ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

474

A CAPACITY INEQUALITY FOR QUASIREGULAR MAPPINGS

BY

O. MARTIO

HELSINKI 1970
SUOMALAINEN TIEDEAKATEMIA

doi:10.5186/aasfm.1971.474

Communicated 13 March 1970 by Olli Lehto

KESKUSKIRJAPAINO
HELSINKI 1970

1. Introduction

In [7] two capacity inequalities for quasiregular mappings are introduced. The first is the outer dilatation capacity inequality

$$\frac{1}{K_O(f)\,N(f\,,A)}\operatorname{cap}E \leqq \operatorname{cap} fE \tag{1.1}$$

where $E = (A\,,C)$ is a normal condenser in the domain G of a non-constant quasiregular mapping f with an outer dilatation $K_O(f)$ and $N(f\,,A) = \sup \operatorname{card}\,(f^{-1}f(x) \cap A)$ over $x \in A$. The second inequality is the inner dilatation $K_I(f)$ capacity inequality

$$\operatorname{cap} fE \leqq K_I(f)\operatorname{cap}E \tag{1.2}$$

where E is any condenser in G. In this paper we show that it is possible, under certain assumptions, to divide $K_I(f)$ in (1.2) by $N(f\,,A)$ corresponding to the situation in (1.1). We also give applications of this theorem.

The main results of this paper were presented in the Roumanian-Finnish seminar on Teichmüller spaces and quasiconformal mappings at Brasov 25—30.8.1969 (see [8]).

2. Notation and terminology

We shall use the same notation and terminology as in [7, 2.1, p. 5] with a few exceptions and additional concepts.

For $A \subset R^n$ we let $\dim A$ denote the topological dimension of A (see [6]).

By $[a\,,b]\,,\,a\,,b \in R^n$, we mean the closed line segment $\{ta + (1-t)b \mid 0 \leqq t \leqq 1\}$ in R^n. If $[a\,,b] \subset R^1$, then we suppose $b > a$, and $(a\,,b)$ denotes the corresponding open segment, the meanings of $[a\,,b)$ and $(a\,,b]$ being obvious. A path $\alpha : [a\,,b] \to R^n\,,\,[a\,,b] \subset R^1$, is a continuous mapping. A curve is an injective path. We let $|\alpha| = \alpha([a\,,b])$. If $x\,,y \in A \subset R^n$ and $\alpha : [a\,,b] \to A$ is a path such that $\alpha(a) = x$ and $\alpha(b) = y$, then α is said to join x and y.

A continuum in R^n is a compact connected non-empty set which is not a point.

3. Lemmas on discrete, open, and sense-preserving mappings

Suppose that $f: G \to R^n$ is discrete, open, and sense-preserving. At first we recall some notation and results in [7].

A domain D is called a normal domain of f if $\bar{D}$ is compact in G and $\partial f D = f \partial D$. A normal neighbourhood of a point $x \in G$ is a normal domain D of f such that $D \cap f^{-1}f(x) = \{x\}$. If $r > 0$ and $x \in G$, then we denote by $U(x, f, r)$ the x-component of $f^{-1}B^n(f(x), r)$. We frequently use the following lemma.

3.1. **Lemma.** [7, Lemma 2.9, p. 9] *If $\bar{U}(x, f, r)$ is compact in G, then $U(x, f, r)$ is a normal domain of f. Furthermore, for every $x \in G$ there exists $\sigma_x > 0$ such that for $0 < r \leqq \sigma_x$*

(1) *$U(x, f, r)$ is a normal neighbourhood of x.*

(2) $U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}B^n(f(x), r)$.

(3) $\bar{U}(x, f, r) = \bar{U}(x, f, \sigma_x) \cap f^{-1}\bar{B}^n(f(x), r)$.

(4) *$d(U(x, f, r)) \to 0$ as $r \to 0$.*

3.2. **Lemma.** $\dim B_f = \dim fB_f \leqq n - 2$.

The important inequality $\dim B_f \leqq n - 2$ is due to Černavskii [1, 2] (for a simpler proof see [13]). The equality $\dim B_f = \dim fB_f$ follows from [3, Theorem 2.2, p. 530].

3.3. *Remark.* If A is a subset of R^n, then the inequality $\dim A \leqq n - 2$ implies: A does not disconnect any domain in R^n [6, Corollary 1, p. 48].

If D is a normal domain of f, then the topological degree $\mu(y, f, D)$ of f is independent of $y \in fD$. This number is denoted by $\mu(f, D)$. For the next lemma we recall that $i(x, f)$ is the local topological degree of f at $x \in G$ and $N(f, D) = \sup N(y, f, D)$ over $y \in R^n$ where $N(y, f, D) = \operatorname{card}(f^{-1}(y) \cap D)$.

3.4. **Lemma.** [7, Lemma 2.12, p. 11] *If D is a normal domain of f, then $\mu(f, D) = N(f, D)$. Furthermore, if D is a normal neighbourhood of $x \in G$, then $i(x, f) = \mu(f, D) = N(f, D)$.*

From the second assertion in Lemma 3.4 we conclude that $x \in B_f$ if and only if $i(x, f) \geqq 2$.

If C is a non-empty and compact subset of G and $y \in fC$, then we set

$$M(y, f, C) = \sum_{x \in f^{-1}(y) \cap C} i(x, f). \tag{3.5}$$

The sum in (3.5) is finite since f is discrete and C compact in G. The number $M(f, C) = \inf M(y, f, C)$ over $y \in fC$ is called the *minimal multiplicity* of f on C.

3.6. **Lemma.** *Suppose that A is open and $C \subset A$ non-empty and compact in G. Then $N(f, A) \geqq M(f, C)$.*

Proof. Let $y \in fC$ and denote $\{x_1, x_2, \ldots, x_k\} = f^{-1}(y) \cap C$. By Lemma 3.1 there exists $r > 0$ such that $U_i = U(x_i, f, r)$ is a normal neighbourhood of x_i and $U_i \subset A$, $1 \leqq i \leqq k$. By Lemma 3.2 there exists $y_0 \in B^n(y, r) \setminus fB_f$. Lemma 3.4 implies

$$N(f, A) \geqq N(y_0, f, A) \geqq \sum_{i=1}^{k} N(y_0, f, U_i) = \sum_{i=1}^{k} i(x_i, f) \geqq M(f, C),$$

and the lemma follows.

