

We now get

$$\sum_{\chi \in X} \log f(\chi) \geqq \sum_{k=1}^{x} \varphi(k) \log k\,, \tag{4}$$

because there exist a most $\varphi(k)$ primitive characters $(\mathrm{mod}\, k)$. (In fact the number of the primitive characters $(\mathrm{mod}\, k)$ is smaller than $\varphi(k)$ if $k \geqq 2$.) We use the relation

$$\varphi(k) > c\, k/\log\log k \;[1] \qquad (k \geqq 2) \tag{5}$$

(cf. [7], p. 24). From (4) we get

$$\sum_{\chi \in X} \log f(\chi) > c \sum_{k=2}^{x} k \log k/\log\log k$$

$$\geqq c(\log\log x)^{-1} \sum_{k=1}^{x} k \log k > c\, x^2 \log x/\log\log x\,.$$

Because

$$\sum_{k \leqq x+1} \varphi(k) = O(x^2)\,,$$

we then get, by (3),

$$0 < n/\log d < c \log\log x/\log x\,.$$

We can thus decide that $n/\log d \to 0$, when $n \to \infty$.

If we for instance have a sequence of cyclotomic fields, then it is also easy to see directly that the conditions of theorem 1 are satisfied. Namely $\log d \geqq \frac{1}{2}\varphi(n)\log n$ (cf. [5], p. 27), and we have

$$0 < n/\log d = O(\log\log n/\log n)\,.$$

Results. In this paper we prove

Theorem 3. *Let* w *be the number of roots of* 1 *in* K *and*

$$G = \begin{cases} \sqrt{d}/2^{n-1} & \text{if } K \text{ is real}\,, \\ w\sqrt{d}/(2\pi)^{n_0} & \text{if } K \text{ is imaginary.} \end{cases}$$

Then

$$c(\varepsilon) f^{-\varepsilon} < hR/G < \exp\left(c(n\Delta + \log\log f + \omega(f))\right),$$

where

$$\Delta = \begin{cases} 0 & \text{if } K \text{ is } P_f \text{ or its real maximal subfield,} \\ \log\log(1 + \varphi(f)/n) + \log\log(\omega(f) + 2) & \text{elsewhere.} \end{cases}$$

We see immediately that theorem 2 is a special case of theorem 3. Further we can prove for the so-called relative class number h^* of K/K_0 the following (K is imaginary).

Theorem 4. *Let* Q *be* 1 *or* 2 *if a fundamental system of units of* K_0 *is also a fundamental system of units of* K *or not respectively. Then*

$$c(\varepsilon) f^{-\varepsilon} \exp(-cn\Delta) < h^*/G' < \exp\left(c(n\Delta + \log\log f + \omega(f))\right),$$

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[1] We want to note that in this paper the constants c and $c(\varepsilon)$ are not necessarily the same in their various occurrences.

where

$$G' = Qw(2\pi)^{-n_0} \Big(\prod_{\chi \in X_1} f(\chi) \Big)^{\frac{1}{2}} . \tag{6}$$

In [5] we have considered the so-called first factor h_1 of the class number of the cyclotomic field P_f. Taking into consideration that then $h^*/Q = h_1$ (cf. [2], p. 13, 71) we found that the result we got (cf. [5], p. 11) is a special case of theorem 4.

At the end of this paper we compare the estimations of theorem 3 with the estimations of theorem 1, if K ranges over a sequence of fields in such a way that $n \to \infty$. We can show that theorem 3 gives such information of the behaviour of hR, which cannot be seen from theorem 1.

