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TOPOLOGICAL AND METRIC PROPERTIES OF QUASIREGULAR MAPPINGS

BY

O. MARTIO, S. RICKMAN and J. VÄISÄLÄ

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SUOMALAINEN TIEDEAKATEMIA

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1. Introduction

1.1. This paper is a continuation to [13] and [14]. Our main interest is centered on the branch set B_f of a quasiregular mapping $f: G \to R^n$. In Section 2 we consider the case $B_f = \emptyset$, which means that f is a local homeomorphism. Section 3 deals with the properties of fB_f in the case $B_f \neq \emptyset$. For example, we show that $\Lambda_{n-2}(fB_f) > 0$ and consider the case where f has an essential isolated singularity. In Section 4 we present relations between the local degree $i(x, f)$ and the dilatation $K(f)$. In Section 5 we improve some results of [13] concerning the linear dilatations.

1.2. *Notation and terminology.* We shall use the same notation and terminology as in [13] and [14]. All point sets are assumed to lie in the compactified n-space $\bar{R}^n = R^n \cup \{\infty\}$. Throughout the paper we assume $n \geqq 2$. The notation $f: G \to R^n$ or $f: G \to \bar{R}^n$ includes the assumptions that G is a domain in R^n or $\bar{R}^n$, respectively, and that f is continuous. If Δ is a segment of line and if $\alpha: \Delta \to \bar{R}^n$ is a path, we let $|\alpha|$ denote the locus $\alpha\Delta$ of α. More generally, if $c = \Sigma m_i \sigma_i$ is a singular 1-chain, then $|c| = \cup |\sigma_i|$. The inner product of two vectors x, y is written as $(x \mid y)$.

2. Local homeomorphisms

2.1. In this section we show that a K-quasiregular local homeomorphism of the unit ball B^n, $n \geqq 3$, is homeomorphic in a smaller ball $B^n(r)$, where r depends only on n and K. This result is applied to give a sufficient condition for the equicontinuity of a family of K-quasiregular local homeomorphisms.

We need some topological results on local homeomorphisms. A set $Q \subset \bar{R}^n$ is said to be *relatively locally connected* if every point of $\bar{Q}$ has arbitrarily small neighborhoods U such that $U \cap Q$ is connected.

2.2. **Lemma.** *Suppose that $f: G \to \bar{R}^n$ is a local homeomorphism, that Q is a simply connected and locally pathwise connected set in $\bar{R}^n$, and that P is a component of $f^{-1}Q$ such that $\bar{P} \subset G$. Then f maps P homeo-*

morphically onto Q *. If, in addition,* Q *is relatively locally connected,* f *maps* $\bar{P}$ *homeomorphically onto* $\bar{Q}$ *.*

Proof. In order to prove the first statement of the theorem, it suffices to show that f defines a covering mapping of P onto Q [9, p. 91]. If $x_0 \in P$, every path in Q starting at $f(x_0)$ may be uniquely lifted to a path in P starting at x_0. Hence $fP = Q$. Let $y \in Q$, and let $P \cap f^{-1}(y) = \{x_1, \ldots, x_k\}$. Choose disjoint neighborhoods U_i of x_i such that $U_i \subset G$ and f is injective in each U_i. Then $F = \bar{P} \setminus (U_1 \cup \cdots \cup U_k)$ is compact, and $y \notin fF$. Choose a neighborhood V of y such that $Q \cap V$ is connected and $V \subset (\cap fU_i) \setminus fF$. Then $P \cap f^{-1}V$ has exactly k components $D_1, \ldots, D_k$, one in each U_i, and f maps D_i homeomorphically onto $Q \cap V$. Thus f defines a covering mapping $P \to Q$.

Assume now that Q is relatively locally connected. We must show that an arbitrary point $y \in \bar{Q}$ has exactly one pre-image in $\bar{P}$. Choose a sequence $V_1 \supset V_2 \supset \cdots$ of neighborhoods of y such that $V_j \cap Q$ is connected for all j and $\cap V_j = \{y\}$. It is easy to see that $\bar{P} \cap f^{-1}(y) = \bigcap_{j=1}^{\infty} \bar{A}_j$ where $A_j = P \cap f^{-1}V_j$. Since f defines a homeomorphism $P \to Q$, every A_j is connected. Hence $\bar{A}_1 \supset \bar{A}_2 \supset \cdots$ is a nested sequence of compact connected sets, which implies that $\bar{P} \cap f^{-1}(y)$ is non-empty and connected. Since $f^{-1}(y)$ is discrete, $\bar{P} \cap f^{-1}(y)$ consists of exactly one point.

2.3 **Theorem.** *If* $n \geqq 3$ *and if* $f: B^n \to R^n$ *is a* K*-quasiregular local homeomorphism, then* f *is injective in a ball* $B^n(\psi(n, K))$*, where* $\psi(n, K)$ *is a positive number depending only on* n *and* K*.*

Proof. We may assume that $f(0) = 0$. As in [13], we let $U(0, f, r)$ denote the component of $f^{-1}B^n(r)$ which contains the origin. Let r_0 be the least upper bound of all positive numbers r such that $\bar{U}(0, f, r) \subset B^n$. Fix $r \in (0, r_0)$, and set $l^* = l^*(0, f, r)$, $L^* = L^*(0, f, r)$ (for notation, see [13, 4.1]). By 2.2, f maps $\bar{U}(0, f, r)$ homeomorphically onto $\bar{B}^n(r)$. Thus f is injective in $B^n(l^*)$. Hence it suffices to find a lower bound for l^*.

