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BOUNDARY BEHAVIOR OF QUASICONFORMAL MAPPINGS IN n-SPACE

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Raimo Näkki

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INTRODUCTION

One of the basic problems in the theory of n-dimensional quasiconformal mappings is to determine whether or not two given homeomorphic domains can be mapped quasiconformally onto each other. The problem is closely related to the possibility of extending quasiconformal mappings of domains, which, in turn, has an immediate connection with the behavior of the mappings near the boundaries.

The mapping and extension problems indicated have mostly been studied in the special case where one of the domains is a ball. (See, for example, the papers [1] — [6] of Gehring or Väisälä.) The reason for this is obvious. Namely, when $n > 2$, there is no analogue of the Riemann mapping theorem, which states that a plane domain can be mapped quasiconformally onto a disc if and only if its boundary is a connected set containing at least two points, and which may be frequently used while discussing the boundary extension of plane quasiconformal mappings.

The starting point of this thesis is, however, more general. For we will study the boundary behavior of n-dimensional quasiconformal mappings between domains about whose boundaries we make hypotheses as weak as possible. Our main interest will be directed to an examination of the extension of such mappings to and over the boundaries, but at times we will also investigate the conditions under which two homeomorphic domains can be mapped quasiconformally onto each other.

We begin in Section 1 by introducing the concepts, associated with the boundary of a domain, upon which almost the entire subsequent theory is essentially based. In Section 2 we study, by means of cluster sets, the local behavior of quasiconformal mappings on the boundary of a domain. Most of the next section deals with questions related to the global boundary behavior of quasiconformal mappings. The developed theory will then be applied to quasiconformal mappings of a ball in Section 4. For example, we characterize those domains D for which every quasiconformal mapping between D and a ball can be extended to a continuous mapping between the closures. Furthermore, by refining a result due to Gehring [2], we show that a Jordan domain D in 3-space can be mapped quasiconformally onto a ball if and only if every point in its boundary has a neighborhood U such that $U \cap D$ can be mapped quasiconformally onto a ball. Some of our results suggest the investigation of the extent to which the presence of a sharp edge in the boundary of a domain destroys the smoothness properties of the domain. The final section is devoted to this investigation.

Notation and terminology. We denote by R^1 the real number system and by $\bar{R}^1 = R^1 \cup \{\infty , -\infty\}$ its two-point compactification. Given two real numbers a and b, $a < b$, we let (a , b) denote the open interval $\{t : a < t < b\}$ and $[a , b]$ the closed interval $\{t : a \leq t \leq b\}$. Unless otherwise stated, all point sets considered in this paper are assumed to lie in $\bar{R}^n = R^n \cup \{\infty\}$, $n \geq 2$, the one-point compactification of the euclidean n-space R^n. For each point $x \in R^n$ we let x_i denote the i-th coordinate of x, taken with respect to a fixed orthonormal basis $(e_1 , \ldots , e_n)$. The subspace $x_n = 0$ of R^n will be identified with R^{n-1}. Sometimes we shall also use cylindrical coordinates (r , φ , z) (polar coordinates (r , φ) if $n = 2$) for a point $x \in R^n$. This means that $r \geq 0$, $0 \leq \varphi < 2\pi$, $z \in R^{n-2}$, and $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $x_i = z_{i-2}$ for $3 \leq i \leq n$. Each point $x \in R^n$ will be treated as a vector with norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

Given a point $x \in R^n$ and a number $r > 0$, we let $B^n(x , r)$ denote the n-dimensional ball $\{y \in R^n : |y - x| < r\}$ and $S^{n-1}(x , r)$ its $(n - 1)$-dimensional boundary sphere $\{y \in R^n : |y - x| = r\}$. We will also employ the abbreviations

$$B^n(r) = B^n(0 , r) , \quad B^n = B^n(1) ,$$

$$S^{n-1}(r) = S^{n-1}(0 , r) , S^{n-1} = S^{n-1}(1) ,$$

where 0 denotes the origin, and write

$$B^n_+(r) = \{x \in B^n(r) : x_n > 0\} , \quad B^n_+ = B^n_+(1) .$$

For each set $E \subset \bar{R}^n$ we let ∂E, $\bar{E}$, and $\complement E$ denote the boundary, closure, and complement of E, all taken with respect to $\bar{R}^n$. Furthermore, given two sets E and F in $\bar{R}^n$, we let $E \setminus F$ denote the difference set $\{x : x \in E , x \notin F\}$ and $d(E , F)$ the euclidean distance between E and F.

As a measure in R^n we use the n-dimensional Lebesgue measure m_n, where the subscript n may be omitted if there is no danger of misunderstanding. The measure of a set $E \subset \bar{R}^n$ is defined as that of $E \setminus \{\infty\}$. Obviously, m_n is also defined for sets in n-dimensional smooth submanifolds of $R^{n'}$, $n' > n$. We abbreviate $\omega_n = m_n(S^n)$.

A *neighborhood* of a set E is an open set containing E. A *domain* is an open connected non-empty set. The notation $f : D \to D'$ includes the assumption that D and D' are domains in $\bar{R}^n$. A domain D is said to be a *Jordan domain* if ∂D is homeomorphic to S^{n-1}. A *ring* is a domain whose complement consists of two components. A *continuum* is a compact connected set which contains more than one point.

Let G be a subset of $\bar{R}^n$. By a *path* in G we mean a continuous mapping $\gamma : \Delta \to G$, where Δ is a closed interval of R^1. Suppose also that E and F are subsets of $\bar{R}^n$. Then we let $\Delta(E, F : G)$ denote the family of all paths which *join* E and F in G; that is, a path $\gamma : [a, b] \to \bar{R}^n$ is an element of $\Delta(E, F : G)$ if and only if one of its end-points $\gamma(a)$, $\gamma(b)$ belongs to E, the other belongs to F, and $\gamma(t) \in G$ for $a < t < b$. The *locus* $|\gamma|$ of a path $\gamma : \Delta \to \bar{R}^n$ is the point set $\gamma\Delta = \{\gamma(t) : t \in \Delta\}$. A *subpath* of a path γ is a restriction of γ to a closed subinterval. A path family Γ_1 is said to be *minorized* by a path family Γ_2 if every path of Γ_1 has a subpath belonging to Γ_2.

Suppose next that Γ is a family of paths in $\bar{R}^n$. We let $F(\Gamma)$ denote the family of all Borel functions $\varrho : \bar{R}^n \to \dot{R}^1$ which are non-negative and for which

$$\int_\gamma \varrho \, ds \geq 1$$

for every rectifiable path $\gamma \in \Gamma$. The *p-modulus* $(p \geq 1)$ of Γ is then defined as

$$M_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^p \, dm_n .$$

If $F(\Gamma) = \emptyset$, we set $M_p(\Gamma) = \infty$. This occurs if and only if Γ contains a constant path. To simplify notations, we write $M(\Gamma)$ instead of $M_n(\Gamma)$ and call it the *modulus* of Γ.

We shall also use surface moduli of path families. Let S be an $(n-1)$-dimensional smooth manifold in R^n. (In this paper, however, we need only the cases where S is a plane, a sphere, or the lateral surface of a right circular cylinder.) If Γ is a path family in S, we again denote by $F(\Gamma)$ the family of all non-negative Borel functions $\varrho : S \to \dot{R}^1$ for which the line integral of ϱ is greater than or equal to one along every rectifiable path $\gamma \in \Gamma$. The *p-modulus of Γ with respect to S* is defined as

$$M_p^S(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_S \varrho^p \, dm_{n-1} .$$

Let D and D' be two domains in $\bar{R}^n$. A homeomorphism $f : D \to D'$ is said to be *K-quasiconformal*, $1 \leq K < \infty$, if it satisfies the double inequality

$$\frac{1}{K} M(\Gamma) \leq M(f\Gamma) \leq K M(\Gamma)$$

for each path family Γ in D. Here $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$. (Sometimes the factor K in the above inequalities is replaced by K^{n-1}, for example in Gehring — Väisälä [4]. This is obviously a matter of notation only.) A homeomorphism f is said to be *quasiconformal* if it is K-quasiconformal for some K. The *maximal dilatation* of f, denoted by $K(f)$, is then defined as the least K for which f is K-quasiconformal. Finally, the domains D and D' are called *quasiconformally equivalent* if there exists a quasiconformal mapping of D onto D'.

1. Classes of boundary points

We begin by defining a number of concepts allowing us to describe the behavior of a domain at a boundary point. Then we give alternative characterizations for some of these concepts and finally determine relationships between them.

Recall that unless otherwise stated, all point sets considered lie in $\bar{R}^n$, $n \geq 2$.

Definitions. Altogether we introduce seven properties. Some of them, however, are nothing but natural generalizations of others. (Cf. Väisälä [8, 17.5].)

1.1. **Connectedness properties.** Let b be a boundary point of a domain D.
(i) D is *locally connected* at b if there exist arbitrarily small neighborhoods U of b such that $U \cap D$ is connected.
(ii) D is *m-connected* at b, $m = 1, 2, \ldots,$ if m is the least integer for which there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components.
(iii) D is *finitely connected* at b if there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of a finite number of components.

1.2. Remark. Obviously a domain is locally connected at a boundary point if and only if it is 1-connected at the point. Furthermore, m-connectedness always implies finite connectedness. The following example shows that the converse is not true.

1.3. Example. Let (r, φ, z) be the cylindrical coordinates in R^n, and let $D = \bar{R}^n \setminus \bigcup_{i=1}^{m} A_i$ where $A_i = \{x = (r, \varphi, z) : 0 \leq r \leq 1/i, \quad \varphi = 1/i, -1/i \leq |z| \leq 1/i\}$. If $m < \infty$, then D is m-connected at the origin b, while if $m = \infty$, then D is still finitely connected at b, but no longer m-connected for any integer m.

1.4. **Quasiconformal collaredness.** Let b be a boundary point of a domain D.

(iv) D is *quasiconformally collared* at b if there exists a neighborhood U of b and a quasiconformal mapping $g : U \cap D \to B^n_+$ such that $\lim_{x \to b} g(x) = 0$ and $\lim_{y \to 0} g^{-1}(y) = b$.

(v) D is *quasiconformally* m*-collared* at b, $m = 1, 2, \ldots,$ if there exists a neighborhood U of b such that $U \cap D$ consists of m components, $E_1, \ldots, E_m$, for each of which there is a quasiconformal mapping $g_i : E_i \to B^n_+$ with $\lim_{x \to b} g_i(x) = 0$ and $\lim_{y \to 0} g_i^{-1}(y) = b$.

1.5. REMARK. Obviously a domain is quasiconformally collared at a boundary point if and only if it is quasiconformally 1-collared at the point. It is also evident that the half-ball B^n_+ could be replaced in 1.4 by several other domains, for example by the ball B^n. The choice of the origin to be the limit at b of the mapping in question is unessential as well. We have, however, given a preference to the formulations in 1.4 because of certain technical advantages.

1.6. EXAMPLE. Let D be the domain in Example 1.3 with a finite m, let (r, φ, z) be the cylindrical coordinates in R^n, and let

$$E_i = \begin{cases} B^n(1/m) \cap \{x = (r, \varphi, z) : \varphi \notin [1/m, 1]\} & \text{if } i = 1, \\ B^n(1/m) \cap \{x = (r, \varphi, z) : 1/i < \varphi < 1/(i-1)\} & \text{if } i = 2, \ldots, m. \end{cases}$$

Then $E_1, \ldots, E_m$ are the components of $D \cap B^n(1/m)$. For $x = (r, \varphi, z) \in E_i$ set $f_i(x) = (r, h_i(\varphi), z)$, where

$$h_i(\varphi) = \begin{cases} \pi(2\pi + \varphi - 1)/(2\pi + 1/m - 1) & \text{if } 0 \le \varphi < 1/\mathrm{m} \text{ and } i = 1, \\ \pi(\varphi - 1)/(2\pi + 1/m - 1) & \text{if } 1 < \varphi < 2\pi \text{ and } i = 1, \\ \pi i(i-1)(\varphi - 1/i) & \text{if } i = 2, \ldots, m. \end{cases}$$

Next for $y \in B^n(1/m)$ let $g(y) = my$. Then $g_i = g \circ f_i$ is a quasiconformal mapping of E_i onto the half-ball $B^n \cap \{x : x_2 > 0\}$ with $\lim_{x \to 0} g_i(x) = 0 = \lim_{y \to 0} g_i^{-1}(y)$. Hence D is quasiconformally m-collared at the origin.

1.7. **Quasiconformal flatness and accessibility.** Let b be a boundary point of a domain D.

(vi) D is *quasiconformally flat* at b if $M(\Delta(F_1, F_2 : D)) = \infty$ whenever F_1 and F_2 are two connected subsets of D with $b \in \bar{F}_1 \cap \bar{F}_2$.

(vii) b is *quasiconformally accessible* from D if, given any neighborhood U of b, there exists a continuum $A \subset D$ and a number $\delta > 0$ such that $M(\Delta(A, F : D)) \geq \delta$ whenever F is a connected subset of D with $b \in \bar{F}$ and $F \cap \partial U \neq \emptyset$.

1.8. Remark. In Väisälä [8], »quasiconformally flat» is called »property P_1» and the property of being »quasiconformally accessible» is similar to »property P_2».

1.9. Example. Let D again be the domain defined in 1.3. Arguments similar to those to be presented in 1.17 and 1.18 show that D is quasiconformally flat at the origin b if and only if $m = 1$, and that b is quasiconformally accessible from D regardless of whether m is finite or infinite.

The following abbreviated expression will be used throughout the paper: If a domain has one of the properties (i) — (vii) at each boundary point, it is said to have the property in question *on the boundary.*

Alternative characterizations. We will now describe alternative ways of defining some of the concepts (i) — (vii). For example, we show, with a view to the study of cluster sets in Section 2, how the topological concepts (i) — (iii) can be defined in terms of sequences of points.