3.7. **Lemma.** *Suppose that D is a normal domain of f. Then $M(f, \bar{D}) = \mu(f, D)$.*

Proof. Clearly $M(f, \bar{D}) \leqq \mu(f, D)$. Let $y \in f\bar{D}$ be such that $M(f, \bar{D}) = M(y, f, \bar{D})$. Choose a neighbourhood U of $\bar{D}$ such that $f^{-1}(y) \cap \bar{U} \setminus \bar{D} = \emptyset$, and let $r < d(y, f\partial U)$ be such that the sets $U_i = U(x_i, f, r)$ are normal neighbourhoods of the points $x_1, x_2, \ldots, x_k$ of $f^{-1}(y) \cap \bar{D}$. By Lemma 3.2 there exists a point $y_0 \in B^n(y, r) \cap fD \setminus fB_f$. Lemma 3.4 yields

$$M(f, \bar{D}) = \sum_{i=1}^{k} i(x_i, f) = \sum_{i=1}^{k} \mu(f, U_i) \geqq \operatorname{card}(f^{-1}(y_0) \cap D) = \mu(f, D).$$

The lemma follows.

In view of Remark 3.3 the proof of the following lemma is clear.

3.8. **Lemma.** *Suppose that $D \subset R^n$ is a domain, x and y are distinct points in D, and $A \subset R^n$ is a closed set such that $\dim A \leqq n-2$. Then there is a curve $\alpha : [a, b] \to D$ which joins x and y, and such that $\alpha((a, b)) \cap A = \emptyset$.*

4. Path joinable points

Suppose that $f : G \to R^n$ is discrete, open, and sense-preserving. Let $D \subset G$ be a domain, y and $\bar{y}$ points in fD, and $\beta : [a, b] \to fD$ a path which joins y and $\bar{y}$. We say that $x \in f^{-1}(y) \cap D$ and $\bar{x} \in f^{-1}(\bar{y}) \cap D$ are *(f, D, β)-joinable points* if there exists a path $\alpha : [a, b] \to D$ such that $f \circ \alpha = \beta$ and α joins x and $\bar{x}$.

4.1. **Lemma.** [7, Lemma 2.7, p. 9] *Suppose that* D *is a normal domain of* f, $\beta : [a, b] \to fD$ *is a path,* $a \leqq t_0 \leqq b$, *and* $x_0 \in D$ *such that* $\beta(t_0) = f(x_0)$. *Then there is a path* $\alpha : [a, b] \to D$ *such that* $\alpha(t_0) = x_0$ *and* $f \circ \alpha = \beta$.

The path α is called a *lift* of β. We need the above lemma only for curves in which case it is essentially due to Whyburn [15, (2.1), p. 186]. Observe that if β is a curve then its lift α is a curve.

4.2. Here we study the case where y and $\bar{y}$, $y \neq \bar{y}$, belong to the image of some normal domain D of f and $\beta : [a, b] \to fD$ is a straight line segment joining y and $\bar{y}$. These are kept fixed in the following discussion.

Let

$$f^{-1}(y) \cap D = \{x_1, x_2, \ldots, x_s\},$$
$$f^{-1}(\bar{y}) \cap D = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{\bar{s}}\}$$

where each point is counted according to its local topological degree. Then $s = \mu(f, D) = \bar{s}$. Denote $S = \{1, 2, \ldots, s\}$.

4.3. **Lemma.** *There is a bijection* $\varphi : S \to S$ *such that if* $i \in S$ *then* x_i *and* $\bar{x}_{\varphi(i)}$ *are* (f, D, β)*-joinable points.*

Proof. By Lemma 3.1 it is possible to cover $|\beta|$ by balls $B^n(y_k, r_k) \subset fD$, $k = 1, 2, \ldots, m$, $m > 2$, such that (1) $y_k \in |\beta|$, (2) $y = y_1$, $\bar{y} = y_m$, (3) $|y - y_2| < |y - y_3| < \ldots < |y - y_m|$, (4) $B^n(x_k, r_k) \cap B^n(x_{k+1}, r_{k+1}) \neq \emptyset$, $1 \leqq k \leqq m - 1$, and (5) if $x \in f^{-1}(y_k) \cap D$ then $U(x, f, r_k)$ is a normal neighbourhood of x. Let $V = \bigcup B^n(y_k, r_k)$.

Lemmas 3.2 and 3.8 show that we can construct a curve $\gamma : [a, b] \to V$ such that it joins y and $\bar{y}$, $\gamma((a, b)) \cap f(B_f \cap D) = \emptyset$, and $|\gamma| \cap B^n(y_k, r_k)$ is connected for every k, $1 \leqq k \leqq m$. Let $z \in B^n(y, r_1) \cap |\gamma| \setminus \{y\}$ and $\bar{z} \in B^n(\bar{y}, r_m) \cap |\gamma| \setminus \{\bar{y}\}$. Then the local topological index at each point of $f^{-1}(z) \cap D$ or $f^{-1}(\bar{z}) \cap D$ is one, hence, by Lemma 3.4, we may write

$$f^{-1}(z) \cap D = \{a_1, a_2, \ldots, a_s\},$$
$$f^{-1}(\bar{z}) \cap D = \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_s\}$$

where all the points a_i and $\bar{a}_i$, $1 \leqq i \leqq s$, are distinct.