Characters. We must first consider characters closely (cf. e.g. [3], pp. 216—224). We need the following

Lemma 1.

$$\sum_{\chi \in X} \chi(a) = \begin{cases} n & \textit{if} \ \bar{a} \in H , \\ 0 & \textit{elsewhere.} \end{cases} \tag{7}$$

In the case of the imaginary field the residue class of -1 does not belong to H. Then the group $H_0 = H \cup (-\bar{1}) H$ corresponds to the field K_0 (the real maximal subfield of K). For that case we have the following

Lemma 2.

$$\sum_{\chi \in X_1} \chi(a) = \begin{cases} n_0 & \textit{if} \ \bar{a} \in H , \\ -n_0 & \textit{if} \ \bar{a} \in (-1) H , \\ 0 & \textit{elsewhere.} \end{cases}$$

Proof. The number of the characters of X_1 and X_0 equals $\frac{1}{2} n = n_0$. We can thus decide that the first and the second condition of our lemma holds. Let us further suppose that $\bar{a} \notin H_0$ $((a, f) = 1)$. This is equivalent to the conditions $\pm \bar{a} \in H$ and therefore, by (7),

$$\sum_{\chi \in X_1} \chi(\pm a) + \sum_{\chi \in X_0} \chi(\pm a) = 0 .$$

Subtracting these two equations (corresponding to the values $+a$ and $-a$ respectively) we get the third condition of our lemma in the case $(a, f) = 1$. If $(a, f) > 1$ then the result of the lemma is trivial.

In the following we denote by χ^* the primitive character equivalent to χ.

Lemma 3. *Let f_j be the greatest divisor of f prime to j. If $\chi^*(j) \neq 0$, then there exists a character $\bar{\chi}$ $(\mathrm{mod}\, f_j)$ equivalent to χ^* and thus $\bar{\chi}(j) = \chi^*(j)$.*

Proof. Suppose that $\chi^*(j) \neq 0$. We find that $\chi^*(j) = 0$ if and only if $f(\chi) | f_j$. Therefore we can find a character $\bar{\chi}$ $(\mathrm{mod}\, f_j)$ equivalent to χ^*. Because $(f(\chi), j) = (f_j, j) = 1$ the result of lemma 3 follows.

Let us use $\approx$ as a notation of equivalence of characters. Directly from the definition of the equivalence follows then

Lemma 4. *If $\chi_1 \approx \bar{\chi}_1$ and $\chi_2 \approx \bar{\chi}_2$, then $\chi_1 \chi_2 \approx \bar{\chi}_1 \bar{\chi}_2$.*

We define the sets X^1 and $\bar{X}$ as follows:

$$X^1 = \{\chi \in X : \chi^*(j) \neq 0\}, \qquad \bar{X} = \{\bar{\chi} : \bar{\chi} \approx \chi, \chi \in X^1\}.$$

We can now formulate

Lemma 5. *The set $\bar{X}$ is a group. If χ runs through the set X^1, then the equivalent character $\bar{\chi}$ runs through the group $\bar{X}$.*

Proof. $\bar{X}$ is a subset of the group of all the characters $(\mathrm{mod}\, f_j)$. Let χ_0 denote the principal character. Because $\chi_0^*(j) = 1 \neq 0$, it follows that $\chi_0 \in X^1$ and therefore $\bar{\chi}_0 \in \bar{X}$. We thus see that $\bar{X}$ is not empty. If $\bar{\chi}_1, \bar{\chi}_2 \in \bar{X}$, then $(\bar{\chi}_1\bar{\chi}_2)^*(j) \neq 0$. Further, by lemma 4, $\chi_1 \chi_2 \approx \bar{\chi}_1 \bar{\chi}_2 \approx (\bar{\chi}_1 \bar{\chi}_2)^* = (\chi_1 \chi_2)^*$. Therefore $\bar{\chi}_1 \bar{\chi}_2 \in \bar{X}$. Thus we can conclude that $\bar{X}$ is a group. Let χ be a character of X^1. Then $\chi^*(j) \neq 0$ and from lemma 3 it follows that there exists a unique character $\bar{\chi}$ such that $\chi \approx \chi^* \approx \bar{\chi}$ and the last sentence of our lemma holds.

Preliminary lemmas. We now focus our attention on the preliminaries we need during the proof of theorems 3 and 4 (cf. also [5]).

Lemma 6. *Let $\chi (\neq \chi_0)$ be a character $(\mathrm{mod}\, f)$. Then*

$$L(1, \chi^*) = L(1, \chi) \prod_{p | f} (1 - \chi^*(p)/p)^{-1},$$

where $L(s, \chi) = \sum_{j=1}^{\infty} \chi(j) j^{-s}$ is the Dirichlet's L-function belonging to χ (cf. [7], p. 127).