1.10. **Theorem.** *Given a domain D and a boundary point b, the following statements are equivalent:*

(1) *D is m-connected at b.*

(2) *There exists a neighborhood U of b such that $U \cap D$ consists of m components each of which is locally connected at b.*

(3) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components each of which is locally connected at b.*

(4) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components, the boundary of each containing b.*

(5) *m is the least integer for which the following condition holds: If $(b_{1,k}), \ldots, (b_{m+1,k})$ are $m + 1$ sequences of points in D converging to b and if U is a neighborhood of b, then there exists a component of $U \cap D$ which contains subsequences of two different sequences.*

Proof. (1) $\Rightarrow$ (2): Let D be m-connected at b. By the definition, there exists a neighborhood U of b such that $U \cap D$ consists of m components, $E_1, \ldots, E_m$, and that $V \cap D$ has at least m components whenever

$V \subset U$ is a neighborhood of b. We claim that every E_i, $i = 1, \ldots, m$, is locally connected at b. Obviously, b is a boundary point of each domain E_i. If some E_i, say E_1, is not locally connected at b, there exists a neighborhood $U_1 \subset U$ of b with $W \cap E_1$ containing at least two components whenever $W \subset U_1$ is a neighborhood of b. But then $W \cap D$ has at least $m + 1$ components for all such neighborhoods W, which contradicts the m-connectedness property of D at b.

(2) $\Rightarrow$ (3): Let U and $E_1, \ldots, E_m$ be as above and let W be a neighborhood of b. Since E_i, $i = 1, \ldots, m$, is locally connected at b, for each i there is a neighborhood $U_i \subset W$ of b such that $U_i \cap E_i$ is connected. Then $U^* = (U_1 \cap \ldots \cap U_m) \cup (U_1 \cap E_1) \cup \ldots \cup (U_m \cap E_m)$ is a neighborhood of b, $U^* \subset W$, and $U^* \cap D$ consists of components $U_1 \cap E_1, \ldots, U_m \cap E_m$. Since $V \cap D$ has at least m components whenever $V \subset U^*$ is a neighborhood of b, we see as above that $U_i \cap E_i$, $i = 1, \ldots, m$, is locally connected at b.

(3) $\Rightarrow$ (4): This implication is trivial.

(4) $\Rightarrow$ (5): Let U be a neighborhood of b and let $V \subset U$ be a neighborhood of b such that $V \cap D$ consists of components $E_1, \ldots, E_m$ for which $b \in \partial E_i$, $i = 1, \ldots, m$. Next let $(b_{1,k}), \ldots, (b_{p+1,k})$ be $p + 1$ sequences of points in D converging to b. If $p \geq m$, then at least one E_i contains subsequences of two different sequences mentioned above. These subsequences are thus contained in a single component of $U \cap D$. Consequently, if m_0 is the smallest number for which the condition in (5) holds, then $m_0 \leq m$. To prove that $m_0 \geq m$ one only need choose for $i = 1, \ldots, m$ a sequence $(b_{i,k})$ in such a way that $b_{i,k} \in E_i$ for all k and $b_{i,k} \to b$ as $k \to \infty$.

(5) $\Rightarrow$ (1): Assume that (5) holds but (1) does not. Then, by what was proved above, D cannot be p-connected at b for any p, $1 \leq p \leq m$. Thus there exists a neighborhood U of b such that $V \cap D$ contains points of at least $m + 1$ different components of $U \cap D$ whenever V is a neighborhood of b. Assume, for convenience, that $b \neq \infty$. For $i = 1, \ldots, m + 1$ choose a point $b_{i,k} \in D \cap B^n(b, 1/k)$ so that if $i \neq j$, then $b_{i,k}$ and $b_{j,l}$, $k = 1, 2, \ldots$, $l = 1, 2, \ldots$, belong to different components of $U \cap D$. The sequences $(b_{1,k}), \ldots, (b_{m+1,k})$ converge to b, but the condition in (5) is not satisfied, contrary to the hypothesis. The proof is thus complete.

The next theorem offers an analogue of Theorem 1.10 in case of finite connectedness. Since its proof follows the same reasoning as that of Theorem 1.10, being even simpler, it may be omitted. (See also Väisälä [8, Theorem 17.7].)

1.11. **Theorem.** *Given a domain D and a boundary point b, the following statements are equivalent:*

(1) *D is finitely connected at b.*

(2) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of a finite number of components, the boundary of each containing b.*

(3) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of a finite number of components each of which is finitely connected at b.*

(4) *If (b_k) is a sequence of points in D converging to b and if U is a neighborhood of b, then at least one of the components of $U \cap D$ contains a subsequence of (b_k).*

The next theorem is a quasiconformal analogue of Theorem 1.10. Its simple proof also may be omitted.

1.12. **Theorem.** *Given a domain D and a boundary point b, the following statements are equivalent:*

(1) *D is quasiconformally m-collared at b.*

(2) *There exists a neighborhood U of b such that $U \cap D$ consists of m components each of which is quasiconformally collared at b.*

(3) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components each of which is quasiconformally collared at b.*

(4) *There exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components, $E_1, \ldots, E_m$, for each of which there is a quasiconformal mapping $g_i : E_i \to B_+^n$ with $\lim\limits_{x \to b} g_i(x) = 0$ and $\lim\limits_{y \to 0} g_i^{-1}(y) = b$.*

1.13. Remark. In the statement (4) of Theorem 1.12, if the requirement concerning the existence of the limits of g_i and g_i^{-1} is replaced by the mere requirement $b \in \partial E_i$, then g_i has a limit b_i' at b, which, observing that E_i is locally connected at b, can be proved essentially in the same way as in the case where the image domain is B^n. (See, for example, the proof of Väisälä [6, Theorem 1].) However, g_i^{-1} may fail to possess a limit at b_i'. This is seen, for example, by considering the domain $D = B_+^2(2) \setminus \bigcup\limits_{k=1}^{\infty} I_k$, where $I_k = \{x = (r, \varphi) : 0 \leq r \leq 1,\ \varphi = 1/k\}$, at the point $b = e_1$.

Among the concepts (i) — (vii) introduced above, the quasiconformal flatness and accessibility properties are not readily perceived, particularly the latter. We will next show that the choice of a continuum is of secondary consideration in the definition of quasiconformal accessibility (see 1.7). That is, given a domain D, a point $b \in \partial D$, and a continuum $A \subset D$, the point b either is or fails to be quasiconformally accessible from D, independently of A. (It is therefore justified to use the expression »b is quasiconformally accessible from D» without explicit reference to any continuum.) In order to prove this, we must establish a strengthened version (Lemma 1.15) of the following well-known result: If A and A^* are two disjoint connected compact sets in $\bar{R}^n$, then $M(\Delta(A, A^* : \bar{R}^n)) > 0$ provided that neither A nor A^* degenerates into a single point. For this purpose, an auxiliary lemma is needed.

1.14. **Lemma.** *Let a_0 and a_0' be two points in a domain D and let D_0 be a subdomain of D containing a_0 and a_0'. Then there exists a homeomorphism $f : \bar{D} \to \bar{D}$ such that f is quasiconformal in D, $f(a_0) = a_0'$, and $f(x) = x$ for $x \in \bar{D} \setminus D_0$.*

Proof. Without restriction it may be assumed that $a_0 \neq \infty \neq a_0'$. Let L_0 be a polygonal arc with successive vertices $a_0, a_1, \ldots, a_p, a_0'$ joining a_0 and a_0' in D_0. We will first construct a homeomorphism $f_0 : \bar{D} \to \bar{D}$ so that f_0 is quasiconformal in D, $f_0(a_0) = a_1$, and $f_0(x) = x$ for $x \in \bar{D} \setminus D_0$.

Let $0 < d_0 < d(L_0, \partial D_0)/n^{1/2}$. Performing a preliminary similarity transformation, we may assume that $a_0 = d_0 e_n$ and $a_1 = d_1 e_n$ with $0 < d_0 < d_1$. Denote

$$C = \{x : 0 \leq |x - x_n e_n| \leq d_0, \quad 0 \leq x_n \leq d_0 + d_1\},$$
$$C' = \{x : 0 \leq |x - x_n e_n| \leq d_0 - x_n, \quad 0 \leq x_n \leq d_0\},$$
$$C'' = \{x : 0 \leq |x - x_n e_n| \leq d_0(x_n - d_0)/d_1, \quad d_0 \leq x_n \leq d_0 + d_1\},$$

and set

$$f_0(x) = \begin{cases} x & \text{if } x \in \bar{D} \setminus C, \\ x + \dfrac{d_1 - d_0}{d_0}(d - |x - x_n e_n|) e_n & \text{if } x \in C \setminus (C' \cup C''), \\ x + \dfrac{d_1 - d_0}{d_0} x_n e_n & \text{if } x \in C', \\ x + \dfrac{d_1 - d_0}{d_1}(d_0 + d_1 - x_n) e_n & \text{if } x \in C''. \end{cases}$$

Then f_0 is a piecewise differentiable homeomorphism of $\bar{D}$ onto itself with $f_0 D = D$. A simple calculation shows that f_0 is $(d_1/d_0)^{n-1}$-quasiconformal in D. Moreover, $f_0(a_0) = a_1$ and $f_0(x) = x$ for $x \in \bar{D} \setminus D_0$, as desired.

Similarly, for $i = 1, \ldots, p$ there exists a homeomorphism $f_i : \bar{D} \to \bar{D}$ such that f_i is quasiconformal in D, $f_i(x) = x$ for $x \in \bar{D} \setminus D_0$, and $f_i(a_i) = a_{i+1}$ where $a_{p+1} = a_0'$. The mapping

$$f = f_p \circ \ldots \circ f_0$$

then satisfies the conditions of the lemma.

1.15. **Lemma.** *If A and A^* are two continua in a domain D, then $M(\Delta(A, A^* : D)) > 0$.*

Proof. We may assume that $A \cap A^* = \emptyset$ and that $D \neq \bar{R}^n$, for otherwise $M(\Delta(A, A^* : D)) = \infty$ or the assertion is a well-known result. Choose points $a_0 \in A$, $a^* \in A^*$, so that $|a_0 - a^*| = d(A, A^*)$, and set

$$2r = \min \{d(A, A^*), \ d(A^*, \partial D), \ \max_{x \in A^*} d(x, a^*)\} .$$

Next choose a point $a_0' \in D \setminus A^*$ with $d(a_0', a^*) = r$ and a subdomain D_0 of D with $a_0 \in D_0$, $a_0' \in D_0$, $A \cap \complement D_0 \neq \emptyset$, and $A^* \cap D_0 = \emptyset$. If f is a mapping of the preceding lemma, then $S^{n-1}(a^*, t)$ meets both fA and $fA^* = A^*$ for $r < t < 2r$. Since $B^n(a^*, 2r) \subset D$, we obtain, by Väisälä [8, Theorem 10.12],

$$M(\Delta(A, A^* : D)) \geq c_n \log 2/K(f) ,$$

where $c_n > 0$ is the n-modulus of the family of all paths joining e_n and $-e_n$ in S^{n-1}, and where $K(f) < \infty$ is the maximal dilatation of f in D. The lemma is thus proved.

We are now in a position to present a stronger form of the definition of quasiconformal accessibility.

1.16. **Theorem.** *A boundary point b of a domain D is quasiconformally accessible from D if and only if the following condition is satisfied: Given any neighborhood U of b and any continuum A^* in D, there exists a positive number δ^* such that $M(\Delta(A^*, F : D)) \geq \delta^*$ whenever F is a connected subset of D with $b \in \bar{F}$ and $F \cap \partial U \neq \emptyset$.*

Proof. It evidently suffices to prove the necessity part. Let A^* be an arbitrary continuum in D, let U be a neighborhood of b, and let A and δ be the quantities appearing in 1.7.(vii), the definition of quasiconformal accessibility, corresponding to the neighborhood U. We have to find a constant $\delta^* > 0$ for which the modulus condition in the present theorem is satisfied.

Assume first that $A \cap A^* = \emptyset$, and, for convenience, that $\infty \notin A$. Let

$$4r = \min \{d(A , A^*) , d(A , \partial D)\} ,$$

let $A_1 , \ldots , A_p$ be a finite covering of A by closed balls with centers $a_i \in A$, $i = 1 , \ldots , p$, and radii r, and let

$$M(\Gamma_i^*) = \delta_i , \tag{1.16.1}$$

where $\Gamma_i^* = \Delta(A_i , A^* : D)$. By Lemma 1.15, each $\delta_i > 0$. We claim that

$$\delta^* = 3^{-n} \min \{\delta/p , \delta_1 , \ldots , \delta_p , c_n \log 2\} , \tag{1.16.2}$$

where $c_n > 0$ is as defined in Väisälä [8, (10,11)], can be chosen for the desired positive constant.

Now let F be a connected set in D such that $b \in \bar{F}$ and $F \cap \partial U \neq \emptyset$. Set

$$\Gamma = \Delta(A , F : D) , \quad \Gamma_i = \Delta(A_i , F : D) , \quad \Gamma^* = \Delta(A^* , F : D) .$$

Since the modulus is a monotone and subadditive function,

$$\delta \leq M(\Gamma) \leq M(\Delta(\bigcup_{i=1}^{p} A_i , F : D)) \leq \sum_{i=1}^{p} M(\Gamma_i) .$$

Thus for some i, say $i = 1$,

$$M(\Gamma_1) \geq \delta/p . \tag{1.16.3}$$

We must show that $M(\Gamma^*) \geq \delta^*$. It is sufficient to consider the case where $A^* \cap F = \emptyset$, for any path family containing a constant path has infinite modulus.

Choose $\varrho \in F(\Gamma^*)$. If

$$\int_{\gamma_1} \varrho \, ds \geq \frac{1}{3} \quad \text{or} \quad \int_{\gamma_1^*} \varrho \, ds \geq \frac{1}{3} \tag{1.16.4}$$

for every rectifiable path $\gamma_1 \in \Gamma_1 , \gamma_1^* \in \Gamma_1^*$, then $3\varrho \in F(\Gamma_1)$ or $3\varrho \in F(\Gamma_1^*)$. This implies

$$\int_{R^n} \varrho^n \, dm \geq 3^{-n} \min \{M(\Gamma_1) , M(\Gamma_1^*)\} . \tag{1.16.5}$$

Assume now that there exist rectifiable paths $\gamma_1 \in \Gamma_1$ and $\gamma_1^* \in \Gamma_1^*$

for which (1.16.4) is not true. Suppose first that $F \cap B^n(a_1, 2r) = \emptyset$. Let R_1 be the ring $r < |x - a_1| < 2r$ and set $\Delta_1 = \Delta(|\gamma_1|, |\gamma_1^*| : R_1)$. Since $\varrho \in F(\Gamma^*)$,

$$\int_{\alpha_1} \varrho \, ds > \frac{1}{3} \tag{1.16.6}$$

for every rectifiable path $\alpha_1 \in \Delta_1$. Thus $3\varrho \in F(\Delta_1)$. Since every $S^{n-1}(a_1, t)$ meets both $|\gamma_1|$ and $|\gamma_1^*|$ for $r < t < 2r$ and since $B^n(a_1, 2r) \subset D$,

$$\int_{R^n} \varrho^n dm \geq 3^{-n} c_n \log 2 \tag{1.16.7}$$

according to Väisälä [8, Theorem 10.12]. Suppose finally that $F \cap B^n(a_1, 2r) \neq \emptyset$. Let R_1^* be the ring $2r < |x - a_1| < 4r$ and set $\Delta_1^* = \Delta(F, |\gamma_1^*| : R_1^*)$. Taking into account the fact that $b \in \bar{F}$, we see that also in this case (1.16.6) holds, with α_1^* in place of α_1, for every rectifiable path $\alpha_1^* \in \Delta_1^*$. Hence we conclude as above that (1.16.7) holds. Consequently, since $\varrho \in F(\Gamma^*)$ was arbitrary and since either (1.16.5) or (1.16.7) is true, we obtain $M(\Gamma^*) \geq \delta^*$ by combining (1.16.1) — (1.16.3), as desired.