Define the mapping $\varphi : S \to S$ as follows: If $i \in S$ then $a_i \in f^{-1}(z) \cap D$. By Lemma 4.1 we lift the curve γ to a curve $\gamma' : [a, b] \to D$ such that $a_i \in |\gamma'|$ and $f \circ \gamma' = \gamma$. Then some $\bar{a}_j \in f^{-1}(\bar{z}) \cap D$ belongs to $|\gamma'|$. Since $i(x, f) = 1$ for $x \in \gamma'((a, b))$, there is only one lift γ' of γ such that $a_i \in |\gamma'|$. Thus $\bar{a}_j$ is uniquely determined. Set $\varphi(i) = j$.

Finally we construct the required lift for the segment β. Let γ_i, $i \in S$, and γ' be as above. For each k, $1 \leqq k \leqq m$, let $U_k = U(x_0^k , f , r_k)$ be the only normal neighbourhood of the normal neighbourhoods $U(x , f , r_k)$, $x \in f^{-1}(y_k) \cap D$, which meets $|\gamma'|$. For if there exist two such normal neighbourhoods, say U_k and U'_k, then, since $|\gamma| \cap B^n(y_k , r_k)$ is connected, we obtain

$$f(|\gamma'| \cap U_k) = |\gamma| \cap B^n(y_k , r_k) = f(|\gamma'| \cap U'_k)$$

which is impossible. Let $W = \cup U_k$. The set W is a domain which contains $|\gamma'|$ and the points x_i and $\bar{x}_{q(i)}$. Pick points $y'_k \in B^n(y_k , r_k) \cap B^n(y_{k+1} , r_{k+1})$, $k = 1 , 2 , \ldots , m-1$, so that they lie between y_k and y_{k+1} on $|\beta|$. Divide the segment $|\beta|$ into non-overlapping segments $|\beta'_1| = [y , y'_1]$, $|\beta_2| = [y'_1 , y_2]$, $|\beta'_2| = [y_2 , y'_2] , \ldots , |\beta_m| = [y'_{m-1} , \bar{y}]$. Let $x'_k \in f^{-1}(y'_k) \cap U_k \cap U_{k+1}$ and suppose that $\alpha_k : [a , b] \to U_k$ (resp. $\alpha'_k : [a , b] \to U_k$) is a lift of β_k (resp. β'_k) such that $\alpha_k(a) = x'_{k-1}$ for $2 \leqq k \leqq m$ (resp. $\alpha'_k(b) = x'_k$ for $1 \leqq k \leqq m-1$). Since $f^{-1}(y_k) \cap U_k = \{x_0^k\}$, $\alpha_k(b) = \alpha'_k(a) = x_0^k$ for $2 \leqq k \leqq m-1$ and $\alpha'_1(a) = x_i$, $\alpha_m(b) = \bar{x}_{q(i)}$. Joining these curves, in the order $\alpha'_1 , \alpha_2 , \alpha'_2 , \ldots , \alpha_m$, into one single curve and performing an obvious change of the parameter we obtain the required lift of β. The lemma follows.

5. Capacity inequality

Suppose that $f : G \to R^n$ is a non-constant quasiregular mapping. From Rešetnjak's results [9, 10], f is discrete, open, and sense-preserving, hence we may use the results obtained in the sections 3 and 4.

Following [7, 5.2, p. 24] we call a pair $E = (A , C)$ a condenser if $A \subset R^n$ is open and C is non-empty and compact in A. A condenser E is said to be in G if $A \subset G$. E is a normal condenser if A is a normal domain of f. The image of a condenser E in G is a condenser $fE = (fA , fC)$. The capacity of E is defined as

$$\operatorname{cap} E = \operatorname{cap} (A , C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm$$

where $W_0(E) = W_0(A , C)$ is the set of all non-negative real-valued functions $u \in C_0(A)$ such that u is ACL and $u \mid C \geqq 1$. It is not difficult to show (see [7, Lemma 5.5, p. 25]) that

$$\operatorname{cap} E = \inf_{u \in W_0^\infty(E)} \int_A |\nabla u|^n \, dm$$

where $W_0^\infty(E) = W_0(E) \cap C_0^\infty(A)$.

5.1. **Theorem.** *Suppose that* $f: G \to R^n$ *is a non-constant quasiregular mapping. If* $E = (A, C)$ *is a condenser in* G *and* $N(f, A) < \infty$, *then*

$$\text{(5.2)} \qquad \operatorname{cap} fE \leqq \frac{K_I(f)\, N(f, A)^{n-1}}{M(f, C)^n} \operatorname{cap} E .$$

The proof of this result is closely related to that of the formula (1.2) [7, Theorem 7.1, p. 29]. At first we prove two lemmas.

Suppose that $u \in W_0^\infty(E)$. Define $v: fA \to R^1$ by

$$\text{(5.3)} \qquad v(y) = \frac{1}{M(f, C)} \sum_{x \in f^{-1}(y)} i(x, f)\, u(x) .$$

The sum in (5.3) contains only a finite number of non-zero terms since f is discrete and $\operatorname{spt} u$ is compact in G.

5.4. **Lemma.** *The function* v *has the properties*

(1) $\operatorname{spt} v$ *is compact in* fA, *moreover* $f(\operatorname{spt} u) = \operatorname{spt} v$.

(2) $v(y) \geqq 1$, $y \in fC$.

(3) v *is continuous.*

Proof. Set $U = \{x \in A \mid u(x) \neq 0\}$ and $V = \{x \in fA \mid v(x) \neq 0\}$. Then $fU = V$. By the continuity of f, $f(\operatorname{spt} u) = f\bar{U} = \bar{V} = \operatorname{spt} v$. This implies that $\operatorname{spt} v$ is compact in fA.

To prove (2) let $y \in fC$ and denote $\{x_1, x_2, \ldots, x_k\} = C \cap f^{-1}(y)$. Then

$$v(y) = \frac{1}{M(f, C)} \sum_{x \in f^{-1}(y)} i(x, f)\, u(x) \geqq \frac{1}{M(f, C)} \sum_{i=1}^{k} i(x_i, f)\, u(x_i)$$

$$\geqq \frac{1}{M(f, C)} \sum_{i=1}^{k} i(x_i, f) \geqq 1 ,$$

since every $u(x_i) \geqq 1$.