Let $l = l(0, f, l^*)$, and suppose $l < r$. Then $A = U(0, f, r) \setminus \bar{U}(0, f, l)$ is a ring, and f maps A K-quasiconformally onto the spherical ring $B^n(r) \setminus \bar{B}^n(l)$. Since both boundary components of A meet the sphere $S^{n-1}(l^*)$, it follows from a well-known estimate [26, 11.7] that $\operatorname{cap} A \geqq a_n > 0$, where a_n depends only on n. Thus

$$a_n \leqq \operatorname{cap} A \leqq K \operatorname{cap} fA = K\omega_{n-1} (\log (r/l))^{1-n}.$$

This gives an inequality

$$r/l \leqq \varkappa(n, K), \tag{2.4}$$

which is true also in the case $r = l$.

We shall now make use of a method of Zorič [28] and Agard-Marden [1]. Choose $x_0 \in \partial U(0, f, r)$ such that $|x_0| = L^*$, and set $y_0 = f(x_0)$. Then $|y_0| = r$. For $t \in (r, r + l)$ and for $\varphi \in (0, \pi]$ let

$$C(t, \varphi) = \{y \mid |y - y_0| = t, (y_0 - y|y_0) > rt \cos \varphi\}.$$

Thus $C(t, \varphi)$ is a spherical cap (possibly a punctured sphere), which is symmetric with respect to the line segment $J = \{sy_0 | -l/r < s < 0\}$ and meets J at $z_t = (r - t)y_0/r$. Let z_t^* be the unique point in $U(0, f, r) \cap f^{-1}(z_t)$, and let $C^*(t, \varphi)$ be the z_t^*-component of $f^{-1}C(t, \varphi)$. Let φ_t be the least upper bound of all $\varphi \in (0, \pi]$ such that f maps $C^*(t, \varphi)$ homeomorphically onto $C(t, \varphi)$. We show that $C^*(t, \varphi_t)$ meets $S^{n-1}(L^*)$ for every $t \in (r, r + l)$. Suppose that this is false. Then $C^*(t, \varphi_t) \subset B^n(L^*)$ for some t. By 2.2, f maps $\bar{C}^*(t, \varphi_t)$ homeomorphically onto $\bar{C}(t, \varphi_t)$. Note that for $n = 2$, the proof breaks down here, since $C(t, \pi)$ is then not relatively locally connected. By [28, Remark 1, p. 422] or by Corollary 3.8, f is injective in a neighborhood of $\bar{C}^*(t, \varphi_t)$. In view of the definition of φ_t, this implies $\varphi_t = \pi$, which means that $\bar{C}^*(t, \varphi_t)$ is a topological sphere. The bounded component D of $\complement\bar{C}^*(t, \varphi_t)$ is contained in $B^n(L^*)$. Since $\partial fD \subset S^{n-1}(y_0, t)$, $fD = B^n(y_0, t)$. Thus D is a component of $f^{-1}B^n(y_0, t)$. By 2.2, f maps $\bar{D}$ homeomorphically onto $\bar{B}^n(y_0, t)$. Since $z_t^* \in \bar{D} \cap \bar{U}(0, f, r)$ and since $\bar{B}^n(y_0, t) \cap \bar{B}^n(r)$ is connected, it follows from [28, Remark 2, p. 422] that f is injective in $\bar{D} \cup \bar{U}(0, f, r)$. Since $y_0 \in fD$, this implies that $x_0 \in D$, which is impossible, because $D \subset B^n(L^*)$. Thus $C^*(t, \varphi_t)$ meets $S^{n-1}(L^*)$ for all $t \in (r, r + l)$.

Set $V = \bigcup C(t, \varphi_t)$ and $V^* = \bigcup C^*(t, \varphi_t)$, where the unions are taken over $t \in (r, r + l)$. Arguing as in [28, p. 425] we see that V^* and V are domains and that f maps V^* homeomorphically onto V. For each $t \in (r, r + l)$ choose a point $x_t^* \in C^*(t, \varphi_t) \cap S^{n-1}(L^*)$. Let $\Gamma(t)$ be the family of all paths joining x_t^* and z_t^* in $C^*(t, \varphi_t)$, and set $\Gamma = \bigcup \Gamma(t)$. Since $|z_t^*| \leqq l^*$,

$$M(\Gamma) \leqq \omega_{n-1} (\log (L^*/l^*))^{1-n}. \tag{2.5}$$

On the other hand, a well-known modulus estimate [26, 10.12] yields

$$M(f\Gamma) \geqq b_n \log (1 + l/r), \tag{2.6}$$

where the positive constant b_n depends only on n. Since $M(f\Gamma) \leqq KM(\Gamma)$, we obtain from (2.4), (2.5), and (2.6) an inequality $l^* \geqq L^*\psi(n, K)$, where $\psi(n, K) > 0$ depends only on n and K. Since $L^* \to 1$ as $r \to r_0$, this proves the theorem.

2.7. **Corollary.** *Suppose that* $n \geqq 3$, *that* $f : G \to R^n$ *is a K-quasiregular mapping and that* $x_0 \in G \setminus B_f$. *Then* f *is injective in the ball* $B^n(x_0, r)$ *where* $r = \psi(n, K)d(x_0, B_f \cup \partial G)$.