Lemma 7. *If $x \geqq 3$ then*

$$\sum_{p \leqq x} p^{-1} = O(\log \log x)$$

(cf. [7], p. 20).

Lemma 8. *Let $(f, a) = 1$ and $0 < a < f$. If $\pi(x, f, a)$ is the number of primes $\equiv a \,(\mathrm{mod}\, f)$ not exceeding x, then*

$$\pi(x, f, a) = O(\varphi(f)^{-1} x / \log (x/f)) \qquad (x > f)$$

(cf. [7], p. 44).

There exists a constant c such that in the region

$$\sigma \geqq 1 - c/\log (f(|t| + 2)) \geqq 3/4\,, \quad t \text{ arbitrary}$$

always $L(s\,,\chi) \neq 0$ $(s = \sigma + it)$ for all the characters χ (mod f), disregarding one possible exception (cf. [7], p. 130). If such an exceptional character exists it is a real one and in the following we denote it by χ'.

Lemma 9. *Let* $f \leqq \exp(\log^{\frac{1}{2}} x)$ *and* $\chi \neq \chi_0\,, \chi'$. *Then*

$$\sum_{j \leqq x} \chi(j)\, \Lambda(j) = O(x \exp(-c \log^{\frac{1}{2}} x))\,,$$

where

$$\Lambda(j) = \begin{cases} \log p & \text{if } j = p^k \ (p \text{ is a prime and } k \geqq 1)\,, \\ 0 & \text{otherwise} \end{cases}$$

(cf. [7], pp. 133—136).

In the following we denote by Z either the set X or X_1. Further $Z^0 = Z - \{\chi_0\}$ and $Z' = Z - \{\chi_0\,, \chi'\}$.

Lemma 10. *Let*

$$U(x) = \sum_{\chi \in Z'} \sum_{j \leqq x} \chi(j)\, \Lambda(j)\,.$$

If $x \geqq \exp(\log^3 f)$ *then*

$$U(x) = O(x/\log x)\,.$$

Proof. Since $\log f \leqq \log^{1/3} x$, we get, by lemma 9,

$$U(x) = O(nx \exp(-c \log^{\frac{1}{2}} x)) = O(x \exp(\log f - c \log^{\frac{1}{2}} x)) = O(x/\log x)\,.$$

Lemma 11.

$$\omega(f) = O(\log f/\log \log f)$$

(cf. [8], p. 108).

Lemma 12. *If* j *is a natural number and* a *is an integer such that* $(a\,,f) = 1$, *then the number of solutions of*

$$x^j \equiv a \pmod f$$

is at most $j^{\omega(f)+1}$ (cf. e.g. [5], p. 22).

Lemma 13. *If* $(a\,,f) = 1\,, 0 < a < f$ *and*

$$p^j \equiv a \pmod f\,, \quad p^j \geqq \tfrac{1}{2} f\,,$$

then

$$\sum_{\geqq 2} \sum_{p} (jp^j)^{-1} = O(\omega(f)/f)\,.$$

Proof. First write the series in the form

$$\sum = \sum_{j \geqq 2} j^{-1} \sum_{\substack{p<f \\ p^j \equiv a(\bmod f),\, p^j \geqq \frac{1}{2} f}} p^{-j} + O(f^{-1}) .$$

Let us denote

$$p^j = v_{jk} f + a , \tag{8}$$

where k takes, by lemma 12, at most $N(j) = j^{\omega(f)+1}$ values. Further for a fixed j

$$\sum_{\substack{p<f \\ p^j \equiv a \,(\bmod f),\, p^j \geqq \frac{1}{2} f}} p^{-j} = \sum_{(k)} (v_{jk} f + a)^{-1} , \tag{9}$$