For (3), let $y \in fA$ and $\varepsilon > 0$. We may suppose that $y \in \operatorname{spt} v$ since otherwise (3) is trivial. Choose a neighbourhood U of $\operatorname{spt} u$ such that $\bar{U}$ is compact in A and $y \notin f\partial U$. Let $\{x_1, x_2, \ldots, x_k\} = f^{-1}(y) \cap U$. By Lemma 3.1 and the continuity of u, there exists a number r such that $0 < r < d(y, f\partial U)$ and the normal neighbourhoods $U_i = U(x_i, f, r)$, $1 \leqq i \leqq k$, satisfy the conditions: (i) $\bar{U}_i \subset U$ and (ii) $|u(x) - u(x_i)| < \varepsilon$, $x \in U_i$. Denote $U_0 = \bigcup U_i$. Let $z \in B^n(y, r)$. Then $f^{-1}(z) \cap U \subset U_0$. Fix i, $1 \leqq i \leqq k$, and let $f^{-1}(z) \cap U_i = \{z_1^i, z_2^i, \ldots, z_{l(i)}^i\}$.

Since f is discrete, there exists $C_1 > 0$ such that

$$\sum_{x \in f^{-1}(z_1) \cap U_0} i(x, f) \leqq C_1$$

for all $z_1 \in R^n$. Because U_i is a normal neighbourhood of x_i, we obtain

$$i(x_i , f) = \mu(f , U_i) = \sum_{j=1}^{l(i)} i(z_j^i , f) .$$

This yields

$$|i(x_i , f)\, u(x_i) - \sum_{j=1}^{l(i)} i(z_j^i , f)\, u(z_j^i)| = |\sum_{j=1}^{l(i)} i(z_j^i , f)\, u(x_i) - \sum_{j=1}^{l(i)} i(z_j^i , f)\, u(z_j^i)|$$

$$\leqq \sum_{j=1}^{l(i)} i(z_j^i , f)\, |u(x_i) - u(z_j^i)| < C_1 \varepsilon .$$

Summing these inequalities over i gives

$$|v(y) - v(z)| = \frac{1}{M(f , C)} \left| \sum_{i=1}^{k} \left[i(x_i , f)\, u(x_i) - \sum_{j=1}^{l(i)} i(z_j^i , f)\, u(z_j^i) \right] \right|$$

$$< \frac{k\, C_1}{M(f , C)} \varepsilon \leqq \frac{C_1^2}{M(f , C)} \varepsilon .$$

The inequality holds for every $z \in B^n(y , r)$, hence the lemma is proved.

5.5. **Lemma.** v *is* ACL.

Proof. It is enough to prove that v is ACL in a neighbourhood of each point of spt v. Fix $y_0 \in \operatorname{spt} v$ and let $f^{-1}(y_0) \cap \operatorname{spt} u = \{x_1 , x_2 , \ldots , x_q\}$. Choose r_0 such that $0 < r_0 < d(y_0 , \partial f A)$ and such that the domains $U(x_j , f , r_0)$ are normal neighbourhoods of x_j. Next choose a positive number $r_1 \leqq r_0$ such that

$$B^n(y_0 , r_1) \cap f(\operatorname{spt} u \setminus \bigcup_{j=1}^{q} U(x_j , f , r_0)) = \emptyset .$$

Then the components of $f^{-1}B^n(y_0 , r_1)$ which meet spt u are the sets $U_j = U(x_j , f , r_1)$. Set $U = \bigcup U_j$. We have

$$v(y) = \frac{1}{M(f , C)} \sum_{x \in f^{-1}(y) \cap U} i(x , f)\, u(x) \tag{5.6}$$

for every $y \in B^n(y_0 , r_1)$.

Let Q be an open n-interval with closure in $B^n(y_0 , r_1)$. Write $Q = Q_0 \times J$ where Q_0 is an $(n-1)$-interval in R^{n-1} and J is an open segment in some x_i-axis, $1 \leqq i \leqq n$. For each Borel set $F \subset Q_0$ put $\varphi(F) = m(U \cap f^{-1}(F \times J))$. Then φ is a completely additive set function in the family of all Borel sets in Q_0. By Legesgue's theorem $\overline{\varphi}'(z) < \infty$ for almost all $z \in Q_0$. Fix such z and set $J_z = \{z\} \times J$. It is sufficient to show that v is absolutely continuous on $\bar{J}_z$.

Let Φ denote the set of all continuous mappings $g : \bar{J}_z \to U$ such that $f \circ g$ is the identity mapping of $\bar{J}_z$. Observe that every $g\bar{J}_z$ is contained in some U_j.

Let $[y_1, \bar{y}_1], [y_2, \bar{y}_2], \ldots, [y_p, \bar{y}_p]$ be a collection of disjoint closed segments on $\bar{J}_z$. Fix r, $1 \leqq r \leqq p$, and j, $1 \leqq j \leqq q$. Using the notation of 4.2 with $D = U_j$, $y = y_r$, $\bar{y} = \bar{y}_r$, $|\beta| = [y_r, \bar{y}_r]$, and

$$f^{-1}(y_r) \cap U_j = \{x_i^{rj} \mid i = 1, 2, \ldots, s(j)\},$$

$$f^{-1}(\bar{y}_r) \cap U_j = \{\bar{x}_i^{rj} \mid i = 1, 2, \ldots, s(j)\}$$

where each point is counted according to its local topological degree, we obtain

$$(5.7) \qquad \sum_{x \in f^{-1}(y_r) \cap U_j} i(x, f)\, u(x) - \sum_{x \in f^{-1}(\bar{y}_r) \cap U_j} i(x, f)\, u(x) = \sum_{i=1}^{s(j)} (u(x_i^{rj}) - u(\bar{x}_{\varphi(i)}^{rj}))$$

where $\varphi = \varphi_{rj}$ is the mapping given by Lemma 4.3. Since x_i^{rj} and $\bar{x}_{\varphi(i)}^{rj}$ are (f, U_j, β)-joinable points, there exists $g_i^{rj} \in \Phi$ such that $g_i^{rj}(y_r) = x_i^{rj}$ and $g_i^{rj}(\bar{y}_r) = \bar{x}_{\varphi(i)}^{rj}$. Summing over j and r in (5.7) we have by (5.6)

$$(5.8) \qquad \sum_{r=1}^{p} |v(y_r) - v(\bar{y}_r)| \leqq \sum_{r=1}^{p} \sum_{j=1}^{q} \sum_{i=1}^{s(j)} |u(g_i^{rj}(y_r)) - u(g_i^{rj}(\bar{y}_r))| .$$