In the preceding argument we assumed that $A \cap A^* = \emptyset$. If A meets A^*, we may choose a continuum $A_1^* \subset D \setminus (A \cup A^*)$ and apply the above procedure first to the continua A, A_1^*, and then to A_1^*, A^*. This completes the proof of the theorem.

Interrelations. We conclude this section by investigating relations between the concepts (i) — (vii). Some trivialities have been pointed out in earlier remarks. We now present some less trivial relationships.

1.17. **Theorem.** *Let D be a domain which is quasiconformally m-collared at a boundary point b. Then*
(1) *D is m-connected at b.*
(2) *D is quasiconformally flat at b if and only if $m = 1$.*
(3) *b is quasiconformally accessible from D.*

Proof. (1) follows from Theorems 1.10.(4) and 1.12.(4). The sufficiency part of (2) is proved like the corresponding assertion in Theorem 17.10 of Väisälä [8], while the necessity part of (2) can be deduced from (1) and from the next theorem. Thus it remains to establish (3).

For this, let U be a neighborhood of b. By Theorem 1.12.(4), we can find a neighborhood $V \subset U$ of b such that $V \cap D$ consists of m components, $E_1, \ldots, E_m$, for each of which there exists a quasiconformal mapping $g_i : E_i \to B_+^n$ with $\lim_{x\to b} g_i(x) = 0$, $\lim_{y\to 0} g_i^{-1}(y) = b$. By the latter limit condition, we may choose a number r, $0 < r < 1$, so that the distance between $g_i^{-1}B_+^n(r)$ and ∂V is positive for each i. Denote by I' the segment $x_1 = \ldots = x_{n-1} = 0$, $r/2 \leq x_n \leq r$, and by I the union of the sets $g_i^{-1}I'$. Next let $A \subset D$ be a continuum which contains I. We will show that the modulus condition in 1.7.(vii), the definition of quasiconformal accessibility, is satisfied by this A and by $\delta = b_n \log 2/K$, where b_n is a positive constant depending only on n and where $K = \max_{1\leq i\leq m} K(g_i)$.

Now let F be a connected set in D with $b \in \bar{F}$ and $F \cap \partial U \neq \emptyset$. Since $\lim_{x\to b} g_i(x) = 0$ for each i, the origin belongs to the closure of $g_i(E_i \cap F)$ for at least one i. Fix such i and set

$$\Gamma' = \Delta(g_i(E_i \cap A)\,,\ g_i(E_i \cap F) : B_+^n)\,.$$

If $g_i(E_i \cap A)$ intersects $g_i(E_i \cap F)$, then $M(\Gamma') = \infty$, and there is nothing to prove. Otherwise choose $\varrho \in F(\Gamma')$. From the choice of r and A and from the connectedness of F we infer that every hemisphere $S_+(t) = S^{n-1}(t) \cap B_+^n$ meets both $g_i(E_i \cap A)$ and $g_i(E_i \cap F)$ for $r/2 < t < r$. Since $\varrho|S_+(t)$, the restriction of ϱ to $S_+(t)$, belongs to the family $F(\Gamma'(t))$ where $\Gamma'(t) = \Delta(g_i(E_i \cap A),\ g_i(E_i \cap F) : S_+(t))$, we obtain

$$\int_{R^n} \varrho^n \, dm \geq \int_{r/2}^{r} dt \int_{S_+(t)} \varrho^n \, dm_{n-1} \geq b_n \log 2$$

by Fubini's theorem and by Väisälä [8, Theorem 10.2]. Hence $M(\Gamma') \geq b_n \log 2$. Finally, the monotoneity of the modulus and the K-quasiconformality of g_i imply

$$M(\Delta(A\,,F : D)) \geq M(g_i^{-1}\Gamma') \geq b_n \log 2/K\,,$$

thus completing the proof of (3).

1.18. **Theorem.** *Let D be a domain which is both quasiconformally flat and finitely connected at a boundary point b. Then D is locally connected at b.*

Proof. Suppose, contrary to the assertion, that D is not locally connected at b. Performing a preliminary inversion if necessary, we may assume that $b \neq \infty$. Since D is finitely connected at b, there exists, by Theorem 1.11.(3), a neighborhood U of b such that $U \cap D$ consists of components $E_1, \ldots, E_p$, $p \geq 2$, each of which is finitely connected at b. Let $d = d(b, \partial U)$. Again by Theorem 1.11, there is for $i = 1, 2$ a connected set $F_i \subset E_i \cap B^n(b, d/2)$ with $b \in \bar{F}_i$. Denote

$$\Gamma = \Delta(F_1, F_2 : D),$$
$$\Delta = \Delta(S^{n-1}(b, d/2), S^{n-1}(b, d) : R),$$

where R is the ring $d/2 < |x - b| < d$. Since the path family Γ is minorized by the family Δ, we obtain

$$M(\Gamma) \leq M(\Delta) = \omega_{n-1}(\log 2)^{1-n} < \infty .$$

But this contradicts the hypothesis that D is quasiconformally flat at b. The theorem is thus proved.

It is generally a laborous task to find out, by direct use of the definitions, whether a given domain D has the implicit properties of being quasiconformally flat and accessible at some boundary point b. In many circumstances it is, however, easy to verify that D is quasiconformally m-collared at b for some integer m (as an example, see 1.6), in which case the conclusions concerning quasiconformal flatness and accessibility follow from Theorem 1.17. As another example we next present a simple geometric condition which implies quasiconformal m-collaredness for $m = 1$ or $m = 2$.

1.19. **Theorem.** *Let $b \neq \infty$ be a boundary point of a domain D and let e be a unit vector. Suppose that b has a neighborhood U such that $S = U \cap \partial D$ is homeomorphic to B^{n-1}, and suppose that for each pair of points b_1, b_2 in S, the acute angle which the segment $b_1 b_2$ makes with e is never less than $\alpha > 0$. Then*
(1) *D is quasiconformally collared at b if $b \in \partial \bar{D}$.*
(2) *D is quasiconformally 2-collared at b if $b \notin \partial \bar{D}$.*

Proof. Let T be the $(n - 1)$-dimensional hyperplane through b which has e as its normal. By hypotheses, there exists a neighborhood $V \subset U$ of b such that every point $x \in V$ has a unique representation of the form $x = s + t_x e$, where $s \in S$ and $t_x \in R^1$. Let $p : S \to T$ be the orthogonal projection and for $x \in V$ set $g(x) = p(s) + t_x e$. By the n-dimensional analogue of Corollary 5.1 in Gehring — Väisälä [4], g is a quasiconformal mapping. Since $g(V \cap S) \subset T$, there exists a neighborhood $W \subset V$ of b such that g maps each of the two components of $W \setminus S$ onto a half-ball. If $b \in \partial \bar{D}$, one of these two components coincides with $W \cap D$, and D is thereby quasiconformally collared at b. Otherwise, the components of $W \setminus S$ are the same as those of $W \cap D$, in which case D is quasiconformally 2-collared at b. The proof is complete.

1.20. REMARKS AND COUNTEREXAMPLES. (1) In Theorem 1.17, quasiconformal collaredness is not a necessary condition for quasiconformal flatness or accessibility. The domain $D = \bar{R}^n \setminus \{b\}$ serves as a counterexample. More generally, by methods similar to those used in the proof of Theorem 1.17 one can show that *a domain D has the quasiconformal flatness and accessibility properties at a boundary point $b \neq \infty$ if there exists a positive number r such that $\Lambda_{n-2}(S^{n-1}(b, t) \cap \partial D) = 0$ $(S^{n-1}(b, t) \cap \partial D = \emptyset$ if $n = 2)$ for almost every $t \in (0, r)$.* Here Λ_{n-2} is the $(n-2)$-dimensional Hausdorff measure.

(2) In Theorem 1.18, quasiconformal flatness does not alone imply local connectedness. For example, the domain $D = B^2_+(2) \setminus \bigcup_{k=1}^{\infty} I_k$, where I_k is as defined in Remark 1.13, is quasiconformally flat, but not locally connected, at points te_1, $0 \leq t < 1$.

(3) In Theorem 1.19, the angle condition is superfluous when $n = 2$. For if $U \cap \partial D$ is an open Jordan arc, there is a neighborhood $V \subset U$ of b such that $V \setminus \partial D$ consists of two simply connected components, and therefore the requirements in the definition of quasiconformal collaredness are readily seen to be satisfied. On the contrary, if $n \geq 3$, then, as we shall see in Section 5, the other hypotheses of the theorem imply, in fact, neither quasiconformal flatness nor accessibility.

2. Cluster sets

In the present section we study the local boundary behavior of quasiconformal mappings. This will be done in terms of the properties possessed by the sets of all their limit points. We therefore introduce the following topological concept:

2.1. **Cluster set.** Let f be a mapping of a domain D into $\bar{R}^n$ and let b be a point in ∂D. The *cluster set* $C(f, b)$ of f at b is the set of all points $b' \in \bar{R}^n$ for which there exists a sequence (b_k) in D such that $b_k \to b$ and $f(b_k) \to b'$. Alternatively, $C(f, b) = \bigcap \overline{f(U \cap D)}$ where U runs through all neighborhoods of b. The cluster set $C(f, E)$ of f on a non-empty set $E \subset \partial D$ is defined as the union of the sets $C(f, b)$, $b \in E$.

2.2. REMARK. Obviously, $C(f, b)$ is a non-empty compact set, f has a limit at b if and only if $C(f, b)$ reduces to a single point, and $C(f, b) \subset \partial fD$ if f is a homeomorphism.

Cluster sets and topological properties. We begin by considering cluster sets at points where a given domain has one of the topological properties (i) — (iii), defined in 1.1.

2.3. **Theorem.** *Let $f: D \to D'$ be a quasiconformal mapping and let D be m-connected at a boundary point b. Then the following holds:*
(1) *$C(f, b)$ contains at most m components.*
(2) *If $C(f, b)$ contains exactly m components, $C_1, \ldots, C_m$, there exist arbitrarily small neighborhoods U of b with $U \cap D$ consisting of m components, $E_1, \ldots, E_m$, such that $C_i = C(f|E_i, b)$, $i = 1, \ldots, m$.*
(3) *In particular, if $C_i = \{b_i'\}$, then $b_i' = \lim_{x \to b} f|E_i(x)$.*
(4) *If D' is quasiconformally flat at every point of $C(f, b)$, then $C(f, b)$ contains at least m points.*
(5) *$C(f, b)$ either contains at most $m - 1$ points, quasiconformally accessible from D', or consists of m points.*
(6) *If D' has the quasiconformal flatness and accessibility properties at every point of $C(f, b)$, then $C(f, b)$ contains exactly m points.*
(7) *If U is a neighborhood of b with $U \cap D$ consisting of components $E_1, \ldots, E_m$ and if b' is a point, quasiconformally accessible from D', belonging to $C(f|E_i, b)$ for $i = 1, \ldots, m$, then $b' = \lim_{x \to b} f(x)$.*

Proof. (1) By the statement (3) of Theorem 1.10 and by the definition of $C(f, b)$, there exists for each positive integer k a neighborhood U_k of b such that $U_{k+1} \subset U_k$, $U_k \cap D$ consists of m components, say $E_{1,k}, \ldots, E_{m,k}$, $E_{i,k+1} \subset E_{i,k}$ for $i = 1, \ldots, m$, and

$$C(f, b) = \bigcup_{i=1}^{m} \bigcap_{k=1}^{\infty} \overline{fE_{i,k}} .$$

The assertion follows now from the fact that for each i, the set $\bigcap_{k=1}^{\infty} \overline{fE_{i,k}}$, as an intersection of a contracting sequence of continua, is either a continuum or a point.

(2) Again by Theorem 1.10.(3), there exist arbitrarily small neighborhoods U of b with $U \cap D$ consisting of m components, $E_1, \ldots, E_m$, each of which is locally connected at b. Since $C(f|E_i, b)$, $i = 1, \ldots, m$, is connected by virtue of (1), we conclude, possibly by relabeling, that $C_i = C(f|E_i, b)$.

(3) follows from (2).

(4) We may assume that $m \geq 2$, for otherwise there is nothing to prove. Composing f with an auxiliary inversion if necessary, we may further assume that $b \neq \infty$. Let U be a neighborhood of b with each of the components, $E_1, \ldots, E_m$, of $U \cap D$ being locally connected at b.

Set $d = d(b, \partial U)$. Again by Theorem 1.10.(3), there is for $i = 1, \ldots, m$ a connected set $F_i \subset E_i \cap B^n(b, d/2)$ with $b \in \bar{F}_i$. Denoting $\Gamma_{i,j} = \Delta(F_i, F_j : D)$, $1 \leq i < j \leq m$, and appealing to the argument given in the proof of Theorem 1.18, we obtain

$$M(\Gamma_{i,j}) \leq \omega_{n-1}(\log 2)^{1-n} < \infty .$$

If $C(f, b)$ contained at most $m - 1$ points, there would exist integers i and j, $1 \leq i < j \leq m$, and a point c in $C(f, b)$ such that $c \in \overline{fF_i} \cap \overline{fF_j}$. But since fF_i and fF_j are connected sets in D' and since D' is quasiconformally flat at c,

$$M(f\Gamma_{i,j}) = M(\Delta(fF_i, fF_j : D')) = \infty ,$$

which contradicts the quasiconformality of f and proves that $C(f, b)$ contains at least m points.