where in $\sum_{(k)}$ k takes the values determined by (8). Hence

$$\sum_{(k)} (v_{jk} f + a)^{-1} \leqq 2 f^{-1} v_j ,$$

where

$$v_j = \sum_{(k)} (v_{jk} + 1)^{-1} .$$

By using these notations in (9) we get

$$\sum = O(f^{-1} \sum_{j \geqq 2} v_j/j) + O(f^{-1}) . \tag{10}$$

We further get

$$\sum_{j=2}^{x} v_j/j = \sum_{i=2}^{x-1} (j(j+1))^{-1} V + V_x/x , \tag{11}$$

where

$$V_j = \sum_{t=2}^{j} v_t \qquad (V_1 = 0) .$$

It is clear, that all the numbers v_{jk} are distinct and therefore

$$V_j < \sum_{t=1}^{T(j)} t^{-1} = O(\log T(j)) ,$$

where $T(j) = \sum_{t=1}^{j} N(t)$. On the other hand,

$$\log T(j) = O\left(\log \left(\int_1^{j+1} \xi^{\omega(f)+1} \, d\xi\right)\right) = O((\omega(f) + 2) \log (j + 1)) .$$

Using these results in (10) and (11) we get

$$\sum = O(f^{-1} (\omega(f) + 2) \sum_{j=1}^{\infty} (j(j+1))^{-1} \log (j+1)) = O(\omega(f)/f) ,$$

and the lemma is proved.

Lemma 14. *If* $\chi \neq \chi_0$ *then*

$$L(1\,,\chi) = \exp\Big(\sum_{j\geqq 2}\chi(j)\,\Lambda(j)\,(j\log j)^{-1}\Big)$$

(cf. [4], p. 449).

Lemma 15. *If in* $\prod_{X^0}\,\chi$ *runs through the characters* $\in X^0$, *then*

$$hR = \begin{cases} 2^{1-n}\sqrt{d}\prod_{X^0} L(1\,,\chi^*) & \text{if } K \text{ is real,} \\ w(2\pi)^{-n_0}\sqrt{d}\prod_{X^0} L(1\,,\chi^*) & \text{if } K \text{ is imaginary} \end{cases}$$

(cf. [2], p. 7).

Lemma 16. *For the exceptional character* χ' *we have*

$$c(\varepsilon)f^{-\varepsilon} < |L(1\,,\chi')| < e^{c\,\log\log f}$$

(cf. [8], p. 105).

Proof of theorems 3 and 4. In order to estimate the product

$$\prod = \prod_{Z^0} L(1\,,\chi^*)\,, \tag{12}$$

where χ runs through the characters $\in Z^0$ we write, by lemma 6,

$$\prod_{Z^0} L(1\,,\chi^*) = \prod_1\prod_2\prod_3\,, \tag{13}$$

where

$$\prod_1 = \prod_{Z^0} L(1\,,\chi)\,,\quad \prod_2 = \prod_{\chi\in Z}\prod_{p|f}(1-\chi^*(p)/p)^{-1}\,,$$

$$\prod_3 = \prod_{p|f}(1-p^{-1})\,.$$

(In the case $Z = X_1$ the product $\prod_3$ is empty.)

Consider first the product $\prod_1$. By lemma 14, we get

$$\prod_{Z'} L(1\,,\chi) = \exp\Big(\sum_{\chi\in Z'}\sum_{j\geqq 2}\chi(j)\,\Lambda(j)\,(j\log j)^{-1}\Big)\,. \tag{14}$$

From the exponent of the equation (14) we distinguish a finite sum being extended over all integers j such that $2 \leqq j \leqq [\exp(\log^3 f)] + 1 = W$. We denote $V = 2f$ and divide this sum into parts as follows:

$$\sum_1 = \sum_{\frac12 f\leqq p<V} p^{-1}\Big(\sum_Z(\chi(p)\Big)\,,\quad \sum_2 = \sum_{V<p\leqq W} p^{-1}\Big(\sum_Z \chi(p)\Big)\,,$$

$$\sum_3 = -\sum_{p\leqq W} p^{-1}(\chi_0(p)+\chi'(p))\,,\quad \sum_4 = \sum_{\frac12 f\leqq p^j\leqq W}\ \sum_{j\geqq 2}(jp^j)^{-1}\Big(\sum_Z \chi(p^j)\Big)\,,$$