Since $u \in C_0^\infty(A)$ there is a constant C_1 such that $|u(x) - u(y)| \leqq C_1 |x - y|$ for all x and y in U. Then (5.8) implies

$$(5.9) \qquad \sum_{r=1}^{p} |v(y_r) - v(\bar{y}_r)| \leqq C_1 \sum_{j=1}^{q} \sum_{i=1}^{s(j)} \sum_{r=1}^{p} |g_i^{rj}(y_r) - g_i^{rj}(\bar{y}_r)| .$$

By [7, Lemma 7.10, p. 33] for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each j, $1 \leqq j \leqq q$, and i, $1 \leqq i \leqq s(j)$,

$$\sum_{r=1}^{p} |g_i^{rj}(y_r) - g_i^{rj}(\bar{y}_r)| < \varepsilon / q\, s(j)\, C_1$$

if $\sum |y_r - \bar{y}_r| < \delta$. Hence the left side of (5.9) is smaller than ε if $\sum |y_r - \bar{y}_r| < \delta$. The lemma is proved.

5.10. *Proof for Theorem* 5.1. Lemmas 5.4 and 5.5 show that for each $u \in W_0^\infty(E)$ v belongs to $W_0(fE)$. Let $y_0 \in \operatorname{spt} v \setminus f(\operatorname{spt} u \cap B_f)$. By the same method as in [7, Lemma 7.15, p. 35] we can find a neighbourhood V_0 of y_0 such that for every connected neighbourhood $V \subset V_0$ of y_0 the following conditions hold:

(1) $V \cap f(\operatorname{spt} u \cap B_f) = \emptyset$.

(2) The components of $f^{-1}V$ which meet $\operatorname{spt} u$ form a finite collection $D_1, D_2, \ldots, D_k$.

(3) f defines a homeomorphism $f_i : D_i \to V$, i.e. $f_i = f \mid D_i$ is a quasiconformal mapping, $1 \leqq i \leqq k$.

Furthermore we may suppose that

(4) $d(D_i) < d(\operatorname{spt} u, \partial A)$, $1 \leqq i \leqq k$.

Set $g_i = f_i^{-1}$. Since $i(x, f) = 1$ for $x \in G \setminus B_f$,

$$v(y) = \frac{1}{M(f, C)} \sum_{i=1}^{k} u(g_i(y))$$

for all $y \in V$. Thus

$$|\nabla v(y)| \leqq \frac{1}{M(f, C)} \sum_{i=1}^{k} |\nabla u(g_i(y))| \, |g_i'(y)| \tag{5.11}$$

a.e. in V because every g_i is a.e. differentiable in V.

There is a countable number of open disjoint cubes $Q_1, Q_2, \ldots$ such that $fA \setminus f(\operatorname{spt} u \cap B_f) = \cup \bar{Q}_j$ and such that if Q_j meets $\operatorname{spt} v$ then the conditions (1) — (4) are satisfied with $V = Q_j$. By [7, Lemma 2.27, p. 15] $m(fB_f) = 0$, hence

$$\operatorname{cap} fE \leqq \sum_{j=i}^{\infty} \int_{Q_j} |\nabla v|^n \, dm . \tag{5.12}$$

Fix Q_j. If it does not meet $\operatorname{spt} v$, then

$$\int_{Q_j} |\nabla v|^n \, dm = 0 .$$

If Q_j meets $\operatorname{spt} v$, let $g_i : Q_j \to D_i$, $1 \leqq i \leqq k$, denote the inverse mappings given by (3). Minkowski's inequality yields by (5.11)

$$\begin{aligned}\Big(\int_{Q_j} |\nabla v|^n \, dm\Big)^{1/n} &\leqq \frac{1}{M(f, C)} \sum_{i=1}^{k} \Big(\int_{Q_j} |\nabla u(g_i(y))|^n \, |g_i'(y)|^n \, dm(y)\Big)^{1/n} \\ &\leqq \frac{K_I(f)^{1/n}}{M(f, C)} \sum_{i=1}^{k} \Big(\int_{g_i Q_j} |\nabla u|^n \, dm\Big)^{1/n}\end{aligned}$$

where we have used the quasiconformality of g_i's in the last step. It follows from (4) that $k \leqq N(f, A)$, hence Hölder's inequality gives

$$\begin{aligned}\Big(\int_{Q_j} |\nabla v|^n \, dm\Big)^{1/n} &\leqq \frac{K_I(f)^{1/n} \, k^{1-1/n}}{M(f, C)} \Big(\sum_{i=1}^{k} \int_{g_i Q_j} |\nabla u|^n \, dm\Big)^{1/n} \\ &\leqq \frac{K_I(f)^{1/n} \, N(f, A)^{1-1/n}}{M(f, C)} \Big(\int_{f^{-1} Q_j} |\nabla u|^n \, dm\Big)^{1/n} .\end{aligned}$$

By (5.12)

$$\operatorname{cap} fE \leqq \frac{K_I(f)\, N(f\,,A)^{n-1}}{M(f\,,C)^n} \sum_{j=1}^{\infty} \int_{f^{-1}Q_j} |\nabla u|^n \, dm$$

$$\leqq \frac{K_I(f)\, N(f\,,A)^{n-1}}{M(f\,,C)^n} \int_A |\nabla u|^n \, dm \, .$$

This holds for every $u \in W_0^\infty(E)$, hence the formula (5.2) is proved.