(5) Suppose, contrary to the assertion, that $C(f, b)$ contains $m + 1$ distinct points $b'_1, \ldots, b'_{m+1}$ and that $b'_1, \ldots, b'_m$ are quasiconformally accessible from D'. In order to avoid technical difficulties we assume, as we obviously may, that both b and each b'_j, $j = 1, \ldots, m + 1$, are finite points.

Let $2r = \min |b'_i - b'_j|$, $1 \leq i < j \leq m + 1$, and choose a continuum $A' \subset D'$. By Theorem 1.16, there exists for each $i = 1, \ldots, m$ a constant $\delta_i > 0$ such that

$$M(\Delta(A', F' : D')) \geq \delta_i$$

whenever F' is a connected set in D' with $b'_i \in \bar{F}'$ and $F' \cap S^{n-1}(b'_i, r) \neq \emptyset$. Let

$$\delta = \min_{1 \leq i \leq m} \delta_i , d = d(A, \partial D) ,$$

where $A = f^{-1}A'$. Then δ and d are both positive. For $j = 1, \ldots, m + 1$ choose a sequence $(b_{j,k})$ in D so that $b_{j,k} \to b$, $f(b_{j,k}) \to b'_j$. Fix ε, $0 < \varepsilon < d$. Since D is m-connected at b, there exists, by Theorem 1.10.(5), a component F of $B^n(b, \varepsilon) \cap D$ and integers i and j, $1 \leq i < j \leq m + 1$, such that F contains subsequences of $(b_{i,k})$ and $(b_{j,k})$. Set $F' = fF$, $\Gamma = \Delta(A, F : D)$, $\Gamma' = \Delta(A', F' : D')$. Since F' is connected and $\bar{F}'$ contains the points b'_i and b'_j,

$$M(\Gamma') \geq \delta .$$

On the other hand, the path family Γ is minorized by the family $\Delta(S^{n-1}(b, \varepsilon), S^{n-1}(b, d) : D)$. Consequently,

$$M(\Gamma) \leq \omega_{n-1}\left(\log \frac{d}{\varepsilon}\right)^{1-n} .$$

These inequalities for $M(\Gamma')$ and $M(\Gamma)$ must hold for all ε, $0 < \varepsilon < d$. Letting $\varepsilon \to 0$ leads to the desired contradiction.

(6) follows from (4) and (5).

(7) By Theorem 1.10, we may assume that each E_i is locally connected at b. The method used in the proof of (5) then shows that $b' = \lim_{x \to b} f | E_i(x)$. Thus $b' = \lim_{x \to b} f(x)$, and the proof is complete.

The next theorem, proposition (2) of which will be frequently referred to in the sequel, is an immediate consequence of propositions (1) and (5) in Theorem 2.3.

2.4. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be locally connected at a boundary point b. Then the following holds:*

(1) *$C(f, b)$ is either a continuum or a point.*

(2) *If there is a point b' in $C(f, b)$ which is quasiconformally accessible from D', then $b' = \lim_{x \to b} f(x)$.*

An argument similar to that employed in the proof of Theorem 2.3.(4) yields the following result:

2.5. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be finitely connected at a boundary point b without being m-connected for any integer m. If D' is quasiconformally flat at every point of $C(f, b)$, then $C(f, b)$ is infinite.*

2.6. Remark. Propositions (1) — (3) in Theorem 2.3 hold for every continuous mapping, as is evident from their proofs. The same is true of the first statement in Theorem 2.4.

Cluster sets and quasiconformal properties. We next consider cluster sets at points where a given domain has one of the quasiconformal properties (iv) — (vi), defined in 1.4 and 1.7.

2.7. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be quasiconformally m-collared at a boundary point b. Then, in addition to the statements of Theorem 2.3, the following holds:*

(1) *$C(f, b)$ contains at most m points at which D' is finitely connected.*

(2) *$C(f, b)$ contains exactly m points if D' is locally connected at every point of $C(f, b)$.*

(3) *If D' is finitely connected at every point of $C(f, b)$, if U is a neighbor hood of b with $U \cap D$ consisting of components $E_1, \ldots, E_m$, and*

if there is a point b' *belonging to* $C(f|E_i, b)$ *for* $i = 1, \ldots, m$, *then* $b' = \lim_{x \to b} f(x)$.

Proof. (1) Suppose, contrary to the assertion, that D' is finitely connected at $m+1$ distinct points $b'_1, \ldots, b'_{m+1}$ of $C(f, b)$. Composing f with an auxiliary inversion if necessary, we may assume that each $b'_j \neq \infty$, $j = 1, \ldots, m+1$. Let $2d = \min |b'_i - b'_j|$, $1 \leq i < j \leq m+1$, and choose for $j = 1, \ldots, m+1$ a sequence $(b_{j,k})$ in D so that $b_{j,k} \to b$, $f(b_{j,k}) \to b'_j$. Since D' is finitely connected at b'_j, there exists, by virtue of Theorem 1.11.(4), a component F'_j of $D' \cap B^n(b'_j, d/2)$ which contains a subsequence of $(f(b_{j,k}))$. Thus $b'_j \in \bar{F}'_j$, and it follows that

$$M(\Delta(F'_i, F'_j : D')) \leq \omega_{n-1}(\log 2)^{1-n} < \infty$$

whenever $1 \leq i < j \leq m+1$.

On the other hand, since D is quasiconformally m-collared at b, there exists, by Theorem 1.12.(2), a neighborhood U of b such that $U \cap D$ consists of m components, $E_1, \ldots, E_m$, each of which is quasiconformally collared at b. Next, since the set $F_j = f^{-1}F'_j$, $j = 1, \ldots, m+1$, is connected and $b \in \bar{F}_j$, and since each E_l, $l = 1, \ldots, m$, is quasiconformally flat at b according to Theorem 1.17.(2), there exist integers i, j, and l, $1 \leq i < j \leq m+1$, $1 \leq l \leq m$, for which

$$M(\Delta(F_i, F_j : D)) \geq M(\Delta(F_i \cap E_l, F_j \cap E_l : E_l)) = \infty .$$

This contradicts the quasiconformality of f.

(2) Local connectedness implies finite connectedness; hence, by (1), $C(f, b)$ contains at most m points. We thus claim that $C(f, b)$ contains at least m points. The case $m = 1$ is immediate, because $C(f, b) \neq \emptyset$. Suppose that $m \geq 2$ and that $C(f, b)$ contains at most $m - 1$ points. Let U and $E_1, \ldots, E_m$ be as above. By Theorem 1.17.(1), each E_i, $i = 1, \ldots, m$, is locally connected at b. Thus, by Theorem 2.4.(1), $C(f|E_i, b)$ reduces to a single point. Consequently, there exist integers i and j, $1 \leq i < j \leq m$, and a point $b' \in C(f, b)$ such that

$$\lim_{x \to b} f|E_i(x) = b' = \lim_{x \to b} f|E_j(x) .$$

On the other hand, D' is locally connected at b' and, by virtue of Theorem 1.17.(3), b is quasiconformally accessible from D. Hence, by Theorem 2.4.(2), $b = \lim_{y \to b'} f^{-1}(y)$. We may therefore choose a neighborhood V' of b' such that $V' \cap D'$ is connected and $f^{-1}(V' \cap D') \subset U$. But $f^{-1}(V' \cap D')$ is also connected and must thus be included in one of the components E_i and E_j. This is a contradiction.

(3) By Theorem 1.12, we may assume that each E_i is quasiconformally collared at b. The method used in the proof of (1) then shows that $b' = \lim_{x \to b} f|E_i(x)$. Thus $b' = \lim_{x \to b} f(x)$, and the proof is complete.

2.8. Remark. In the proof of proposition (1), it would have been sufficient to assume, instead of quasiconformal m-collaredness of D at b, that m represents the greatest integer for which the following condition is satisfied: There exist connected sets $F_1, \ldots, F_m$ in D with $b \in \bar{F}_1 \cap \ldots \cap \bar{F}_m$ such that $M(\Delta(F_i, F_j : D)) < \infty$ whenever $1 \leq i < j \leq m$. In the case $m = 1$ this yields the following result (Theorem 17.13 of Väisälä [8]):

2.9. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be quasiconformally flat at a boundary point b. Then $C(f, b)$ contains at most one point at which D' is finitely connected. In particular, if D' is finitely connected at every point of $C(f, b)$, then f has a limit at b.*

2.10. Remark. Observe the following difference between Theorems 2.4.(2) and 2.9: Let $f : D \to D'$ be a quasiconformal mapping and let $b \in \partial D$, $b' \in C(f, b)$. If D is locally connected at b and if b' is quasiconformally accessible from D', then $b' = \lim_{x \to b} f(x)$. On the other hand, if D is quasiconformally flat (or collared) at b and if D' is finitely connected (or locally connected) at b', then f need not necessarily possess a limit at b. This is seen, for example, by choosing B^n for D, the n-dimensional analogue of the domain described in Gehring — Väisälä [4, 10.7] for D', the origin for b', and $\lim_{y \to b'} f^{-1}(y)$ for b. Indeed, $C(f, b)$ consists of the segment $x_1 = \ldots = x_{n-1} = 0$, $0 \leq x_n \leq 1$.

3. Boundary extension

In the present section we study the global boundary behavior of quasiconformal mappings. This, however, is closely related to the local one. Accordingly, we shall begin by considering the possibility of extending quasiconformal mappings to one boundary point.

Väisälä proved in [6] that if a domain is locally connected on the boundary, then every quasiconformal mapping of it onto a ball can be extended to a homeomorphism between the closures. In the middle part of the section we discuss the same extension problem for domains more general than a ball. (The extension of a quasiconformal mapping to a continuous mapping between the closures of the domains in question will not be dealt with before Section 4, and there only in the special case where one of the domains is a ball.) This section will be concluded with an investigation of the extension of quasiconformal mappings over boundary surfaces.

Extension to one boundary point. Let $f: D \to D'$ be a quasiconformal mapping and b a point in ∂D. Theorems 2.3.(7), 2.4.(2), 2.7.(3), and 2.9 provide certain conditions under which f has a limit at b, and under which f therefore can be extended to a continuous mapping of $D \cup \{b\}$. We consider now aspects pertinent to our demand that this extended mapping be a homeomorphism.

3.1. **Theorem.** *Let b and b' be boundary points of domains D and D', respectively, and let $f: D \cup \{b\} \to D' \cup \{b'\}$ be a homeomorphism which is quasiconformal in D. If D has any one of the properties* (i) — (vii) *at b, then D' has the same property at b'.*

Proof. (i) Local connectedness: A special case of (ii).

(ii) m-connectedness: Let U' be a neighborhood of b'. By hypotheses, there exist a neighborhood V of b and a neighborhood V' of b' such that $V \cap D$ consists of m components, say $E_1, \ldots, E_m$, that $f(V \cap D) \subset U'$, that $V' \subset U'$, and that $f^{-1}(V' \cap D') \subset V$. Then $W' = V' \cup f(V \cap D)$ is a neighborhood of b', $W' \subset U'$, and $W' \cap D' = f(V \cap D)$ consists of m components, namely $fE_1, \ldots, fE_m$. Consequently, D' is m'-connected at b' for some m', $1 \leq m' \leq m$. Considering likewise the inverse mapping f^{-1}, we conclude $m \leq m'$.

(iii) Finite connectedness: The proof is similar to that of (ii).

(iv) Quasiconformal collaredness: A special case of (v).

(v) Quasiconformal m-collaredness: Let U be a neighborhood of b such that $U \cap D$ consists of m components, $E_1, \ldots, E_m$, for each of which there exists a quasiconformal mapping $g_i: E_i \to B^n_+$ with $\lim_{x \to b} g_i(x) = 0$, $\lim_{y \to 0} g_i^{-1}(y) = b$. Choose a neighborhood V' of b' with $f^{-1}(V' \cap D') \subset U$. For $i = 1, \ldots, m$ set $E_i' = fE_i$. Then $U' = V' \cup E_1' \cup \ldots \cup E_m'$ is a neighborhood of b' with $U' \cap D'$ consisting of components $E_1', \ldots, E_m'$. Set $f_i = f|E_i$ and $h_i = g_i \circ f_i^{-1}$. Since for each i, $h_i(E_i') = B^n_+$, $\lim_{x \to b'} h_i(x) = 0$, $\lim_{y \to 0} h_i^{-1}(y) = b'$, and since h_i as a composed mapping of two quasiconformal mappings is itself also quasiconformal, D' is quasiconformally m-collared at b'.

(vi) Quasiconformal flatness: Let F_1' and F_2' be two connected sets in D' with $b' \in \bar{F}_1' \cap \bar{F}_2'$. Set $F_1 = f^{-1}F_1'$, $F_2 = f^{-1}F_2'$, $\Gamma = \Delta(F_1, F_2 : D)$, $\Gamma' = \Delta(F_1', F_2' : D')$. By hypotheses, F_1 and F_2 are connected sets in D with $b \in \bar{F}_1 \cap \bar{F}_2$. Thus $M(\Gamma) = \infty$, which implies $M(\Gamma') = \infty$.

(vii) Quasiconformal accessibility: Let U' be a neighborhood of b', let U be a neighborhood of b with $f(U \cap D) \subset U'$, and let A and δ be as in 1.7.(vii), the definition of quasiconformal accessibility. We show that in the same definition, $A' = fA$ and $\delta/K(f)$ may be chosen for the quantities corresponding to the domain D', the point b', and the neighborhood U'. To this end, let F' be a connected set in D' such that $b' \in \bar{F}'$ and $F' \cap \partial U' \neq \emptyset$. By hypotheses, the set $F = f^{-1}F'$ is connected, $b \in \bar{F}$, and $F \cap \partial U \neq \emptyset$. Hence $M(\Delta(A, F : D)) \geq \delta$, which implies $M(\Delta(A', F' : D')) \geq \delta/K(f)$. The proof of the theorem is thus complete.

3.2. Remarks. (1) Theorem 3.1 shows that all of the properties (i) — (vii) are quasiconformal invariants. The first three of these are, in fact, topological invariants.

(2) Theorem 3.1 does not apply to every continuous mapping which is quasiconformal in D. To see this, let f be a quasiconformal mapping of $D = B^2$ onto $D' = \bar{R}^2 \setminus \overline{\bigcup_{k=1}^{\infty} I_k}$ where I_k is as defined in Remark 1.13, and let $b' = e_1/2$. It is not difficult to find a point $b \in \partial D$ for which $\lim_{x \to b} f(x) = b'$. The domain D has all of the properties (i) — (vii) at b, while D' has none of these at b'.