$$\sum_5 = -\sum_{p^j\leqq W}\ \sum_{j\geqq 2}(jp^j)^{-1}(\chi_0(p^j)+\chi'(p^j))\,,$$

$$\sum_6 = \sum_{p^j<\frac12 f}\ \sum_{j\geqq 1}(jp^j)^{-1}\Big(\sum_Z \chi(p^j)\Big)\,,$$

where in $\sum_Z$ χ runs through all the characters of Z. It should be noted that if $Z = X_1$, then in $\sum_3$ and $\sum_5$ there can exist only χ'. If in this case there is no exceptional character, then the sums in question are empty. Let further

$$\sum\nolimits_7 = \sum\nolimits_{Z'} \sum_{j>W} \chi(j)\, \Lambda(j)\, (j \log j)^{-1} ,$$

where in $\sum_{Z'}$ χ runs through all the characters of Z'.

We first get, by lemma 1 and 2,

$$\sum\nolimits_1 = O\Big(n \sum_{\substack{\frac{1}{2}f \leqq p < V \\ \bar{p} \in H_0}} p^{-1}\Big) = O\Big(n \sum\nolimits_a \sum_{\substack{j \\ \frac{1}{2}f \leqq fj + a < V}} (fj + a)^{-1}\Big) =$$

$$O\Big(n \sum\nolimits_a f^{-1}\Big) = O(1) ,$$

where in $\sum_a$ a takes all the values $1 \leqq a \leqq f - 1$, for which $\bar{a} \in H_0$. (If K is real then $H_0 = H \cup (-\bar{1})\, H = H$.) Further we get, by lemma 8,

$$\sum\nolimits_2 = O\Big(n \sum_{\substack{V < p \leqq W \\ \bar{p} \in H_0}} p^{-1}\Big) = O\Big(n \sum\nolimits_a \sum_{V < j \leqq W-1} \pi(j\,, f\,, a)\, (j(j+1))^{-1}\Big)$$

$$= O\left(\int_V^W (\xi \log\, (\xi/f))^{-1}\, d\xi\right) = O(\log \log f) .$$

From lemma 7, it follows that

$$\sum\nolimits_3 = O(\log \log f) .$$

By using lemma 13, we can write

$$\sum\nolimits_4 = O\Big(n \sum\nolimits_a \sum_{\substack{j \geqq 2 \quad p \\ p^j \equiv a\, (\mathrm{mod}\, f),\, p^j \geqq \frac{1}{2} f}} \sum (jp^j)^{-1}\Big) = O\Big(n \sum\nolimits_a \omega(f)/f\Big) = O(\omega(f)) .$$

It is easy to see that

$$\sum\nolimits_5 = O(1) .$$

Further denote

$$\sum\nolimits_6 = \begin{cases} \sum_6^1 & \text{if } Z = X ,\\ \sum_6^2 & \text{if } Z = X_1 . \end{cases}$$

For $\sum_6^1$ we have

$$\sum\nolimits_6^1 = n \sum_{\substack{p^j < \frac{1}{2} f \\ \bar{p}^j \in H}} \sum_{j \geqq 1} (jp^j)^{-1} . \tag{15}$$

Let us define

$$\delta = \begin{cases} 0 \text{ if } K \text{ is } P_f \text{ or its real maximal subfield,} \\ \log\log(1 + \varphi(f)/n) \quad \text{elsewhere.} \end{cases}$$

From (15) we see that

$$0 \leqq \sum\nolimits_6^1 \leqq cn\,\delta\,.$$

For $\sum_6^2$ we get

$$\sum\nolimits_6^2 = O\Big(n_0 \sum_{\substack{p^j < \frac{1}{2}f \\ \bar{p}^j \in H_0}} \sum_{j \geqq 1} (jp^j)^{-1}\Big)\,.$$