It is easy to give an example of a quasiregular mapping $f: G \to R^n$ and a condenser $E = (A\,,C)$ in G such that $N(f\,,A)^{n-1}/M(f\,,C)^n > 1$. For example, consider the analytic function $f: R^2 \to R^2$, $z \mapsto z^2$, and the condenser $(A\,,C) = (B^2\,,\{1/2\})$, both given in complex notation. Then $N(f\,,A)/M(f\,,C)^2 = 2/1 = 2 > 1$. Hence the inequality (5.2) may be worse than (1.2). We also remark that by Lemma 3.6 the inequality $N(f\,,A) \geqq M(f\,,C)$ is satisfied for any condenser $(A\,,C)$ in G. However, in some important cases the condition of the following corollary holds.

5.13. **Corollary.** *Suppose that* $f: G \to R^n$ *is a non-constant quasiregular mapping and* $E = (A\,,C)$ *a condenser in* G *such that* $N(f\,,A) = M(f\,,C)$. *Then*

$$\operatorname{cap} fE \leqq \frac{K_I(f)}{N(f\,,A)} \operatorname{cap} E \, .$$

The next corollary gives a very precise estimate for the variation of the capacity.

5.14. **Corollary.** *Suppose that* $f: G \to R^n$ *is a non-constant quasiregular mapping and* $E = (A\,,C)$ *a normal condenser in* G *such that* $N(f\,,A) = M(f\,,C)$. *Then*

$$\frac{1}{K_O(f)\, N(f\,,A)} \operatorname{cap} E \leqq \operatorname{cap} fE \leqq \frac{K_I(f)}{N(f\,,A)} \operatorname{cap} E \, .$$

This follows immediately from (1.1) and Corollary 5.13.

5.15. *Remark.* We give an example of the case $N(f\,,A) = M(f\,,C)$. Let $f: G \to R^n$ be discrete, open, and sense-preserving. Suppose that $A = U(x\,,f\,,r_1)$ is a normal neighbourhood of $x \in G$. Let $0 < r < r_1$ and denote $C = \bar{U}(x\,,f\,,r)$. Then $E = (A\,,C)$ is a condenser in G, and by Lemmas 3.4 and 3.7,

$$N(f\,,A) = i(x\,,f) = \mu(f\,,A) = M(f\,,C) \, .$$

6. Applications

The first theorem generalizes the well-known fact: If $f: G \to R^2$ is an analytic function, then $f'(x) = 0$ for $x \in B_f$.

6.1. **Theorem.** *Suppose that* $f: G \to R^n$ *is quasiregular and* $x \in G$. *If* $K_I(f) < i(x, f)$, *then* f *is differentiable at* x *and* $f'(x) = 0$.

Proof. We may suppose that f is not a constant. Let $U_0 = U(x, f, r_0)$ be a normal neighbourhood of $x \in G$. Suppose that $r_1 > 0$ is so small that $U_0 \supset \bar{B}^n(x, r_1)$. By Lemma 3.1 there exists a positive number r_2 such that for $0 < r \leqq r_2$ $U(x, f, r) \subset B^n(x, r_1)$. Let $y \in U(x, f, r_2)$ and suppose $y \neq x$. Set $r = |f(x) - f(y)| > 0$. Define a condenser E in G as $(U_0, \bar{U}(x, f, r))$. By Remark 5.15 we have

$$i(x, f) = M(f, \bar{U}(x, f, r)) = N(f, U_0) .$$

Suppose that the condition $K_I(f) < i(x, f)$ is satisfied. By Corollary 5.13

$$\text{(6.2)} \qquad \frac{\omega_{n-1}}{(\log(r_0/r))^{n-1}} = \operatorname{cap} fE \leqq \frac{K_I(f)}{i(x, f)} \operatorname{cap} E = \frac{1}{\alpha} \operatorname{cap} E$$

where $\alpha > 1$. Let

$$L^*(r) = \inf \{\beta > 0 \mid B^n(x, \beta) \supset U(x, f, r)\},$$

$$l^*(r) = \sup \{\beta > 0 \mid B^n(x, \beta) \subset U(x, f, r)\}$$

(see [7, 4.1, p. 17]). Since the condenser $(B^n(x, r_1), \bar{B}^n(x, L^*(r)))$ separates E, i.e. $U_0 \supset B^n(x, r_1)$ and $\bar{U}(x, f, r) \subset \bar{B}^n(x, L^*(r))$, we have

$$\operatorname{cap} E \leqq \operatorname{cap}(B^n(x, r_1), \bar{B}^n(x, L^*(r))) = \frac{\omega_{n-1}}{(\log(r_1/L^*(r)))^{n-1}} .$$

The inequality (6.2) yields

$$\text{(6.3)} \qquad r \leqq C_1 L^*(r)^{\alpha^{1/(n-1)}}$$

where $C_1 = r_0/r_1^{\alpha^{1/(n-1)}}$. By the metric characterization of quasiregular mappings [7, Theorem 4.6, p. 19]

$$\frac{L^*(r)}{l^*(r)} \leqq C_2$$

for $0 < r \leqq r_2$. But

$$\frac{L^*(r)}{|x - y|} \leqq \frac{L^*(r)}{l^*(r)} \leqq C_2$$

which implies, by (6.3),

$$|f(x) - f(y)| = r \leqq C_1 C_2^{\alpha^{1/(n-1)}} |x - y|^{\alpha^{1/(n-1)}} .$$

Since this holds for every $y \in U(x, f, r_2)$ and $\alpha^{1/(n-1)} > 1$, f is differentiable at x and $f'(x) = 0$. The theorem follows.