2.8. **Corollary.** (*Theorem of Zorič* [28]) *If* $n \geqq 3$ *and if* $f : R^n \to R^n$ *is a quasiregular local homeomorphism, then* f *is a homeomorphism.*

2.9 **Theorem.** *Suppose that* G *is a domain in* $\bar{R}^n$, $n \geqq 3$, *that* $K \geqq 1$, *and that* $r > 0$. *If* W *is a family of K-quasimeromorphic local homeomorphisms* $f : G \to \bar{R}^n$ *such that every* $f \in W$ *omits two points* $a_f, b_f \in \bar{R}^n$ *with* $q(a_f, b_f) \geqq r$, *then* W *is equicontinuous. Here* q *is the spherical metric* [14, 3.10].

Proof. For $f \in W$ let T_f be a Möbius transformation such that $T_f(b_f) = \infty$. By 2.7, every point in G has a neighborhood in which every $T_f \circ f$, and hence f, is injective. The theorem follows from the corresponding result for quasiconformal mappings [26, 19.2].

2.10. **Corollary.** *If* W *is a family of K-quasimeromorphic local homeomorphisms of a domain* $G \subset \bar{R}^n$, $n \geqq 3$, *and if every* $f \in W$ *omits two fixed points in* $\bar{R}^n$, *then* W *is equicontinuous.*

2.11. *Remark.* The results 2.3, 2.7, 2.8, 2.9, and 2.10 fail to be true for $n = 2$. The mappings $f_j(z) = e^{jz}$ serve as a counterexample in every case.

2.12. *Path families and quasimeromorphic local homeomorphisms.* Suppose that $f : G \to \bar{R}^n$ is quasimeromorphic and that Γ is a path family in G. In [13] we conjectured that

$$M(f\Gamma) \leqq K_I(f)M(\Gamma) . \tag{2.13}$$

We shall now prove this inequality for local homeomorphisms. This result is needed in Section 3.

2.14. **Lemma.** *If* $f : G \to \bar{R}^n$ *is a quasimeromorphic local homeomorphism and if* Γ *is a path family in* G, *then* (2.13) *is true.*

Proof. Since the family of all paths through a given point is of modulus zero, we may assume that $\infty \notin G$ and $\infty \notin fG$. We cover G with a countable number of domains U_i such that $\bar{U}_i \subset G$ and such that f defines a homeomorphism $f_i : U_i \to fU_i$. Set $g_i = f_i^{-1}$. Let Γ^* be the family of all paths $\gamma \in \Gamma$ such that $f \circ \gamma$ is locally rectifiable. Let Γ_i be the family of all paths $\gamma \in \Gamma^*$ which have a closed subpath β such that $|\beta| \subset U_i$ and such that g_i is not absolutely continuous on $f \circ \beta$. By Fuglede's theorem [26, 28.2] $M(f\Gamma_i) = 0$. Setting $\Gamma_0 = \Gamma^* \setminus (\cup \Gamma_i)$ we thus have $M(f\Gamma_0) = M(f\Gamma)$. Hence it suffices to show that

$$M(f\Gamma_0) \leqq K_I(f) M(\Gamma) .$$

Let E be the set of all $x \in G$ such that f is differentiable at x and $J(x, f) > 0$. Then E is a Borel set and $m(G \setminus E) = 0$. Suppose that $\varrho \in F(\Gamma)$. Define $\sigma : G \to \dot{R}^1$ by

$$\begin{aligned} \sigma(x) &= \varrho(x)/l(f'(x)) \quad \text{for } x \in E , \\ \sigma(x) &= \infty \quad \text{for } x \in G \setminus E . \end{aligned}$$

Then σ is a Borel function. Next define $\varrho' : R^n \to \dot{R}^1$ by

$$\begin{aligned} \varrho'(y) &= \sup_{x \in f^{-1}(y)} \sigma(x) \quad \text{for } y \in fG , \\ \varrho'(y) &= 0 \quad \text{for } y \in \complement fG . \end{aligned}$$

Then

$$\varrho'(y) \geqq \sigma(g_i(y)) \geqq \varrho(g_i(y)) L(y , g_i) \tag{2.15}$$

for $y \in fU_i$, where $L(y , g_i) = \limsup |g_i(y + h) - g_i(y)|/|h|$, cf. [13, p. 16]. We shall show that $\varrho' \in F(f\Gamma_0)$.