In the same way as above we get

$$\sum\nolimits_6^2 = O(n\delta)\,.$$

If we consider the series $\sum_7$, we get

$$\sum\nolimits_7 = O\Big(\sum_{j > W} U(j)\left(j^{-1}/\log j - (j+1)^{-1}/\log(j+1)\right) -$$

$$U(W)\,(W+1)^{-1}/\log(W+1)\Big) = O\Big(\sum_{j > W} (j\log^2 j)^{-1}\Big) + O(1) = O(1)\,.$$

Here we have made use of lemma 10. From lemma 11, it follows that

$$(16) \qquad \exp\left(-c(\log\log f + \omega(f)\right) > f^{-c/\log\log f} > c(\varepsilon)\,f^{-\varepsilon}\,.$$

Combining the above estimations and taking into consideration lemma 16 and the relation (16) we get

$$\exp\left(c(n\delta + \log\log f + \omega(f)\right) > \prod\nolimits_1 > \begin{cases} c(\varepsilon)\,f^{-\varepsilon} \quad \text{if} \quad Z = X\,, \\ c(\varepsilon) f^{-\varepsilon} \exp(-cn\delta) \quad \text{if} \quad Z = X_1\,. \end{cases}$$

In order to estimate the product $\prod_2$ we write

$$\prod\nolimits_2 = \exp\Big(\sum\nolimits_Z \sum_{p \mid f} \sum_{j=1}^{\infty} \chi^*(p^j)\,(jp^j)^{-1}\Big)\,.$$

Let f_p denote the greatest divisor of f prime to p. Further denote

$$Z^1 = \{\chi \in Z : \chi^*(p^j) \neq 0\}\,.$$

Hence, by lemma 3 and lemma 5,

$$\sum\nolimits_Z \chi^*(p^j) = \sum\nolimits_{Z^1} \chi^*(p^j) = \sum_{\bar{\chi} \in \bar{Z}} \bar{\chi}(p^j)\,,$$

where $\bar{Z}$ is the set of the characters $(\bmod f_p)$, which are equivalent to the characters χ of Z^1. If $Z = X$, then it follows from lemma 5 that $\bar{Z}$ is subgroup $\bar{X}$ of the group of all the characters $(\bmod f_p)$. If $Z = X_1$, then $\bar{Z}$ is the set $\bar{X}_1$ of the odd characters of $\bar{X}$. We can thus apply lemmas 1 and 2. Let n_p be the order of $\bar{X}$, and denote by β the residue class $(\mathrm{mod}\, f_p)$, which contains p. Then

$$(17) \qquad \sum_Z \chi^*(p^j) = \begin{cases} n_p & \text{if } \beta^j \in \bar{H} \text{ and } Z = X\,, \\ \pm \frac{1}{2} n_p & \text{if } \beta^j \in \bar{H}_0 \text{ and } Z = X_1 \quad (f_p > 1)\,, \\ 0 & \text{elsewhere,} \end{cases}$$

where $\bar{H}$ and $\bar{H}_0$ denote the corresponding subgroups of G_{f_p} (see introduction). Let us denote by $\sum_8$ the series in the exponent of $\prod_2$. Applying (17) we get

$$\sum\nolimits_8 = \begin{cases} \sum\limits_{p\mid f} n_p \sum\limits_{\substack{j=1 \\ \beta^j \in \bar{H}}}^{\infty} (jp^j)^{-1} & \text{if } Z = X\,. \\ O\Big(\sum\limits_{p\mid f} n_p \sum\limits_{\substack{j=1 \\ \beta^j \in \bar{H}}}^{\infty} (jp^j)^{-1}\Big) & \text{if } Z = X_1\,, \end{cases}$$

Let ξ be the least positive exponent such that $\beta^\xi \in \bar{H}$. If

$$(18) \qquad \beta^\nu \in \bar{H}\,,$$

then $\xi \mid \nu$ and, on the other hand, if $\xi \mid \nu$ then (18) holds. Since we can write

$$\sum_{j=1}^{\infty} (p^{-\xi})^j = (p^\xi - 1)^{-1}\,,$$

we have, if $Z = X$,

$$0 \leqq \sum\nolimits_8 \leqq \sum_{\substack{p \mid f \\ \beta^\xi \in \bar{H}}} n_p (p^\xi - 1)^{-1}\,.$$