6.4. *Remark.* The inequality $K_I(f) < i(x, f)$ in Theorem 6.1 is best possible, i.e. there exists a quasiregular mapping f such that $K_I(f) = i(x, f)$ and f is not differentiable at x. The winding mapping $f: R^3 \to R^3$, $(r, \varphi, z) \mapsto (r, 2\varphi, z)$ in cylindral coordinates, gives an example since $i(0, f) = K_I(f) = 2$ and it is not differentiable at 0. Similar examples exist in all dimensions $n \geqq 2$. A question arises: Does there exist any quasiregular mapping $f: G \to R^n$, $n \geqq 3$, such that $K_I(f) < i(x, f)$ for some $x \in G$? However, the example of Rickman [11] shows that $i(x, f)$ has no upper bound in terms of $K_I(f)$.

6.5. **Corollary.** *Suppose that $f: G \to R^n$ is a non-constant quasiregular mapping and $\alpha : [a, b] \to B_f$ a rectifiable curve. Then*

$$K_I(f) \geqq \inf_{x \in |\alpha|} i(x, f) .$$

The bound is best possible.

Proof. Integrate $f'(x)$ along $|\alpha|$ and use Theorem 6.1 and Remark 6.4.

6.6. **Corollary.** *If $f: G \to R^n$ is a non-constant quasiregular mapping and $\alpha : [a, b] \to B_f$ is a rectifiable curve, then $K_I(f) \geqq 2$. The bound is best possible.*

6.7. The author conjectures that Corollaries 6.5 and 6.6 hold if α is any continuum in B_f. Anyway, we obtain the following weaker result.

6.8. **Theorem.** *Suppose that $f: G \to R^n$ is a non-constant quasiregular mapping and F is a compact set in B_f such that $\Lambda_p(fF) > 0$. Then*

$$\frac{K_I(f)}{\inf_{x \in F} i(x, f)} > \left(\frac{p}{n}\right)^{n-1} .$$

Proof. Following [7, 4.1, p. 17] we set for $x \in G$ and $0 < r < d(f(x), \partial fG)$

$$L^*(x, f, r) = \sup |x - z| ,$$

$$l^*(x, f, r) = \inf |x - z|$$

over $z \in \partial U(x, f, r)$ and for $0 < r < d(x, \partial G)$

$$L(x, f, r) = \sup |f(x) - f(y)|$$

over $|x - y| = r$. By [7, Theorem 4.6, p. 19] there exists $a > 0$ such that

$$H^*(x, f) = \overline{\lim_{r \to 0}} \frac{L^*(x, f, r)}{l^*(x, f, r)} < a \tag{6.9}$$

for all $x \in F$. By [7, Lemma 4.8, p. 20] we may find an interval $(0, r_1)$ such that the mapping $(x, r) \mapsto L^*(x, f, r)$ is lower semicontinuous and the mapping $(x, r) \mapsto l^*(x, f, r)$ is continuous on $F \times (0, r_1)$. Hence for $1/i < r_1$ the sets

$$C_i = \{x \in F \mid L^*(x, f, r)/l^*(x, f, r) \leqq a \text{ for all } 0 < r < 1/i\}$$

are compact. Furthermore $\cup C_i = F$, thus it is possible to fix i such that $\Lambda_p(fC_i) > 0$. Since the mapping $x \mapsto i(x, f)$ is upper semicontinuous, the sets $C_{ij} = \{x \in C_i \mid i(x, f) = j\}$, $j = 2, 3, \ldots$, are Borel sets. Fix j such that $\Lambda_p(fC_{ij}) > 0$. Pick $x_0 \in C_{ij}$ and r_0 with the properties (i) $0 < r_0 < 1/i$, (ii) $U_0 = U(x_0, f, r_0)$ is a normal neighbourhood of x_0, and (iii) $\Lambda_p(f(C_{ij} \cap U)) > 0$ for all $U = U(x_0, f, r)$, $0 < r \leqq r_0$. Choose $r_0' > 0$ such that $\bar{B}^n(x_0, 2r_0') \subset U_0$ and then $r_0'' > 0$ such that $\bar{U}(x_0, f, r_0'') \subset B^n(x_0, r_0')$.

By the continuity of f, the function

$$\sigma(r) = \sup_{x \in F} L(x, f, r),$$

$r \in (0, d(F, \partial G))$, has the property $\sigma(r) \to 0$ as $r \to 0$. Suppose that $t_1 \in (0, d(F, \partial G))$ is such that $r \in (0, t_1]$ implies

$$\sigma(r) < d(fS^{n-1}(x_0, r_0'), B^n(f(x_0), r_0'')) > 0.$$

Assume now that the condition

$$\frac{K_I(f)}{\inf\limits_{x \in F} i(x, f)} \leqq \left(\frac{p}{n}\right)^{n-1} \tag{6.10}$$

holds. Denote $F_1 = C_{ij} \cap U(x_0, f, r_0'')$. We show that $\Lambda_p(fF_1) = 0$. Let $t > 0$ and $\varepsilon > 0$. By [7, Theorem 8.3, p. 38] $m(B_f) = 0$, and so $\Lambda_n(F_1) = 0$. Hence it is possible to cover F_1 by balls $B^n(x_k, r_k)$, $k = 1, 2, \ldots$, such that (1) $x_k \in F_1$, (2) $r_k < t_1$, (3) $\sigma(r_k) < t/2$, and (4) $\sum r_k^n < \varepsilon$. Fix k and define $E_k = (U_0, \bar{U}(x_k, f, L_k))$ where $L_k = L(x_k, f, r_k)$. Then E_k is a condenser in G since, by (2), $\bar{U}(x_k, f, L_k) \subset B^n(x_0, r_0') \subset U_0$. Because $i(x_0, f) = i(x_k, f)$, the set $U(x_k, f, L_k)$ is a normal neighbourhood of x_k. Thus Lemmas 3.7 and 3.4 yield

$M(f, \bar{U}(x_k, f, L_k)) = \mu(f, U(x_k, f, L_k)) = i(x_k, f) = i(x_0, f) = N(f, U_0)$.