(3) In Theorem 3.1, if f is continuous, and quasiconformal in D, and if D and D' possess any one of the properties (i) — (vii), except (iv) or (v), at b and b', respectively, then f need not necessarily be a homeomorphism, as is readily seen by means of examples. However, if f preserves either (iv) or (v), then it is a homeomorphism, as the proof of the next theorem will show.

3.3. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be quasiconformally m-collared at a boundary point b. Suppose that U is a neighborhood of b, appearing in the definition of quasiconformal m-colláredness (see 1.4), $E_1, \ldots, E_m$ being the components of $U \cap D$. Then f can be extended to a homeomorphism of $D \cup \{b\}$ if and only if D' is quasiconformally m-collared at some point belonging to $C(f|E_i, b)$ for $i = 1, \ldots, m$.*

Proof. The necessity of the condition was established in Theorem 3.1. For the sufficiency, let b' be the point of $\partial D'$ at which D' is quasiconformally m-collared and which belongs to $C(f|E_i, b)$ for $i = 1, \ldots, m$. Since D is m-connected at b and since b' is quasiconformally accessible from D' (Theorem 1.17), f can be extended to a continuous mapping of

$D \cup \{b\}$ onto $D' \cup \{b'\}$ according to Theorem 2.3.(7). In order to prove that this extended mapping is a homeomorphism, we must show that $b = \lim_{y \to b'} f^{-1}(y)$.

If $m = 1$, the assertion follows directly from Theorem 2.4.(2), because, again by Theorem 1.17, D' is locally connected at b' and b is quasiconformally accessible from D. Assume next that $m \geq 2$, and, for convenience of notation, that $b \neq \infty$. Since $b' = \lim_{x \to b} f(x)$, we can use Theorem 1.12.(3) to find a neighborhood U of b and a neighborhood U' of b' such that $f(U \cap D) \subset U'$, $U \cap D$ consists of components $E_1, \ldots, E_m$, $U' \cap D'$ consists of components $G'_1, \ldots, G'_m$, each E_i is quasiconformally collared at b, and each G'_i is quasiconformally collared at b', $i = 1, \ldots, m$. Let $d = d(b, \partial U)$, and choose for each i a connected set $F_i \subset E_i \cap B^n(b, d/2)$ with $b \in \bar{F}_i$. Appealing to the argument given in the proof of Theorem 2.3.(4), we infer that $M(\Gamma_{i,j}) < \infty$ whenever $1 \leq i < j \leq m$. Here $\Gamma_{i,j} = \Delta(F_i, F_j : D)$. Accordingly, since $b' \in \overline{fF_i}$ for $i = 1, \ldots, m$ and since each G'_i is quasiconformally flat at b' (Theorem 1.17.(2)), the sets fF_i and fF_j must belong to different components of $U' \cap D'$ whenever $i \neq j$. Thus $b \in C(f^{-1}|G'_i, b')$ for each i. Theorems 1.17 and 2.3.(7) now imply that $b = \lim_{y \to b'} f^{-1}(y)$, as desired.

As an immediate corollary we obtain

3.4. **Theorem.** *Let $f: D \to D'$ be a quasiconformal mapping and let D be quasiconformally collared at a boundary point b. Then f can be extended to a homeomorphism of $D \cup \{b\}$ if and only if D' is quasiconformally collared at some point of $C(f, b)$.*

Extension to the whole boundary. We now proceed to investigate the *homeomorphic* extension of a quasiconformal mapping to the entire boundary. Before establishing our main theorem in this subsection, namely a generalization of Theorems 1 and 2 in Väisälä [6], we state without proof a simple topological lemma which describes the relation between the local and the global boundary extension. With a view to the discussion in Section 4, where not only the homeomorphic but also the continuous extension of a quasiconformal mapping to the whole boundary will be dealt with, we formulate the lemma as follows:

3.5. **Lemma.** *Let $f: D \to D'$ be a homeomorphism. Then*

(1) *f can be extended to a continuous mapping $\bar{f}: \bar{D} \to \bar{D}'$ if and only if $\lim_{x \to b} f(x)$ exists for every $b \in \partial D$.*

(2) *f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$, if and only if $\lim_{x \to b} f(x)$ and $\lim_{y \to b'} f^{-1}(y)$ exist for every $b \in \partial D$, $b' \in \partial D'$.*

3.6. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let D be locally connected on the boundary.*

(1) *If D is quasiconformally collared on the boundary, then f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$ if and only if D' also is quasiconformally collared on the boundary.*

(2) *If D is quasiconformally flat on the boundary, then f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$ if and only if D' also is locally connected and quasiconformally flat on the boundary.*

(3) *If D has the quasiconformal accessibility property on the boundary, then f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$ if and only if D' also is locally connected and has the quasiconformal accessibility property on the boundary.*

Proof. (1) is an immediate corollary of Theorem 3.4 and Lemma 3.5.(2). Propositions (2) and (3) follow from Theorems 2.4.(2), 2.9, 3.1, and Lemma 3.5.(2).

As an application we show that domains of certain types are not quasiconformally equivalent.

3.7. **Theorem.** *Let D be a domain which is locally connected on the boundary.*

(1) *If D is not quasiconformally collared at every point of ∂D, then it cannot be mapped quasiconformally onto any domain which is quasiconformally collared on the boundary.*

(2) *If D does not have the quasiconformal flatness and accessibility properties at every point of ∂D, then it cannot be mapped quasiconformally onto any domain which has both of these properties on the boundary.*

Proof. (1) If $f : D \to D'$ is a quasiconformal mapping and if D' is quasiconformally collared on the boundary, we may use Theorems 1.17, 2.4.(2), and 2.9, in conjunction with Lemma 3.5.(2), to infer that f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{D}'$. Thus, by Theorem 3.6.(1), D is quasiconformally collared on the boundary.

(2) The assertion follows from Theorems 2.4.(2), 2.9, 3.6.(2), 3.6.(3), and Lemma 3.5.(2).

3.8. Remarks. (1) Lemma 3.5 as well as Theorems 3.6.(1) and 3.6.(3) remain valid if one insists that the hypotheses concerning boundary points be satisfied at every point of any sets $E \subset \partial D$, $C(f, E) \subset \partial D'$. The same is not true of Theorem 3.6.(2) (see the example in 2.10).

(2) It is not known to the writer if Theorem 3.6 holds without the additional assumptions, that is, if a quasiconformal mapping between two domains which are locally connected on the boundaries is always extendable to a homeomorphism between the closures. The same question remains open also in case of Jordan domains. For plane Jordan domains, however, as is very well known, the extension is possible.

Extension over boundary surfaces. We conclude this section by investigating the possibility of extending quasiconformal mappings over quasiconformal spheres. A set S is said to be a *quasiconformal sphere* if there exists a quasiconformal mapping f of a domain $D \supset S$ with $fS = S^{n-1}$. From a result due to Gehring [2] it follows that D can actually be chosen to be the entire space $\bar{R}^n$. We generalize Theorem 3 of Väisälä [7] to n dimensions. The proof we will give also applies to the case $n = 2$ and is slightly different from that of Väisälä.

3.9. **Theorem.** *Let $f: D \to D'$ be a quasiconformal mapping and let ∂D consist of a quasiconformal sphere S and of a compact set (possibly $\emptyset$) not meeting S. Denote by D_1 the component of $\complement S$ for which $D \cap D_1 = \emptyset$ and by S' the cluster set $C(f, S)$. Then f can be extended to a quasiconformal mapping of $D \cup S \cup D_1$ if and only if S' is a quasiconformal sphere.*

Proof. The necessity of the condition is trivial. For the sufficiency, assume that S' is a quasiconformal sphere. We will first show that there is a neighborhood U of S, a neighborhood U' of S', and a quasiconformal mapping $h: U \to U'$ such that $h(x) = f(x)$ for $x \in U \cap D$. Then we apply Theorem 2 of Gehring [2].

Since D and D' are quasiconformally collared at all points of S and S', respectively, f can be extended, by Remark 3.8.(1), to a homeomorphism $f^*: D \cup S \to D' \cup S'$. For $0 < r_1 < r_2 < \infty$ let $R(r_1, r_2)$ denote the spherical ring $B^n(r_2) \setminus \bar{B}^n(r_1)$. Then, by virtue of hypotheses, there is a neighborhood V' of S' and a quasiconformal mapping $g': V' \to R(1/2, 2)$ such that $g'(V' \cap D') = R(1, 2)$. Similarly, there is a neighborhood U of S and a quasiconformal mapping $g: U \to R(1/2, 2)$ such that $g(U \cap D) = R(1, 2)$ and $f(U \cap D) \subset V'$. For $x \in S^{n-1} \cup R(1, 2)$ set $f_1(x) = g' \circ f^* \circ g^{-1}(x)$. Then f_1 is a homeomorphism of $S^{n-1} \cup R(1, 2)$ onto $g' \circ f^*(U \cap \bar{D})$, and it is quasiconformal in $R(1, 2)$. We can therefore extend f_1 by reflection to obtain a quasiconformal mapping f_2 of $R(1/2, 2)$ onto a domain $G \subset R(1/2, 2)$. Set $U' = g'^{-1}G$ and $h = g'^{-1} \circ f_2 \circ g$. Then U' is a neighborhood of S', and $h: U \to U'$ is a quasiconformal mapping agreeing with f in $U \cap D$.

Now let D'_1 be the component of $\complement S'$ which does not meet D'. Applying Theorem 2 of Gehring [2] to the domains D_1 and D'_1, to the neighborhoods U and U', and to the mappings $g|U \cap D_1$ and $g\,|U' \cap D'_1$, respectively, we find neighborhoods U_1 of $S = \partial D_1$, U'_1 of $S' = \partial D_1$, and quasiconformal mappings $h_1 : D_1 \to B^n$, $h'_1 : D'_1 \to B^n$, such that $h_1(x) = g(x)$ for $x \in U_1 \cap D_1$ and $h'_1(x) = g'(x)$ for $x \in U'_1 \cap D'_1$. The same theorem applied to the domain B^n, to the neighborhood $g(U \cap U_1)$, and to the mapping $h'_1 \circ h \circ h_1^{-1}|g(U \cap U_1) \cap B^n$ yields a neighborhood W of S^{n-1} and a quasiconformal mapping $h_2 : B^n \to B^n$ such that $h_2(x) = h'_1 \circ h \circ h_1^{-1}(x)$ for $x \in W \cap B^n$. Set

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in D\,, \\ f^*(x) & \text{if } x \in S\,, \\ h_1'^{-1} \circ h_2 \circ h_1(x) & \text{if } x \in D_1\,. \end{cases}$$

Then $\hat{f}$ is the desired extension of f.

By repetition of the above argument we obtain the following result:

3.10. **Theorem.** *Let $f : D \to D'$ be a quasiconformal mapping and let ∂D consist of a finite number of disjoint quasiconformal spheres. Then f can be extended to a quasiconformal mapping $\hat{f} : \bar{R}^n \to \bar{R}^n$ if and only if also the components of $\partial D'$ are quasiconformal spheres.*

4. Quasiconformal mappings of a ball

In order to apply and illustrate the results of Sections 2 and 3, many of which are not always apparent because of their fairly general nature, in this section we restrict our discussion to quasiconformal mappings between two domains one of which is a ball. We begin by considering cluster sets of a quasiconformal mapping $f : D \to B^n$ and the correspondence of the boundaries induced by f. We postulate, for example, a condition which describes the behavior of D at a point $b \in \partial D$ and which is both necessary and sufficient that the cluster set of f at b contain exactly m points ($m = 1\,, 2\,, \ldots$). We also characterize those domains D for which either f or f^{-1} (or both) admits an extension to a continuous mapping between the closures.

In the latter part of the section we discuss the possibility of mapping a domain quasiconformally onto a ball. As our main result there, we show, eliminating a superfluous condition in a result due to Gehring [2], that a Jordan domain D in $\bar{R}^3$ can be mapped quasiconformally onto B^3 if and only if every point in ∂D has a neighborhood U such that $U \cap D$ can be mapped quasiconformally onto B^3.

Cluster sets and boundary extension. In the following two theorems we summarize a number of results related to those presented in Sections 2 and 3. (See also Gehring [3, Theorem 1] and Väisälä [6].)

4.1. **Theorem.** *Let $f : D \to B^n$ be a quasiconformal mapping and let $b \in \partial D$. Then*

(1) *$C(f, b)$ reduces to a single point, i.e., f has a limit at b and therefore can be extended to a continuous mapping of $D \cup \{b\}$, if and only if D is quasiconformally flat at b.*

(2) *$C(f, b)$ reduces to a single point with b the cluster set of f^{-1} at that point, i.e., f can be extended to a homeomorphism of $D \cup \{b\}$, if and only if D has the quasiconformal flatness and accessibility properties at b.*

(3) *$C(f, b)$ contains exactly m points, $m = 2, 3, \ldots$, if and only if m is the greatest integer for which the following condition holds: There exist connected sets $F_1, \ldots, F_m$ in D with $b \in \bar{F}_1 \cap \ldots \cap \bar{F}_m$ such that $M(\Delta(F_i, F_j : D)) < \infty$ whenever $1 \leq i < j \leq m$.*

(4) *$C(f, b)$ is infinite if and only if the condition in* (3) *holds for every positive integer m.*

(5) *In particular, if D is m-connected at b, $m = 1, 2, \ldots$, then $C(f, b)$ contains exactly m points.*

(6) *In particular, if D is finitely connected at b without being m-connected for any integer m, then $C(f, b)$ is infinite.*

Proof. Because (5) and (6) can be deduced from Theorems 1.17, 2.3.(6), and 2.5, it is unnecessary to show that they are special cases of (1) — (4). The necessity part of (2) follows, for example, from Theorems 1.17 and 3.4, whereas its sufficiency part is due to proposition (1) and Theorem 2.4.(2). Since (1) is equivalent to (3) with $m = 1$ and since (3), combined with (1), implies (4), it remains to verify (3) for every positive integer m.

Fix such m. To prove the sufficiency part, note that $C(f, b)$ contains at most m points by virtue of Theorem 2.7.(1) and Remark 2.8. We thus claim that $C(f, b)$ contains at least m points. The argument parallels the one given in the proof of Theorem 2.3.(4). Since $C(f, b) \neq \emptyset$, it is sufficient to consider the case $m \geq 2$. If $C(f, b)$ contained at most $m - 1$ points, there would exist connected sets F_i, F_j in D and a point c in $C(f, b)$ such that $b \in \bar{F}_i \cap \bar{F}_j$, $M(\Delta(F_i, F_j : D)) < \infty$, and $c \in \overline{fF_i} \cap \overline{fF_j}$. But this is impossible, because B^n is quasiconformally flat at c.