Let us define

$$\delta' = \begin{cases} \omega(f) & \text{if } K \text{ is } P_f \text{ or its real maximal subfield,} \\ n \log\log(\omega(f) + 2) & \text{elsewhere.} \end{cases}$$

If K is P_f or its real maximal subfield, then $\bar{X}$ contains all the characters $(\mathrm{mod}\, f_p)$ or all the even characters $(\mathrm{mod}\, f_p)$ respectively. Therefore $p^\xi - 1 \geqq c f_p$, and we get

$$0 \leqq \sum\nolimits_8 \leqq c\delta' \text{ if } Z = X\,.$$

In the case $Z = X_1$ the series can be estimated analogously and we get

$$\sum\nolimits_8 = O(\delta') \text{ if } Z = X_1\,.$$

The above results yield

$$\exp(c\delta') \geqq |\prod_2| \geqq \begin{cases} 1 & \text{if } Z = X, \\ \exp(-c\delta') & \text{if } Z = X_1. \end{cases}$$

For $\prod_3$ we finally have

$$2^{-\omega(f)} \leqq \prod_3 < 1.$$

Combining the estimations of $\prod_1$, $\prod_2$, and $\prod_3$ we can write, by (12) and (13),

$$\text{(19)}\quad \exp(c(n\Delta + \log\log f + \omega(f))) > |\prod| > \begin{cases} c(\varepsilon) f^{-\varepsilon} & \text{if } Z = X, \\ c(\varepsilon) f^{-\varepsilon} \exp(-cn\Delta) & \text{if } Z = X_1. \end{cases}$$

Comparing (19) with lemma 15 we see immediately that theorem 3 holds. Consider the case of imaginariy K. From lemma 15 we get

$$\text{(20)}\qquad h_0 R_0 = 2^{1-n_0} \sqrt{d_0} \prod_{X_0^0} L(1, \chi^*),$$

where h_0, R_0 and d_0 denote the class number, the regulator and the absolute value of the discriminant of the real maximal subfield K_0 of K respectively. In $\prod_{X_0^0}$ χ runs through the characters $X_0 - \{\chi_0\}$. Taking into consideration that

$$QR = 2^{n_0 - 1} R_0$$

(cf. [2], p. 11) we get, by (1), (20) and lemma 15,

$$h^* = G' \prod_{X_1} L(1, \chi^*).$$

Applying the result (19) in this expression we get theorem 4.

Comparisons of theorems 1 and 3. Let us write

$$\text{(21)}\qquad \log(hR/\sqrt{d}) \begin{cases} \leqq n \log(c(1+\gamma)) + (\log\sqrt{d})/\gamma, \\ \geqq -nc(\varepsilon) - \varepsilon \log\sqrt{d} \end{cases}$$

(cf. [1], p. 740, 744), where $\gamma(\geqq 1)$ is a parameter. These inequalities are the inequalities which yield theorem 1. If we write the result of theorem 3 in the corresponding forms we get, if K is real,

$$\text{(22)}\quad \log(hR/\sqrt{d}) \begin{cases} < -(n-1)\log 2 + c(n\Delta + \log\log f + \omega(f)), \\ > -(n-1)\log 2 - \varepsilon \log f + \log c(\varepsilon). \end{cases}$$

On the other hand, if K is imaginary, then

$$\text{(23)}\quad \log(hR/\sqrt{d}) \begin{cases} < \log w - n_0 \log(2\pi) + c(n\Delta + \log\log f + \omega(f)), \\ > \log w - n_0 \log(2\pi) - \varepsilon \log f + \log c(\varepsilon). \end{cases}$$

Because $f \leqq d$ and

$$2 \leqq w \leqq c\, n \log \log (n + 2)$$

(cf. [1], p. 740) we see immediately that the lower bounds of (22) and (23) are sharper than the one of (21). As regards the upper bounds, we can choose sequences of fields, where also the upper bounds of (22) and (23) are sharper than the one of (21). For instance, we can take a sequence of fields, where the order of H and $\omega(f)$ (consequently Δ) are restricted.

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