Corollary 5.13 and (6.10) imply

$$\operatorname{cap} fE_k \leqq \frac{K_I(f)}{i(x_0, f)} \operatorname{cap} E_k \leqq \left(\frac{p}{n}\right)^{n-1} \operatorname{cap} E_k .$$

On the other hand

$$\operatorname{cap} fE_k \geqq \frac{\omega_{n-1}}{(\log (2r_0/L_k))^{n-1}}$$

since the condenser fE_k separates the condenser $(B^n(f(x_k), 2r_0), \bar{B}^n(f(x_k), L_k))$ and

$$\operatorname{cap} E_k \leqq \frac{\omega_{n-1}}{(\log (r_0'/L_k^*))^{n-1}}$$

since $(B^n(x_k, r_0'), \bar{B}^n(x_k, L_k^*))$, $L_k^* = L^*(x_k, f, L_k)$, separates E_k. But $U(x_k, f, L_k)$ is a normal neighbourhood of x_k, hence we conclude from [7, Lemma 4.3, p. 18] that $l_k^* = l^*(x_k, f, L_k) = r_k$. Thus the above three inequalities yield

$$(6.11) \qquad L_k \leqq \frac{2r_0}{r_0'^{n/p}} L_k^{*n/p} \leqq \frac{2r_0 a^{n/p}}{r_0'^{n/p}} l_k^{*n/p} = b\, r_k^{n/p} .$$

The set fF_1 is covered by $\bigcup fB^n(x_k, r_k)$ and, by (3), $d(fB^n(x_k, r_k)) < t$, hence we obtain from (6.11) and (4)

$$\Lambda_p^t(fF_1) \leqq \sum d(fB^n(x_k, r_k))^p \leqq 2^p \sum L_k^p \leqq 2^p b^p \sum r_k^n < 2^p b^p \varepsilon .$$

Thus $\Lambda_p(fF_1) = 0$, a contradiction of (iii). The theorem follows.

6.12. **Corollary.** *Suppose that* $f: G \to R^n$ *is a non-constant quasiregular mapping and* $F \subset B_f$ *is a continuum. Then*

$$\frac{K_I(f)}{\inf\limits_{x \in F} i(x, f)} > n^{1-n} .$$

Proof. Since f is light, $\Lambda_1(fF) > 0$, hence the corollary follows from Theorem 6.8.

6.13. *Definition.* A non-constant quasiregular mapping is said to be of *minimal multiplicity* q if $B_f \neq \emptyset$ and $i(x, f) \geqq q$ for all $x \in B_f$.

6.14. **Theorem.** *Suppose that* $f: G \to R^3$ *is of minimal multiplicity* q. *Then* $K_I(f) > q/9$.

Proof. By [12] $\dim B_f = 1$, hence there is a continuum $F \subset B_f$ ([6, II 4, p. 20]). Thus the theorem follows from Corollary 6.12.

6.15. *Remark.* We have been able to prove that $K_I(f) > q(n/(n-2))^{1-n}$ for every quasiregular mapping $f: G \to R^n$ of minimal multiplicity q in any dimension $n \geqq 3$. The authors of [7] intend to return to related questions in a later paper.

6.16. *Remark.* Theorem 6.14 gives a rather good asymptotic estimate for the growth of $K_I(f)$ in terms of q. In fact, the winding mapping $f: R^3 \to R^3$, $(r, \varphi, z) \mapsto (r, q\varphi, z)$ in cylindrical coordinates, is of minimal multiplicity q and $K_I(f) = q$. Hence

$$q \geqq \inf_{f \in W_q} K_I(f) \geqq q/9$$

where W_q is the family of all quasiregular mappings into R^3 of minimal multiplicity q. If the conjecture in 6.7 is true, then

$$\inf_{f \in W_q} K_I(f) = q ,$$

in particular, there would not exist a non-constant quasiregular mapping $f: G \to R^3$ with non-empty B_f such that $K_I(f) < 2$.

Univesity of Helsinki
Helsinki, Finland

References

1. ČERNAVSKII, A. V. (Чернавский, А. В.): Конечнократные открытые отображения многообразий. - Mat. Sbornik 65 (1964), 357–369.
2. —»— Дополнение к статье «О конечнократных открытых отображениях многообразий». - Mat. Sbornik 66 (1965), 471–472.
3. CHURCH, P. T. and HEMMINGSEN, E.: Light open maps on n-manifolds. - Duke Math. J. 27 (1960), 527–536.
4. GEHRING, F. W.: Symmetrization of rings in space. - Trans. Amer. Math. Soc. 101 (1961), 499–519.
5. —»— Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103 (1962), 353–393.
6. HUREWICZ, W. and WALLMAN, H.: Dimension theory. - Princeton University Press, 1941.
7. MARTIO, O., RICKMAN, S. and VÄISÄLÄ, J.: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. A I 448 (1969), 1–40.
8. MARTIO, O.: Behaviour of capacities under quasiregular mappings. - Proceedings of the Roumanian-Finnish seminar on Teichmüller spaces and quasiconformal mappings (to appear).
9. REŠETNJAK, J. G. (Решетняк, Ю. Г.): Пространственные отображения с ограниченным искажением. - Sibirsk. Mat. Ž. 8 (1967), 629–658.
10. —»— Об условии ограниченности индекса для отображений с ограниченным искажением. - Sibirsk. Mat. Ž. 9 (1968), 368–374.
11. RICKMAN, S.: Quasiregular mappings. - Proceedings of the Roumanian-Finnish seminar on Teichmüller spaces and quasiconformal mappings (to appear).
12. TROHIMČUK, J. J. (Трохимчук, Ю. Ю.): О непрерывных отображениях областей эвклидова пространства. - Ukrain. Mat. Ž. 16 (1964), 196–211.
13. VÄISÄLÄ, J.: Discrete open mappings on manifolds. - Ann. Acad. Sci. Fenn. A I 392 (1966), 1–10.
14. —»— Lectures on n-dimensional quasiconformal mappings. - Van Nostrand (to appear).
15. WHYBURN, G. T.: Analytic topology. - Amer. Math. Soc. Colloquium Publications, 1942.

Printed October 1970