Finally, to prove the necessity part, suppose that $C(f, b)$ consists of m distinct points $b'_1, \ldots, b'_m$. Let $F_1, \ldots, F_p$ be connected sets in D with $b \in \bar{F}_1 \cap \ldots \cap \bar{F}_p$ such that $M(\Delta(F_i, F_j : D)) < \infty$ whenever $1 \leq i < j \leq p$. If $p > m$, we conclude as above that $M(\Delta(fF_i, fF_j : B^n)) = \infty$ for some $i \neq j$. Consequently, the greatest number for which the condition in (3) holds cannot exceed m. On the other hand, in view of Theorem 2.7.(1) and Remark 2.8, it can neither be less than m. The theorem is proved.

4.2. **Theorem.** *Let $f : D \to B^n$ be a quasiconformal mapping. Then*
(1) *f can be extended to a continuous mapping $\bar{f} : \bar{D} \to \bar{B}^n$ if and only if D is quasiconformally flat on the boundary.*
(2) *f^{-1} can be extended to a continuous mapping $\bar{f}^{-1} : \bar{B}^n \to \bar{D}$ if and only if every point in ∂D is quasiconformally accessible from D.*
(3) *f^{-1} can be extended to a continuous mapping $\bar{f}^{-1} : \bar{B}^n \to \bar{D}$ if and only if D is finitely connected on the boundary.*
(4) *f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{B}^n$ if and only if D is locally connected on the boundary.*
(5) *f can be extended to a homeomorphism $f^* : \bar{D} \to \bar{B}^n$ if and only if D is a Jordan domain which is quasiconformally collared on the boundary.*
(6) *f can be extended to a quasiconformal mapping $\hat{f} : \bar{R}^n \to \bar{R}^n$ if and only if ∂D is a quasiconformal sphere.*

Proof. Propositions (1), (5), (6), and the sufficiency parts of (2) and (3) follow from Theorems 4.1.(1), 3.6.(1), 3.10, 2.4.(2), and 2.9, (plus Theorem 1.17 and Lemma 3.5), respectively. A direct proof for (4) is given in Väisälä [6]. The result can also be deduced from Theorems 3.6.(1) and 3.7.(1) in this paper. Thus it remains to establish the necessity parts of (2) and (3).

In order to do this for (2), let $\bar{f}^{-1} : \bar{B}^n \to \bar{D}$ be continuous, let $b \in \partial D$, let U be a neighborhood of b, and let $A = f^{-1}\bar{B}^n(1/2)$. We must find a positive number δ such that $M(\Delta(A, F : D)) \geq \delta$ whenever F is a connected set in D with $b \in \bar{F}$ and $F \cap \partial U \neq \emptyset$. For this, let V' be a bounded neighborhood of the compact set $C(f, b)$ with $f^{-1}(V' \cap B^n) \subset U$. Because of the continuity of $\bar{f}^{-1}$, such a neighborhood can be chosen. Set $r = d(C(f, b), \partial V')$. Since each point of ∂B^n is quasiconformally accessible, we infer from Theorem 1.16, on the basis of symmetry, that there exists a positive number δ' such that $M(\Delta(fA, fF : B^n)) \geq \delta'$ whenever $fF \subset B^n$ is a connected set whose closure contains a point $b' \in S^{n-1}$ for which $fF \cap S^{n-1}(b', r) \neq \emptyset$. Consequently, $\delta'/K(f)$ serves as the desired number δ.

Finally, in order to prove the necessity part of (3), let $\bar{f}^{-1}$, b, and U be as above, and let (b_k) be a sequence of points in D converging to b. In view of Theorem 1.11.(4), it is sufficient to find a component of $U \cap D$ which contains a subsequence of (b_k). To this end, choose a converging subsequence (b'_j) of $(f(b_k))$ and let $b' = \lim_{j \to \infty} b'_j$. By hypothesis, $b = \lim_{y \to b'} f^{-1}(y)$. Thus there exists a number $r > 0$ such that $f^{-1}(B^n \cap B^n(b', r)) \subset U$. But since $f^{-1}(B^n \cap B^n(b', r))$ is connected, it must be included in a single component of $U \cap D$. Consequently, this component contains a subsequence of (b_k). The proof is complete.

4.3. EXAMPLE. The domain described in Gehring — Väisälä [4, 10.7] is quasiconformally flat, but not locally connected, on the boundary. Any quasiconformal mapping of it onto a ball can therefore be extended to a continuous mapping, but not to a homeomorphism, between the closures.

Mapping theorems. The Riemann mapping theorem states that a plane domain D can be mapped quasiconformally onto B^2 if and only if ∂D is a connected set which contains at least two points. This geometric condition is necessary but not sufficient for a domain $D \subset \bar{R}^n$, $n \geq 3$, to be quasiconformally equivalent to B^n. Furthermore, in contrast to the case $n = 2$, in higher dimensions one cannot generally conclude whether or not a domain D is quasiconformally equivalent to B^n by looking only at ∂D. In [2] Gehring showed, however, that to draw such a conclusion one need only look at the part of D near ∂D. That is, a domain $D \subset \bar{R}^n$ can be mapped quasiconformally onto B^n if and only if ∂D has a neighborhood U such that $U \cap D$ can be mapped quasiconformally into B^n with ∂D corresponding to ∂B^n. This characterization is of global nature. It is therefore natural to seek for a local characterization. In the rest of this section we investigate local conditions for those domains that possess certain connectedness properties on the boundaries so as to be quasiconformally equivalent to B^n. We begin with a necessary condition, and then show in one special case that this condition will also be sufficient.

4.4. **Theorem.** *Let D be a domain such that for each point $b \in \partial D$, D is $m(b)$-connected at b for some positive integer $m(b)$. If D is quasiconformally equivalent to B^n, then D is quasiconformally $m(b)$-collared at b.*

Proof. Let $f : D \to B^n$ be a quasiconformal mapping and b a point in ∂D. By hypothesis, D is m-connected at b for some integer m. We must find a neighborhood U of b as in 1.4.(v), the definition of quasiconformal m-collaredness.

By Theorem 4.1.(5), $C(f, b)$ contains exactly m points, $b'_1, \ldots, b'_m$. For $i = 1, \ldots, m$ set $E'_i = B^n \cap B^n(b'_i, r)$, where $r > 0$ is so small that $E'_j \cap E'_k = \emptyset$ if $j \neq k$. Next, for each i let $h_i : E'_i \to B^n_+$ be a quasiconformal mapping with $\lim_{y \to b_i'} h_i(y) = 0$, $\lim_{z \to 0} h_i^{-1}(z) = b_i'$. Finally, set $E_i = f^{-1}E_i'$ and choose a neighborhood V of b so that $f(V \cap D) \subset \bigcup_{i=1}^{m} E_i'$. Then $U = V \cup E_1 \cup \ldots \cup E_m$ is a neighborhood of b, $U \cap D$ consists of components $E_1, \ldots, E_m$, and $g_i = h_i \circ f|E_i$ is a quasiconformal mapping of E_i onto B^n_+. Since $b_i' = \lim_{x \to b} f|E_i(x)$ and $b = \lim_{y \to b_i'} f^{-1}(y)$ by virtue of Theorems 2.3.(3) and 2.9, respectively, it follows that $\lim_{x \to b} g_i(x) = 0$ and $\lim_{z \to 0} g_i^{-1}(z) = b$. The proof is thus complete.

In the case where $m(b) = 1$ for each point $b \in \partial D$, Theorem 4.4 (or Theorems 4.2.(4) and 4.2.(5)) provides a slightly strengthened version of Theorem 3 in Väisälä [6]:

4.5. **Theorem.** *Let D be a domain which is locally connected on the boundary and quasiconformally equivalent to B^n. Then D is a Jordan domain which is quasiconformally collared on the boundary.*

4.6. Remark. Gehring showed in [2, Theorem 4] that a Jordan domain $D \subset \bar{R}^3$ is quasiconformally equivalent to B^3 if it is quasiconformally collared on the boundary in the following stronger sense: Each point $b \in \partial D$ has a neighborhood U such that there exists a quasiconformal mapping $g : U \cap D \to B^3_+$ which can be extended to a homeomorphism $g^* : U \cap \bar{D} \to B^3_+ \cup B^2$. From an aesthetical point of view, and also for the purpose of obtaining analogous results for domains more general than Jordan domains, it would seem desirable to remove the requirement on the extendability of g in the above condition. In the next theorem we show that this really can be done. In order to make clear that in 3-space the necessary conditions given in Theorem 4.5 are also sufficient, we formulate the result as follows:

4.7. **Theorem.** *Let $D \subset \bar{R}^3$ be a domain which is locally connected on the boundary. Then D can be mapped quasiconformally onto B^3 if and only if the following conditions are satisfied:*

(1) *D is a Jordan domain.*

(2) *Every point in ∂D has a neighborhood U such that $U \cap D$ can be mapped quasiconformally onto B^3.*

Proof. The necessity part follows from Theorem 4.5. In order to prove the sufficiency part, we only need show that D is quasiconformally collared on the boundary in the sense stated in Remark 4.6.

For this purpose, fix a point $b \in \partial D$. By (2), b has a neighborhood

U such that there exists a quasiconformal mapping $g : U \cap D \to B^3_+$. Since each point of ∂B^3_+ is quasiconformally accessible from B^3_+ (Theorem 1.17.(3)) and since $U \cap D$ is locally connected at b, $\lim_{x \to b} g(x)$ exists according to Theorem 2.4.(2). We may assume, without restriction, that $\lim_{x \to b} g(x) = 0$. Next, since B^3_+ is quasiconformally flat at the origin (Theorem 1.17.(2)), since $C(g^{-1}, 0)$ is connected (Theorem 2.4.(1)), and since $U \cap D$ is locally connected, a fortiori, finitely connected at each point of $U \cap \partial D$, Theorem 2.9 implies that $b = \lim_{y \to 0} g^{-1}(y)$. Choose r, $0 < r < 1$, so that $C(g^{-1}, B^2(r)) \subset U \cap \partial D$. Arguing as above we conclude that for every point $c' \in B^2(r)$, $\lim_{y \to c'} g^{-1}(y) = c$ exists and that $c' = \lim_{x \to c} g(x)$. Set

$$V = g^{-1}B^3_+(r) \cup C(g^{-1}, B^2(r)) \cup \complement \bar{D} .$$

Then V is a neighborhood of b and $g | V \cap D$ can be extended to a homeomorphism $(g | V \cap D)^* : V \cap \bar{D} \to B^3_+(r) \cup B^2(r)$. Composing $(g | V \cap D)^*$ with the mapping h, defined by $h(y) = y/r$, we obtain a homeomorphism of $V \cap \bar{D}$ onto $B^3_+ \cup B^2$ which is quasiconformal in $V \cap D$. The proof of the theorem is thus complete.

Gehring showed in [1] that a quasiconformal mapping $f : D \to B^3$ can be extended to a quasiconformal mapping $\hat{f} : \bar{R}^3 \to \bar{R}^3$ if and only if D is a Jordan domain whose exterior is quasiconformally equivalent to B^3. Together with the previous theorem this yields:

4.8. **Theorem.** *A domain $D \subset \bar{R}^3$ can be mapped onto B^3 by means of a quasiconformal mapping of $\bar{R}^3$ if and only if D is a Jordan domain and every point in ∂D has a neighborhood U such that $U \cap D$ and $U \cap \complement \bar{D}$ can be mapped quasiconformally onto B^3.*

4.9. REMARKS. (1) It is not known whether the n-dimensional analogue of Theorem 4.7 is true for $n > 3$, but it is well known that the analogue holds in the case $n = 2$. (Indeed, the condition (2) can be disregarded in the plane.)

(2) It is not known whether the n-dimensional analogue of Theorem 4.8 is true for $n > 3$, but it is well known that the analogue is false in the case $n = 2$.

4.10. EXAMPLE. Let $D \subset \bar{R}^3$ be a Jordan domain for which the simple angle condition of Theorem 1.19 is satisfied at every boundary point. Then D can be mapped quasiconformally onto B^3. Moreover, each such quasiconformal mapping can be extended to a quasiconformal mapping of the whole space.

5. Quasiconformal ridges

The results of Sections 3 and 4 indicate that the smoothness (together with the similarity) of the boundaries usually determines whether two given domains can be mapped quasiconformally onto each other, and, if such a mapping exists, whether it can be extended to or over the boundaries. Theorem 1.19 offers a simple geometric condition which (combined with a topological condition) implies that the boundary of a domain D is »quasiconformally smooth» in a neighborhood U of a point of $\partial \bar{D}$; that is, D (as well as a component of $\complement \bar{D}$) has the quasiconformal collaredness, flatness, and accessibility properties at all points of $\partial D \cap U$. This poses a question concerning the properties of D at the boundary points for which this geometric condition is not satisfied. In Remark 1.20.(3) we observed that the condition is immaterial for plane domains, and claimed, without any conclusive proof, that such is not the case in higher dimensions. We shall now state arguments to support this assertion, through a detailed examination of the domains whose boundaries contain sharp edges. For the sake of clarity, we will restrict ourselves to 3-space $\bar{R}^3$, although the ensuing results can be carried over to n-space $\bar{R}^n$, $n > 3$, without difficulty.

5.1. **Quasiconformal ridges.** Generalizing Gehring — Väisälä [4, 10.9] we say that a point set S in $\bar{R}^3$ is a *quasiconformal g-ridge* if there exists a quasiconformal mapping f_0 of $\bar{R}^3$ which sends S onto

$$S_0 = \{x : |x_2| = g(x_1) \, , \, 0 \leq x_1 < a \, , \, |x_3| < b\} \, ,$$

where $a < \infty$, $b \leq \infty$, and the function g is subject to the following restrictions:

$$(5.1.1) \quad \begin{cases} 1^\circ.\ g \text{ is continuous in } [0\,,a]\,, g(0) = 0\,, \text{ and } g(u) > 0 \text{ for } \\ \quad\ 0 < u \leq a\,. \\ 2^\circ.\ g' \text{ is continuous and non-decreasing in } (0\,,a)\,. \\ 3^\circ.\ \lim\limits_{u \to 0} g'(u) = 0\,. \end{cases}$$

The pre-image of the set

$$E_0 = \{x : x_1 = x_2 = 0 \, , \ |x_3| < b\}$$

under f_0 is called the *edge* of S. (By Theorem 1.19.(2) in this paper and by Theorem 10.5 in Gehring — Väisälä [4], the domain $\bar{R}^3 \setminus \bar{S}_0$ is quasiconformally 2-collared at each point of $S_0 \setminus E_0$ but at no point of E_0. Thus, by Theorem 3.1, the edge of S does not depend on the choice of the mapping f_0.) A set S is called a *quasiconformal ridge* if it is a quasiconformal g-ridge for some function g satisfying (5.1.1).