Since

$$\{y | \varrho'(y) > t\} = \bigcup_{i=1}^{\infty} \{y \in fU_i | \sigma(g_i(y)) > t\}$$

for all $t > 0$, ϱ' is a Borel function. Suppose that $\gamma \in \Gamma_0$. Let $\varepsilon > 0$, and let $\gamma_\varepsilon : [a , b] \to G$ be a closed subpath of γ such that

$$\int_{\gamma_\varepsilon} \varrho ds > 1 - \varepsilon .$$

Divide [a , b] into non-overlapping subintervals $[a_0 , a_1] , \ldots , [a_{k-1} , a_k]$ such that each $\gamma_j = \gamma | [a_{j-1} , a_j]$ is a path in some U_i. Then (2.15) and a transformation formula [26, 5.3] imply

$$\int_{f \circ \gamma_j} \varrho' ds \geqq \int_{f \circ \gamma_j} \varrho(g_i(y)) L(y , g_i) |dy| \geqq \int_{\gamma_j} \varrho ds .$$

Summing over j yields

$$\int_{f \circ \gamma} \varrho' ds \geqq \int_{f \circ \gamma_\varepsilon} \varrho' ds \geqq \int_{\gamma_\varepsilon} \varrho ds > 1 - \varepsilon .$$

Since ε was arbitrary, this implies

$$\int_{f\circ\gamma} \varrho' ds \geqq 1 .$$

Hence $\varrho' \in F(f\Gamma_0)$. We thus obtain the inequality

$$M(f\Gamma_0) \leqq \int \varrho'^n dm . \tag{2.16}$$

Choose an increasing sequence of Borel functions $\varrho_j : R^n \to R^1$ such that either $0 = \varrho_j(y) = \varrho'(y)$ or $0 < \varrho_j(y) < \varrho'(y)$ for all $y \in R^n$ and such that $\varrho_j(y) \to \varrho'(y)$ for all $y \in R^n$. Let A_j be the set of all $x \in G$ such that $\varrho_j(f(x)) \leqq \sigma(x)$. From the definition of ϱ' it follows that $fA_j = fG$. In other words, $N(y, f, A_j) \geqq 1$ for all $y \in fG$. Using a transformation formula in [16, Theorem 3, p. 364] we obtain

$$\begin{aligned}\int \varrho_j^n dm &\leqq \int \varrho_j(y)^n N(y, f, A_j) dm(y) \\ &= \int_{A_j} \varrho_j(f(x))^n J(x, f) dm(x) \\ &\leqq K_I(f) \int_{A_j} \sigma(x)^n l(f'(x))^n dm(x) \\ &\leqq K_I(f) \int \varrho^n dm .\end{aligned}$$

Letting $j \to \infty$ and using (2.16) we obtain

$$M(f\Gamma_0) \leqq K_I(f) \int \varrho^n dm .$$

This proves (2.13).

3. Properties of fB_f

First we prove a relation between Hausdorff measure and simple connectedness. This is used to show that $\Lambda_{n-2}(fB_f) > 0$ for a discrete and open mapping $f : G \to \bar{R}^n$ with $B_f \neq \emptyset$. Next we examine the behavior of a quasimeromorphic mapping f with an isolated essential singularity. We prove that the set of asymptotic values and $\complement fG$ are contained in the closure of fB_f. We also give an outline of a proof for Iversen's theorem which states that $\complement fG$ is contained in the set of asymptotic values. In the rest of the section we consider for $n = 3$ the case where fB_f is contained in a tamely embedded arc.

3.1. **Lemma.** *Let* $T \subset R^n$ *be a 2-dimensional plane and let* $A \subset R^n$ *be such that* $\Lambda_{n-2}(A) = 0$. *Then* $m(A + T) = 0$.

Proof. It is sufficient to show that $m(A + Q) = 0$ whenever $Q \subset T$ is a closed square. Let $\varepsilon > 0$. Denote by h the length of a side of Q. Choose a covering of A by balls $B_i = B^n(x_i, r_i)$ such that $r_i < 1$ and $\Sigma r_i^{n-2} < \varepsilon$. Then $A + Q \subset \bigcup (B_i + Q)$. On the other hand,

$$m(B_i + Q) \leqq (h + 2r_i)^2 2^{n-2} r_i^{n-2} < (h + 2)^2 2^{n-2} r_i^{n-2},$$

hence

$$m(A + Q) \leqq \Sigma m(B_i + Q) \leqq (h + 2)^2 2^{n-2} \varepsilon,$$

and the lemma follows.

3.2. **Lemma.** *Let* $A \subset R^n$ *be such that* $\Lambda_{n-2}(A) = 0$, *and let* $E \subset R^n$ *be a set which has a countable covering by 2-dimensional planes* $T_1, T_2, \ldots$ *Then* $(E + y) \cap A = \emptyset$ *for almost every* $y \in R^n$.

Proof. Denote $H = \{y \in R^n | (E + y) \cap A \neq \emptyset\}$. If $y \in H$, there exists $z \in E$ such that $z + y \in A$. Hence $y \in A - z$, and so $y \in A - E$. This yields $H \subset A - E \subset \bigcup (A - T_i)$. Since 3.1 implies $m(A - T_i) = 0$, we have $m(H) = 0$. The lemma is proved.

The next result is perhaps well-known, but the authors have been unable to find it in literature.