A domain $D \subset \bar{R}^3$ is said to have a quasiconformal ridge in its boundary if some point $Q \in \partial \bar{D}$ has a neighborhood U such that $S = U \cap \partial D$ is a quasiconformal ridge with Q as a point of its edge. Let $f_0 : \bar{R}^3 \to \bar{R}^3$ be a quasiconformal mapping with $f_0 S = S_0$. Then there exists a constant $c > 0$ such that either $f_0^{-1}(te_1) \in D$ for $0 < t < c$ or $f_0^{-1}(te_1) \in \complement \bar{D}$ for $0 < t < c$. The ridge S is said to be *outward directed* in the first case and *inward directed* in the second case.

A domain D whose boundary contains a quasiconformal ridge S is evidently locally connected at each point of S, irrespective of whether S is outward or inward directed. Hence, with regard to the concepts (i) — (vii) introduced in Section 1, only the quasiconformal properties (iv), (vi), and (vii) are of interest in relation to the properties of D at the points of S. By virtue of Theorems 1.19.(1) and 3.1, D is quasiconformally collared at all points of S other than edge-points. Whether this is also the case at edge-points depends on the *direction* of S with respect to D.

5.2. **Theorem.** *A domain $D \subset \bar{R}^3$ whose boundary contains a quasiconformal ridge S is quasiconformally collared at the edge-points of S if and only if S is inward directed.*

Proof. Since B^3 and B^3_+ are quasiconformally equivalent, the necessity part follows immediately from Theorem 10.5 of Gehring — Väisälä [4]. Conversely, assume that S is inward directed. Let f_0 be as in 5.1. By Theorem 10.7 of Gehring — Väisälä [4], there exists a domain G, locally connected on the boundary and quasiconformally equivalent to B^3, and an open set U, such that $U \cap f_0 D = U \cap G$ and $S_0 = f_0 S = U \cap \partial f_0 D = U \cap \partial G$. Theorem 4.5 then implies that G, and a fortiori $f_0 D$, is quasiconformally collared at every point of S_0. According to Theorem 3.1, the same is true of D at the points of S. The proof is complete.

A domain D whose boundary contains an inward directed quasiconformal ridge S has all the three quasiconformal properties at the edge-points of S (Theorem 1.17). The situation is different if S is outward directed, not merely in view of quasiconformal collaredness but with regard to the quasiconformal flatness and accessibility properties as well. For, as will soon be seen, whether or not D has these latter properties at the edge-points depends on the *sharpness* of S.

5.3. **Theorem.** *Let $D \subset \bar{R}^3$ be a domain whose boundary contains an outward directed quasiconformal g-ridge S.*

(1) *A necessary condition for D to be quasiconformally flat at the edge-points of S is the existence of a number $d > 0$ such that*

$$\int_0^d \int_0^{\pi/2} g(r \sin \varphi) r^{-2} d\varphi dr = \infty . \tag{5.3.1}$$

(2) *A sufficient condition for D to be quasiconformally flat at the edge-points of S is the existence of a number $d > 0$ such that*

$$\int_0^d \left(\int_0^{\pi/2} g(r \sin \varphi)^{-1/2} r d\varphi \right)^{-2} dr = \infty . \tag{5.3.2}$$

Proof of (1). Suppose that (5.3.1) is not true for any positive number d. We want to show that D is quasiconformally non-flat at every edge-point Q of S. In view of Theorem 3.1, we may assume S to be the ridge S_0 in 5.1. It obviously suffices to prove the assertion for the point $Q = 0$.

For $t > 0$ let

$$C(t) = \{x = (r , \varphi , x_2) : r < t , |x_2| < t\} ,$$

where (r , φ , x_2) are cylindrical coordinates in R^3 with the polar angle φ being measured from the positive half of the x_3-axis. Choose $t_0 > 0$ so that

$$S_0 \cap C(2t_0) = \partial D \cap C(2t_0) ,$$

and set

$$\begin{aligned} F_1 &= D \cap C(t_0) \cap \{x = (r , \varphi , x_2) : 0 < r < t_0 , \varphi = \pi/4\} , \\ F_2 &= D \cap C(t_0) \cap \{x = (r , \varphi , x_2) : 0 < r < t_0 , \varphi = 3\pi/4\} , \\ \Gamma &= \Delta(F_1 , F_2 : D) . \end{aligned}$$

Since F_1 and F_2 are connected sets in D with $0 \in \bar{F}_1 \cap \bar{F}_2$, we only need show that $M(\Gamma) < \infty$.

For $x = (r , \varphi , x_2)$ let

$$\varrho(x) = \begin{cases} \dfrac{1}{r} & \text{if } x \in D \cap C(2t_0) \setminus (C(t_0) \cap \{x = (r , \varphi , x_2) : \\ & \qquad \varphi < \pi/4 \text{ or } \varphi > 3\pi/4\}), \\ 0 & \text{otherwise.} \end{cases}$$

Since each path of Γ has a subpath which joins either the sets F_1 and F_2 or the lateral surfaces of $\partial C(t_0)$ and $\partial C(2t_0)$ in $D \cap C(2t_0)$, $\varrho \in F(\Gamma)$. Consequently,

$$M(\Gamma) \leq \int_{R^3} \varrho^3 dm \leq 4 \int_0^{2t_0} \int_{\pi/4}^{\pi/2} \int_0^{g(r \sin \varphi)} r^{-2}\, dx_2 d\varphi dr$$

$$< 4 \int_0^{2t_0} \int_0^{\pi/2} g(r \sin \varphi) r^{-2}\, d\varphi\, dr \, .$$

With antithesis this implies $M(\Gamma) < \infty$ and thereby proves (1).

Proof of (2). Suppose that (5.3.2) holds. We must show that D is quasiconformally flat at every edge-point Q of S. Because of the quasiconformal invariance of this property, it is sufficient to verify the validity of the modulus condition in 1.7.(vi), the definition of quasiconformal flatness, in the case of the ridge $S = S_0$ and the point $Q = 0$.

Let E_1 and E_2 be two line segments lying one in the positive half and one in the negative half of the x_3-axis with the origin as their common end-point. Choose $t_0 > 0$ so that

$$\begin{aligned} &S_0 \cap C(t_0) = \partial D \cap C(t_0) \, , \\ &\partial C(t_0) \cap E_i \neq \emptyset \, , \quad i = 1 \, , 2 \, , \\ &g'(t_0) \leq 1 \, . \end{aligned} \tag{5.3.3}$$

For $t > 0$ set

$$\Gamma_0(t) = \Delta(E_1 \, , E_2 : D \cap \partial C(t)) \, .$$

As the first step we show that

$$M_3^{\partial C(t)}(\Gamma_0(t)) \geq \left(4 \int_0^{\pi/2} g(t \sin \varphi)^{-1/2} t d\varphi \right)^{-2} \tag{5.3.4}$$

for every $t \in (0 \, , t_0)$.

Fix such t and let $\varrho \in F(\Gamma_0(t))$. For each $v \in (-1 \, , 1)$ define a rectifiable path $\gamma_v : [0 \, , \pi] \to \bar{D} \cap \partial C(t)$ by setting

$$\gamma_v(u) = (t \, , u \, , vg(u)) \, ,$$

where the same cylindrical coordinates have been used as in the proof of (1). Since $\gamma_v \in \Gamma_0(t)$ and since $g'(t) \leq 1$ by (5.1.1) and (5.3.3), Hölder's inequality gives

$$1 \leq \left(\int_{\gamma_v} \varrho \, ds \right)^3 < \left(2 \int_0^{\pi} \varrho \; t d\varphi \right)^3$$

$$\leq 8 \int_0^\pi \varrho^3 g(t \sin \varphi) t d\varphi \left(\int_0^\pi g(t \sin \varphi)^{-1/2} t d\varphi \right)^2 .$$

Integrating with respect to v yields

$$\int_{\partial C(t)} \varrho^3 dm_2 \geq \int_{-1}^{1} dv \int_0^\pi \varrho^3 g(t \sin \varphi) t d\varphi \geq \left(4 \int_0^{\pi/2} g(t \sin \varphi)^{-1/2} t d\varphi \right)^{-2} ,$$

and (5.3.4) follows.

Now let F_1 and F_2 be two arbitrary connected sets in D such that $0 \in \bar{F}_1 \cap \bar{F}_2$, let t_0 be as before, with F_i in place of E_i, and for $t > 0$ set

$$\Gamma(t) = \Delta(F_1 , F_2 : D \cap \partial C(t)) .$$

As the second and crucial step we show that

$$M_3^{\partial C(t)}(\Gamma(t)) \geq \frac{M_3^{\partial C(t)}(\Gamma_0(t))}{27} \tag{5.3.5}$$

for every $t \in (0 , t_0)$.

Fix such t. We may assume that $F(\Gamma(t)) \neq \emptyset$, for otherwise (5.3.5) follows trivially. Let $\varrho \in F(\Gamma(t))$. If

$$\int_{\gamma_0} \varrho ds \geq \frac{1}{3} \tag{5.3.6}$$

for every rectifiable path $\gamma_0 \in \Gamma_0(t)$, then $3\varrho \in F(\Gamma_0(t))$, which implies

$$\int_{\partial C(t)} \varrho^3 dm_2 \geq \frac{M_3^{\partial C(t)}(\Gamma_0(t))}{27} . \tag{5.3.7}$$

Suppose now that (5.3.6) is not true for some rectifiable path $\gamma_0 \in \Gamma_0(t)$. Choose a point $a_i \in F_i \cap \partial C(t)$, $i = 1 , 2$. Then $a_1 \neq a_2$, because $F(\Gamma(t))$ was assumed to be non-empty. Consider first the case where a_1 and a_2 both lie outside $|\gamma_0|$, the locus of γ_0. Define a mapping f from cylindrical coordinates (r , φ , x_2) to orthonormal coordinates (x_1 , x_2 , x_3) by

$$f(r , \varphi , x_2) = (r , x_2 , r(\pi/2 - \varphi)) .$$

Then f carries $\bar{D} \cap \partial C(t)$ onto the closure of a plane domain

$$G(t) \subset X(t) = \{x : x_1 = t\} .$$

Moreover,

$$M_3^{X(t)}(f\Gamma) = M_3^{\partial C(t)}(\Gamma) \tag{5.3.8}$$

for each path family Γ in $\bar{D} \cap \partial C(t)$. Let $a_i' = f(a_i)$, $i = 1, 2$, and $\gamma_0' = f \circ \gamma_0$. Obviously $a_i' \notin |\gamma_0'|$. Next choose a point $b_i' \in |\gamma_0'|$ so that its x_3-coordinate is equal to that of a_i', and set $c_1' = f(te_3)$, $c_2' = f(-te_3)$,

$$\Gamma_0'(t) = \Delta(\{c_1'\}, \{c_2'\} : G(t)) .$$

By (5.1.1) and (5.3.3), for each i there exists a path family

$$\Gamma_i'(t) \subset \Delta(\{a_i'\}, \{b_i'\} : G(t))$$

which is similar to $\Gamma_0'(t)$. Thus, by virtue of Väisälä [8, Theorem 8.2] and by (5.3.8),

$$M_3^{X(t)}(\Gamma_i'(t)) > M_3^{X(t)}(\Gamma_0'(t)) = M_3^{\partial C(t)}(\Gamma_0(t)) . \tag{5.3.9}$$

Next let $\Gamma'(t) = f\Gamma(t)$ and define a function $\varrho' : X(t) \to \dot{R}^1$ by setting $\varrho'(x) = \varrho \circ (f|D \cap \partial C(t))^{-1}(x)$ for $x \in G(t)$ and $\varrho'(x) = 0$ otherwise. Then $\varrho' \in F(\Gamma'(t))$. Since (5.3.6) is false also with ϱ' and γ_0' in place of ϱ and γ_0, it follows that

$$\inf_{\gamma_1'} \int_{\gamma_1'} \varrho' ds \geq \frac{1}{3} \quad \text{or} \quad \inf_{\gamma_2'} \int_{\gamma_2'} \varrho' ds \geq \frac{1}{3} , \tag{5.3.10}$$

where the infima are taken over all rectifiable paths $\gamma_1' \in \Gamma_1'(t)$, $\gamma_2' \in \Gamma_2'(t)$. Hence either $3\varrho' \in F(\Gamma_1'(t))$ or $3\varrho' \in F(\Gamma_2'(t))$. In each case

$$\int_{\partial C(t)} \varrho^3 dm_2 \geq \int_{X(t)} \varrho'^3 dm_2 > \frac{M_3^{\partial C(t)}(\Gamma_0(t))}{27} \tag{5.3.11}$$

by (5.3.9).

In the preceding argument we assumed that both of the points a_1 and a_2 lie outside $|\gamma_0|$. If one, say a_1, belongs to $|\gamma_0|$, then the right hand inequality in (5.3.10) is valid, and (5.3.11) thereby follows. Note that $|\gamma_0|$ cannot contain both a_1 and a_2, because (5.3.6) was assumed to be false. All in all, since $\varrho \in F(\Gamma(t))$ was arbitrary and since either (5.3.7) or (5.3.11) holds, we obtain (5.3.5), as desired.

To complete the proof, set $\Gamma = \Delta(F_1, F_2 : D)$. We must show that $M(\Gamma) = \infty$. If $F_1 \cap F_2 \neq \emptyset$, the assertion follows trivially. Otherwise choose $\varrho \in F(\Gamma)$. Since $\varrho|\partial C(t) \in F(\Gamma(t))$ for $0 < t < t_0$,

$$\int_{R^3} \varrho^3 dm_3 \geq \int_0^{t_0} dt \int_{\partial C(t)} \varrho^3 dm_2 \geq \frac{1}{432} \int_0^{t_0} \left(\int_0^{\pi/2} g(t \sin \varphi)^{-1/2} t d\varphi \right)^{-2} dt = \infty$$

by Fubini's theorem in cylindrical coordinates, by (5.3.5), by (5.3.4), and by (5.3.2). Hence $M(\Gamma) = \infty$, and the proof of (2) is complete.