3.3. **Lemma.** *Let* $G \subset R^n$ *be a simply connected domain and let* A *be closed in* G *such that* $\Lambda_{n-2}(A) = 0$. *Then* $G \setminus A$ *is simply connected.*

Proof. Suppose that $\gamma : I \to G \setminus A$, $I = [0,1]$, is a path with $\gamma(0) = \gamma(1) = x_0$. By assumption there exists a homotopy $h : I^2 \to G$ such that $h(0, t) = \gamma(t)$, $h(1, t) = h(s, 0) = h(s, 1) = x_0$ for all $(s, t) \in I^2$. Let τ be a triangulation of I^2 and let $h_1 : I^2 \to G$ be a simplicial approximation of h with respect to τ. Then $h_1(1, t) = h_1(s, 0) = h_1(s, 1) = x_0$ for $(s, t) \in I^2$. Set $\gamma_1(t) = h_1(0, t)$. If the triangulation τ is sufficiently dense, then $(s, t) \mapsto s\gamma(t) + (1 - s)\gamma_1(t)$ defines a homotopy $\gamma \simeq \gamma_1$ in $G \setminus A$. By 3.2 there exists $y \in R^n$ such that $|y| < \min(d(h_1 I^2, \partial G), d(|\gamma_1|, \partial(G \setminus A)))$ and $(h_1 I^2 + y) \cap A = \emptyset$. Thus we obtain the following homotopies in $G \setminus A : \gamma_1 \simeq \gamma_1 + y$ defined by $(s, t) \mapsto \gamma_1(t) + sy$ and $\gamma_1 + y \simeq x_0 + y$

defined by $(s, t) \mapsto h_1(s, t) + y$. This implies $\gamma \simeq x_0 + y$ in $G \setminus A$, and the lemma follows.

3.4. **Theorem.** *Let* $f : G \to \bar{R}^n$ *be discrete and open. If* $B_f \neq \emptyset$, *then* $\Lambda_{n-2}(fB_f) > 0$.

Proof. If $n = 2$, the theorem is trivial. Suppose $n \geqq 3$. Let $x \in B_f$ and pick $r > 0$ such that $U = U(x, f, r)$ is a normal neighborhood of x. The mapping $f|U$ as a pseudo-covering map in the sense of Church and Hemmingsen [4, p. 529]. Hence [4, Corollary 5.2] implies that $B^n(f(x), r) \setminus f(B_f \cap U)$ is not simply connected. Since $f(B_f \cap U)$ is closed in $B^n(f(x), r)$, Lemma 3.3 yields $\Lambda_{n-2}(fB_f) > 0$.

3.5 *Remark.* It has been conjectured [3, p. 368] that if $f : G \to R^n$ is discrete and open, then either $B_f = \emptyset$ or $\dim B_f = n - 2$. For $n = 3$ this has been proved by Trohimčuk [25]; for $n \geqq 4$ this is an open question. Since $\dim B_f = \dim fB_f$ [4, 2.2] and since $\dim A = k$ implies $\Lambda_k(A) > 0$ [10, p. 104], Theorem 3.4 is a weaker result.

3.6. We next study the set fB_f in the case where f has an essential isolated singularity. For terminology, see [14, 4.2]. It turns out that there are striking differences between the cases $n = 2$ and $n \geqq 3$.

We prove first some topological lemmas. Suppose that $f : G \to \bar{R}^n$ is sense-preserving, discrete, and open. If C is a set such that $\bar{C} \subset G$ and if $y \in \bar{R}^n$, then the set $A = f^{-1}(y) \cap C$ is finite, and we set

$$M(y, f, C) = \sum_{x \in A} i(x, f)$$

(cf. Martio [12, 3.5]). Moreover, we set $M^*(f, C) = \sup M(y, f, C)$ over $y \in \bar{R}^n$. Then $N(f, C) \leqq M^*(f, C)$. Furthermore, covering C with a finite number of normal domains $D_1, \ldots, D_k$ we see that $M^*(f, C) \leqq \Sigma \mu(f, D_j) < \infty$.

3.7. **Lemma.** *Suppose that* $f : G \to \bar{R}^n$ *is sense-preserving, discrete, and open, and that* F *is a compact set in* G. *Then there is a neighborhood* U *of* F *such that* $\bar{U} \subset G$ *and* $M^*(f, U) = M^*(f, F)$.

Proof. Suppose that the lemma is not true. Set $k = M^*(f, F)$. For each positive integer j, we can find a point $y_j \in \bar{R}^n$ and a finite set $A_j \subset f^{-1}(y_j)$ such that $d(x, F) < 1/j$ for $x \in A_j$ and such that $\Sigma\, i(x, f) \geqq k + 1$ over $x \in A_j$. Passing to a subsequence we may assume that $\operatorname{card} A_j = h$ is independent of j. Moreover, we may assume that $A_j = \{a_j^1, \ldots, a_j^h\}$

such that every sequence $a_1^m, a_2^m, \ldots$ converges to a point $a_0^m \in F$. It follows that $f(a_0^m) = y_0 = \lim y_j$. Let $b_1, \ldots, b_s$ be the distinct points in $\{a_0^1, \ldots, a_0^h\}$, and let $Q_r = \{m | a_0^m = b_r\}$, $1 \leqq r \leqq s$. Choose disjoint normal neighborhoods U_r of b_r, $1 \leqq r \leqq s$. Then there is j such that $y_j \in fU_1 \cap \ldots \cap fU_s$ and such that $a_j^m \in U_r$ for all $m \in Q_r$ and $1 \leqq r \leqq s$. We obtain

$$k \geqq M(y_0, f, F) \geqq \sum_{r=1}^{s} i(b_r, f) = \sum_{r=1}^{s} \mu(y_0, f, U_r) =$$

$$\sum_{r=1}^{s} \mu(y_j, f, U_r) \geqq \sum_{r=1}^{s} \sum_{m \in Q_r} i(a_j^m, f) = \sum_{x \in A_j} i(x, f) \geqq k + 1 .$$

This contradiction proves the lemma.

3.8. **Corollary.** (*Zorič* [28, p. 422]) *Suppose that $f: G \to \bar{R}^n$ is a local homeomorphism and that F is a compact set in G such that $f|F$ is injective. Then f is injective in some neighborhood of F.*

3.9. **Lemma.** *Suppose that $f: G \to \bar{R}^n$ is a discrete open mapping and that U is a normal domain of f such that fU is relatively locally connected* (*see* 2.1). *If $y \in f(\partial U \setminus B_f)$, then*

$$N(y, f, \partial U \setminus B_f) = N(f, \partial U \setminus B_f) = N(f, \partial U) = N(f, U) .$$

Proof. We may assume that f is sense-preserving. Let $y \in f(\partial U \setminus B_f)$ and let $x_1, \ldots, x_k$ be the points of $f^{-1}(y) \cap (\partial U \setminus B_f) = f^{-1}(y) \cap \bar{U}$. Thus $k = N(y, f, \partial U \setminus B_f)$. Choose disjoint neighborhoods U_i of x_i such that $f|\bar{U}_i$ is injective, and choose then a neighborhood V of y such that $V \subset \cap fU_i$ and such that $V \cap fU = D$ is connected. By [13, 2.6], f maps every component C of $U \cap f^{-1}D$ onto D. Hence $y \in f\bar{C}$, which implies that the components of $U \cap f^{-1}D$ are the domains $C_i = U_i \cap f^{-1}D$. Choose $y_1 \in D$. Using [13, 2.12] we obtain

$$N(f, U) = \mu(y_1, f, U) = \sum_{i=1}^{k} \mu(y_1, f, C_i) = k = N(y, f, \partial U \setminus B_f)$$
$$\leqq N(f, \partial U \setminus B_f) \leqq N(f, \partial U) .$$

It remains to show that $N(f, \partial U) \leqq N(f, U)$. Pick $z \in \partial fU$ such that $N(z, f, \partial U) = N(f, \partial U) = h$. Let $\{x_1, \ldots, x_h\} = f^{-1}(z) \cap \partial U = f^{-1}(z) \cap \bar{U}$. Choose disjoint neighborhoods U_i of x_i, set $F = \bar{U} \setminus (U_1 \cup \ldots \cup U_h)$, and choose then a neighborhood V of z such that $V \cap fF = \emptyset$ and such that $V \cap fU = D$ is connected. Then each U_i contains a component C_i of $U \cap f^{-1}D$, and $fC_i = D$. Hence every point $z_1 \in D$ has at least h pre-images in U. Thus $N(f, U) \geqq N(z_1, f, U) \geqq h = N(f, \partial U)$. The lemma is proved.

3.10. **Lemma.** (*cf. Agard-Marden* [1, 3.A]) *Suppose that* $f : G \to \bar{R}^n$ *is light, that* $A \subset fG$, *and that* $s : A \to G$ *is a continuous section of* f, *that is,* $f \circ s = \mathrm{id}$. *If* A *is relatively locally connected at a point* $y \in \bar{A}$ *(see 2.1), then the cluster set* $C(s, y)$ *is either a compact connected set in* ∂G *or consists of a single point in* G.

Proof. Choose a basic system $U_1 \supset U_2 \supset \ldots$ of neighborhoods of y such that the sets $D_j = U_j \cap A$ are connected. Then the sets sD_j are connected, which implies that $C(s, y) = \bigcap \overline{sD_j}$ is a compact connected set in $\bar{G}$. If $x \in G \cap C(s, y)$, then $f(x) = y$ by the continuity of f. Thus $G \cap C(s, y) \subset f^{-1}(y)$. Since $f^{-1}(y)$ is totally disconnected, either $G \cap C(s, y) = \emptyset$ or $G \cap C(s, y) = C(s, y)$ consists of a single point.

3.11. *Path lifting.* The path lifting problem for light open mappings has been considered by Stoïlow [23, p. 354], [24, p. 109], Whyburn [27, p. 186], Floyd [5, p. 574], and by us [13, 2.7]. We remark that our result [13, 2.7] is a direct corollary of Floyd's theorem, which was unfortunately overlooked in [13]. We shall now give the global version of this result.

Let $f : G \to \bar{R}^n$ be a mapping, let $\beta : [a, b) \to \bar{R}^n$ be a path, and let $x_0 \in f^{-1}(\beta(a))$. We say that a path $\alpha : [a, c) \to G$ is a *maximal lifting of* β *starting at* x_0 if:

(1) $\alpha(a) = x_0$.

(2) $f \circ \alpha = \beta | [a, c)$.

(3) If $c < c' \leqq b$, then there does not exist a path $\alpha' : [a, c') \to G$ such that $\alpha = \alpha' | [a, c)$ and $f \circ \alpha' = \beta | [a, c')$.

Similarly, we define the maximal lifting of a path $\beta : (a, b] \to \bar{R}^n$ terminating at a point $x_0 \in f^{-1}(\beta(b))$.

If $\beta : [a, b) \to \bar{R}^n$ is a path and if $C \subset \bar{R}^n$, we say that $\beta(t) \to C$ as $t \to b$ if the spherical distance $q(\beta(t), C) \to 0$.