5.4. **Theorem.** *Let* $D \subset \bar{R}^3$ *be a domain whose boundary contains an outward directed quasiconformal* g*-ridge* S*. Then the edge-points of* S *are quasiconformally accessible from* D *if and only if there exists a number* $d > 0$ *such that*

$$\int_0^d \frac{du}{g(u)^{1/2}} < \infty \, . \tag{5.4.1}$$

Proof of the necessity. Suppose that (5.4.1) is not true for any positive number d. We want to show that every edge-point Q of S is quasiconformally non-accessible from D. As in the proof of the preceding theorem, we may, without loss of generality, restrict our consideration to the ridge $S = S_0$ and to the point $Q = 0$. Choose $r > 0$ so that

$$S_0 \cap B^3(2r) = \partial D \cap B^3(2r) \, ,$$

let A be a continuum in D, and fix a positive number δ. To prove the assertion, it suffices to find a connected set $F \subset D$ with $0 \in \bar{F}$, $F \cap S^2(r) \neq \emptyset$, and $M(\Delta(A \, , F : D)) < \delta$.

To this end, choose $\varepsilon \, , 0 < \varepsilon < \min \{2r \, , d(A \, , \partial D)\}$, so that

$$\varepsilon g(\varepsilon) < \delta r^2/8 \, . \tag{5.4.2}$$

By antithesis, there exists a number $s > 1$ for which

$$\int_{\varepsilon/s}^{\varepsilon} \frac{du}{g(u)^{1/2}} = \left(\frac{16r}{\delta}\right)^{1/2} . \tag{5.4.3}$$

Now set

$$\begin{aligned} F &= D \cap \bar{B}^3(r) \cap \{x : x_1 < \varepsilon/s\} \, , \\ \Gamma &= \Delta(A \, , F : D) \, , \end{aligned}$$

and define a function $\varrho : R^3 \to \dot{R}^1$ as follows:

$$\varrho(x) = \begin{cases} \dfrac{1}{r} & \text{if } \; x \in D \cap \bar{B}^3(2r) \cap \{x : x_1 \leq \varepsilon/s\} \, , \\[2ex] \max\left\{\dfrac{1}{r} \, , \left(\dfrac{\delta}{16 r g(x_1)}\right)^{1/2}\right\} & \text{if } \; x \in D \cap \bar{B}^3(2r) \cap \{x : \varepsilon/s < x_1 \leq \varepsilon\} \, , \\[2ex] 0 & \text{otherwise} \, . \end{cases}$$

Since every path of Γ has a subpath which joins either the spheres $S^2(r)$ and $S^2(2r)$ or the planes $x_1 = \varepsilon/s$ and $x_1 = \varepsilon$ in $D \cap \bar{B}^3(2r) \cap \{x : x_1 \leq \varepsilon\}$, we infer, by taking (5.4.3) into account, that $\varrho \in F(\Gamma)$. Consequently, by (5.4.2) and by (5.4.3),

$$M(\Gamma) \leq \int_{R^3} \varrho^3 dm \leq 4 \int_{\varepsilon/s}^{\varepsilon} \int_0^{g(x_1)} \int_0^{2r} \left(\frac{\delta}{16 r g(x_1)}\right)^{3/2} dx_3\, dx_2\, dx_1 +$$

$$\frac{m(D \cap \bar{B}^3(2r) \cap \{x : x_1 \leq \varepsilon\})}{r^3}$$

$$< 8r \left(\frac{\delta}{16r}\right)^{3/2} \int_{\varepsilon/s}^{\varepsilon} \frac{dx_1}{g(x_1)^{1/2}} + \frac{4\varepsilon g(\varepsilon)}{r^2} < \delta .$$

This proves the necessity part of the theorem.

Proof of the sufficiency. Suppose that (5.4.1) holds. We must show that every edge-point Q of S is quasiconformally accessible from D. Because of the quasiconformal invariance of this property, it is again sufficient to consider the ridge $S = S_0$ and the point $Q = 0$.

Let U be a neighborhood of the origin. For $t > 0$ set

$$V(t) = \{x : |x_i| < t\,, \quad i = 1\,, 2\,, 3\} .$$

Next choose a number r, $0 < r \leq d$, so that $V(2r) \subset U$ and

$$S_0 \cap V(2r) = \partial D \cap V(2r) .$$

By the condition 3° in (5.1.1), we may choose another number s, $0 < s < r$, so that

$$\frac{g(s)}{s} < \frac{g(r)}{2r} . \tag{5.4.4}$$

Finally set

$$A_1 = \{x : s/2 \leq x_1 \leq r\,, x_2 = 0\,, |x_3| \leq s\} ,$$
$$A_2 = \{x : x_1 = r\,, \quad |x_2| \leq g(r)/2\,, |x_3| \leq s\} ,$$
$$A = A_1 \cup A_2 ,$$

$$\delta = \min \left\{ \frac{s}{21 g(s)} , \frac{s}{K} \right\} , \tag{5.4.5}$$

where

$$K = \left(\int_0^r (1 + g'(u)^2/4)^{3/4}\, g(u)^{-1/2}\, du \right)^2 . \tag{5.4.6}$$

By (5.1.1) and (5.4.1), K is finite and δ therefore positive. We will show that the modulus condition in 1.7.(vii), the definition of quasiconformal accessibility, is satisfied by the above A and δ.

For this, let F be a connected set in D with $0 \in \bar{F}$ and $F \cap \partial U \neq \emptyset$. We claim that $M(\Gamma) \geq \delta$, where

$$\Gamma = \Delta(A\ , F : D)\ .$$

We may assume that $A \cap F = \emptyset$, for otherwise there is nothing to prove. Set

$$X(t) = \{x : x_1 = t\}\ ,$$
$$Z(t) = \{x : x_3 = t\}\ .$$

Then at least one of the following two conditions holds:

1. $X(t) \cap V(s)$ meets F for $s/2 < t < s$.
2. Either $Z(t) \cap V(s)$ meets F for $0 < t < s$ or $Z(t) \cap V(s)$ meets F for $-s < t < 0$.

We must prove that in each case $M(\Gamma) \geq \delta$.

Case 1. Let $\varrho \in F(\Gamma)$. For every $t \in (s/2\ , s)$ there exists a closed disc $B \subset D \cap X(t)$ of radius less than $g(s)/2$ such that B meets both F and A_1. Thus

$$\int_{X(t)} \varrho^3 dm_2 \geq \int_B \varrho^3 dm_2 \geq \frac{2}{21 g(s)}$$

by Väisälä [5, Theorem 3.5]. Integrating with respect to t and using Fubini's theorem we obtain

$$\int_{R^3} \varrho^3 dm_3 \geq \int_{s/2}^{s} dt \int_{X(t)} \varrho^3 dm_2 \geq \frac{s}{21 g(s)}\ .$$

Together with (5.4.5) this implies $M(\Gamma) \geq \delta$.

Case 2. Assume, for example, that $Z(t) \cap V(s)$ meets F for every $t \in (0\ , s)$. Set

$$G = \{x : 0 < x_1 < r\ , |x_2| < g(x_1)/2\ , x_3 = 0\}\ ,$$
$$\Gamma_0 = \Delta(A_2\ , \{0\} : G)\ .$$

As the first step we show that $M_3^{Z(0)}(\Gamma_0) > 0$.

Let $\varrho \in F(\Gamma_0)$. For each $v \in (-1/2, 1/2)$ define a rectifiable path $\gamma_v : [0, r] \to \bar{G}$ by

$$\gamma_v(u) = (u, vg(u), 0) .$$

Then $\gamma_v \in \Gamma_0$, and Hölder's inequality yields

$$1 \leq \left(\int_{\gamma_v} \varrho ds \right)^3 = \left(\int_0^r \varrho(u, vg(u), 0)(1 + v^2 g'(u)^2)^{1/2} du \right)^3$$

$$\leq \int_0^r \varrho^3 g(u) du \left(\int_0^r (1 + v^2 g'(u)^2)^{3/4} g(u)^{-1/2} du \right)^2 .$$

Integrating with respect to v and taking (5.4.6) into account we obtain

$$\int_{Z(0)} \varrho^3 dm_2 \geq \int_{-1/2}^{1/2} dv \int_0^r \varrho^3 g(u) du \geq \frac{1}{K} .$$

Since this holds for every $\varrho \in F(\Gamma_0)$, we have

(5.4.7) $$M_3^{Z(0)}(\Gamma_0) \geq \frac{1}{K} .$$

For the second and crucial step, set

$$\Gamma(t) = \Delta(A_2, F : D \cap Z(t)) .$$

We will now show, with the aid of (5.4.7), that $M_3^{Z(t)}(\Gamma(t)) \geq 1/K$ for each $t \in (0, s)$.

Fix such t, choose a point $p = (p_1, p_2, t)$ of $F \cap V(s)$, and set

(5.4.8) $$k = \frac{r - p_1}{r} .$$

Since $p_1 < s$ and $|p_2| < g(p_1)$, it follows from (5.4.8), from the condition 2° in (5.1.1), and from (5.4.4) that

(5.4.9) $$\frac{kg(r)}{2} + |p_2| < \frac{(r - p_1)g(r)}{2r} + \frac{p_1 g(r)}{2r} = \frac{g(r)}{2} .$$

Defining a conformal affine mapping $f : Z(0) \to Z(t)$ by

$$f(x) = kx + p ,$$

we thus see that $f(0) = p$, and, if $x = (r, x_2, 0) \in G \cap A_2$, then $f(x) = (x_1', x_2', t)$, where $x_1' = kr + p_1 = r$ by (5.4.8) and where $|x_2'| < kg(r)/2 + |p_2| < g(r)/2$ by (5.4.9). Consequently,

$$\Gamma_0' = \{f \circ \gamma : \gamma \in \Gamma_0\} \subset \Gamma(t) \,.$$

From Theorem 8.2 in Väisälä [8], from (5.4.7), and from (5.4.8) it then follows that

$$M_3^{Z(t)}(\Gamma(t)) \geq M_3^{Z(t)}(\Gamma_0') \geq \frac{1}{k}\; M_3^{Z(0)}(\Gamma_0) \geq \frac{1}{K} \,, \tag{5.4.10}$$

as desired.

To complete the proof, let $\varrho \in F(\Gamma)$. Since $\varrho | Z(t) \in F(\Gamma(t))$ for $0 < t < s$, we obtain, by Fubini's theorem and by (5.4.10),

$$\int\limits_{R^3} \varrho^3 dm_3 \geq \int\limits_0^s dt \int\limits_{Z(t)} \varrho^3 dm_2 \geq M_3^{Z(t)}(\Gamma(t)) \int\limits_0^s dt \geq \frac{s}{K} \,.$$

Together with (5.4.5) this implies $M(\Gamma) \geq \delta$. The proof of the sufficiency part, and thus of the whole theorem, is complete.

5.5. Examples. (1) Let D be a domain whose boundary contains an outward directed quasiconformal ridge S defined by the function $g(u) = u^p$, $p > 1$. Then D is not quasiconformally flat at the edge-points of S. Contrary to this, the edge-points are quasiconformally accessible from D if and only if $1 < p < 2$.

(2) The function $g(u) = u/|\log u|$, $0 < u < 1/2$, defines an outward directed quasiconformal ridge in such a way that the corresponding domain has the quasiconformal flatness and accessibility properties at the edge-points.

(3) The first example implies that there exist ridges such that the edge-points are quasiconformally accessible from the corresponding domain, although the domain is quasiconformally non-flat at the edge-points. We finally show that there also exist ridges such that the domain is quasiconformally flat at the edge-points, although these points are quasiconformally non-accessible from the domain. For this purpose, let $g(u) = u/|\log u|$, $g^*(u) = u^2$, and

$$I(g\,,b\,,c) = \int\limits_b^c \Bigl(\int\limits_0^{\pi/2} g(r \sin\varphi)^{-1/2} r d\varphi\Bigr)^{-2} dr \,,\quad I^*(g^*\,,b\,,c) = \int\limits_b^c g^*(u)^{-1/2} du \,.$$

Then $I(g\,,0\,,1/2) = \infty = I^*(g^*\,,0\,,1/2)$. Choose $1/2 = a_1^* > a_2^* > a_1 > a_2 > \ldots > a_k^* > a_{k+1}^* > a_k > a_{k+1} > \ldots > 0$ so that $I(g\,,a_{2k}\,,a_{2k-1}) = 1 = I^*(g^*\,,a_{2k}^*\,,a_{2k-1}^*)$, $k = 1\,,2\,,\ldots$, and

$$g'(a_{2k-1}) = \frac{g^*(a_{2k}^*) - g(a_{2k-1})}{a_{2k}^* - a_{2k-1}} \,,\quad g'(a_{2k}) = \frac{g(a_{2k}) - g^*(a_{2k+1}^*)}{a_{2k} - a_{2k+1}^*} \,.$$

Define a function $\bar{g} : [0\,,\,1/2] \to R^1$ as follows:

$$\bar{g}(u) = \begin{cases} g^*(u) & \text{if } a_{2k}^* < u \leq a_{2k-1}^*\,, \\ g(a_{2k-1}) + g'(a_{2k-1})(u - a_{2k-1}) & \text{if } a_{2k-1} < u \leq a_{2k}^*\,, \\ g(u) & \text{if } a_{2k} < u \leq a_{2k-1}\,, \\ g^*(a_{2k+1}^*) + g'(a_{2k})(u - a_{2k+1}^*) & \text{if } a_{2k+1}^* < u \leq a_{2k}\,, \\ 0 & \text{if } u = 0\,. \end{cases}$$

Then the conditions $1°-3°$ in (5.1.1) are satisfied with $\bar{g}$ and $1/2$ in place of g and a, except that $\bar{g}'(u)$ does not exist for $u = a_k^*$, $k = 2\,,3\,,\ldots$. However, it is clear in what manner $\bar{g}$ can be modified to produce a function which does not suffer from this deficiency. Since $I(\bar{g}\,,0\,,1/2) = \infty = I^*(\bar{g}\,,0\,,1/2)$, we conclude from Theorems 5.3 and 5.4 that this modified function defines a quasiconformal ridge in such a way that the corresponding domain is quasiconformally flat at the edge-points, although these points are quasiconformally non-accessible from the domain.

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