3.12. **Lemma.** *Suppose that* $f : G \to \bar{R}^n$ *is light and open, that* $x_0 \in G$, *and that* $\beta : [a, b) \to \bar{R}^n$ *is a path such that* $\beta(a) = f(x_0)$ *and such that either* $\lim_{t \to b} \beta(t)$ *exists, or* $\beta(t) \to \partial fG$ *as* $t \to b$. *Then* β *has a maximal lifting* $\alpha : [a, c) \to G$. *If* $\alpha(t) \to x_1 \in G$ as $t \to c$, *then* $c = b$ *and* $f(x_1) = \lim_{t \to b} \beta(t)$. *Otherwise* $\alpha(t) \to \partial G$ *as* $t \to c$. *If* f *is discrete and if the local degree* $i(\alpha(t), f)$ *is constant for* $t \in [a, c)$, *then* α *is the only maximal lifting of* β *starting at* x_0.

Proof. Let P be the set of all pairs (α, c) such that $a < c \leqq b$ and $\alpha : [a, c) \to G$ is a path satisfying the conditions (1) and (2) in 3.11. From [13, 2.7] it follows that $P \neq \emptyset$. Define an ordering in P as follows:

$(\alpha, c) \leqq (\alpha', c')$ if $c \leqq c'$ and $\alpha = \alpha' | [a, c)$. By Zorn's lemma it is easy to see that P contains a maximal element (α, c). Then α is a maximal lifting of β starting at x_0.

Suppose that $\alpha(t) \to x_1 \in G$ as $t \to c$. Since f is continuous, $\beta(t) = f(\alpha(t)) \to f(x_1)$. If $c < b$, we can use [13, 2.7] to construct a lifting $(\alpha', c') > (\alpha, c)$. Hence $c = b$.

Next assume that $\alpha(t)$ does not tend to a point $x_1 \in G$ as $t \to c$. We must show that $\alpha(t) \to \partial G$. Suppose that this is not true. Then there is a compact set $F \subset G$ and a sequence $t_1 < t_2 < \ldots$ such that $t_j \in [a, c)$, $t_j \to c$, and $\alpha(t_j) \in F$. We may assume that there is $x_1 \in F$ such that $\alpha(t_j) \to x_1$. Since $\alpha(t) \not\to x_1$, there is a neighborhood U of x_1 and a sequence (s_j) such that $\bar{U} \subset G$, $t_j < s_j < c$ and $\alpha(s_j) \notin U$. We may assume that $\alpha(s) \in U$ for $s \in [t_j, s_j)$. Let C_j be the continuum $\alpha[t_j, s_j] \subset \bar{U}$. Then $x_1 \in \liminf C_j$. By [8, 2—101, p. 101], $\limsup C_j = C$ is connected. If $\beta(t)$ tends to a point y as $t \to c$, then $fC = \{y\}$. Since f is light, $C = \{x_1\}$. This is impossible, because each C_j meets ∂U. Hence $\beta(t)$ has no limit as $t \to c$. By the hypothesis of the theorem, $c = b$ and $\beta(t) \to \partial fG$ as $t \to b$. Hence there is $s < b$ such that $\beta(t) \notin fF$ for $t \in (s, b)$. On the other hand, $\beta(t_j) = f(\alpha(t_j)) \in fF$. This contradiction shows that $\alpha(t) \to \partial G$ as $t \to c$.

Now suppose that f is discrete and that $i(\alpha(t), f) = k$ is constant for $t \in [a, c)$. We may assume that f is sense-preserving. Let $\alpha' : [a, c') \to G$ be another maximal lifting of β starting at x_0. Suppose that $\alpha \neq \alpha'$. Then $E = \{t \mid \alpha(t) \neq \alpha'(t)\} \neq \emptyset$. Set $v = \inf E$. Since $a \notin E$, E is an open set in (a, c). Hence $v \notin E$. Choose a normal neighborhood V of $\alpha(v) = \alpha'(v)$. Then there is $t \in (v, c)$ such that $\alpha(t) \in V$, $\alpha'(t) \in V$ and $\alpha(t) \neq \alpha'(t)$. Then

$$k = \mu(f, V) = \sum \{ i(x, f) \mid x \in V \cap f^{-1}(\beta(t)) \} \geqq i(\alpha(t), f) + i(\alpha'(t), f) \geqq k + i(\alpha'(t), f).$$

This contradiction completes the proof of the lemma.

3.13. *Definition.* If $f : G \to \bar{R}^n$ is a mapping, a point $z \in \bar{R}^n$ is said to be an *asymptotic value* of f at a boundary point $b \in \partial G$ if there is a path $\alpha : [0, 1) \to G$ such that $\alpha(t) \to b$ and $f(\alpha(t)) \to z$ as $t \to 1$.

3.14. **Theorem.** *Suppose that b is an essential isolated singularity of a quasimeromorphic mapping $f : G \to \bar{R}^n$, $n \geqq 3$. If z is an asymptotic value of f at b, then $z \in \overline{f(B_f \cap U)}$ for every neighborhood U of b.*

Proof. Assume that there is a neighborhood U of b such that z does not belong to the closure of $f(B_f \cap U)$. Using the methods of Zorič [28] and Agard-Marden [1] we show that this leads to a contradiction.

We may assume that $b = 0 = z$. Pick $r_0 > 0$ such that $\bar{B}^n(r_0) \subset U \cap (G \cup \{0\})$ and such that $S^{n-1}(r_0) \cap f^{-1}(0) = \emptyset$. Set $U_0 = B^n(r_0) \setminus \{0\}$ and $g = f \mid U_0$. Choose $r' > 0$ such that $\bar{B}^n(r')$ does not meet $fS^{n-1}(r_0) \cup gB_g$. Since 0 is an asymptotic value, there is a path $\alpha : [0, 1) \to U_0$ such that $\alpha(t) \to 0$ and $\beta(t) = f(\alpha(t)) \to 0$ as $t \to 1$. We may assume that $0 < |\beta(t)| < r'$ for all $t \in (0, 1)$. Thus $|\alpha| \subset U_0 \setminus B_g$. For $0 \leqq t < 1$ and $0 < \varphi \leqq \pi$ we define the spherical caps

$$C(t, \varphi) = \{y \in R^n \mid |y| = |\beta(t)|, (y \mid \beta(t)) > |y|^2 \cos \varphi\}.$$

Let $C^*(t, \varphi)$ be the $\alpha(t)$-component of $g^{-1}C(t, \varphi)$, and let φ_t be the least upper bound of all $\varphi \in (0, \pi]$ such that g maps $C^*(t, \varphi)$ homeomorphically onto $C(t, \varphi)$. Set $C(t) = C(t, \varphi_t)$, $C^*(t) = C^*(t, \varphi_t)$. Then g defines a homeomorphism $g_t : C^*(t) \to C(t)$. We show that for almost every $r \in (0, r')$, $|\beta(t)| = r$ implies $0 \notin \bar{C}^*(t)$.

If $0 \in \bar{C}^*(t)$, it follows from Lemma 3.10 that there is a point $y_t \in \bar{C}(t)$ such that $g_t^{-1}(y) \to 0$ as $y \to y_t$ in $C(t)$. Let $\Gamma(t)$ be the family of all paths which join $\beta(t)$ and y_t in $C(t)$, and let $\Gamma^*(t) = g_t^{-1}\Gamma(t)$ and $\Gamma^* = \cup \Gamma^*(t)$ over all t such that $0 \in \bar{C}^*(t)$. Since all paths of Γ^* converge to 0, $M(\Gamma^*) = 0$. By 2.14, this implies $M(g\Gamma^*) = 0$. On the other hand, a well-known modulus estimate [26, 10.2] yields

$$M(g\Gamma^*) \geqq b_n \int_E \frac{dr}{r}$$

where $b_n > 0$ depends only on n and where $E = \{|\beta(t)| \mid 0 \in \bar{C}^*(t)\}$. Hence $m_1(E) = 0$.

Let $T = \{t \mid 0 \leqq t < 1, |\beta(t)| \notin E\}$. If $t \in T$, then $\bar{C}^*(t) \subset U_0 \setminus B_g$. From Lemma 2.2 it follows that f maps $\bar{C}^*(t)$ homeomorphically onto $\bar{C}(t)$. By 3.8, f is injective in a neighborhood of $\bar{C}^*(t)$. This is possible only if $\varphi_t = \pi$. Hence for every $t \in T$, $\bar{C}^*(t) = \bar{C}^*(t, \pi)$ is a topological sphere in $U_0 \setminus B_g$, and f maps $\bar{C}^*(t)$ homeomorphically onto $\bar{C}(t) = S^{n-1}(|\beta(t)|)$. Let $D(t)$ be the bounded component of $\complement \bar{C}^*(t)$. Let $T_0 = \{t \in T \mid 0 \in D(t)\}$. We divide the rest of the proof into two cases, according as 1 belongs to $\bar{T}_0$ or not.

Case 1. $1 \in \bar{T}_0$. We choose an increasing sequence of numbers $t_j \in T_0$ such that $t_j \to 1$. Let $r_j = |\beta(t_j)|$ and $D_j = D(t_j)$. Passing to a subsequence we may assume that $r_{j+1} < r_j$. Since $\alpha(t_j) \to 0$, we may also assume that $D_{j+1} \subset D_j$. Let A_j be the spherical ring $B^n(r_1) \setminus \bar{B}^n(r_j)$. Since g is injective in a neighborhood of ∂D_1, there is a component A_j^* of $g^{-1}A_j$ such that $\partial A_j^* \supset \partial D_1$. Since $\partial D_j \cap A_j^* = \emptyset$ and since $\bar{A}_j \cap gB_g = \emptyset$, $\bar{A}_j^* \subset U_0 \setminus B_g$. By 2.2, f maps A_j^* homeomorphically onto A_j. Hence there is a section $s_j : A_j \to A_j^*$ of f. Moreover, $s_j = s_k \mid A_j$ for

$k > j$. Hence we obtain a section $s: B^n(r_1) \setminus \{0\} \to U_0 \setminus B_g$. By 3.10, we can extend s to a continuous mapping $\bar{s}$ of $B^n(r_1)$. This is possible only if $\bar{s}(0) = 0$, which implies that 0 is a removable singularity of f. We remark that hitherto our proof is not essentially different form that in Agard-Marden [1].

Case 2. $1 \notin \bar{T}_0$. We can find a path $\bar{\alpha}: [-1,1) \to R^n$ such that $\bar{\alpha}(-1) \in \partial U_0$, $\alpha(-1,1) \subset U_0$, $\bar{\alpha} \mid [0,1) = \alpha$, and $\bar{\beta}(t) = f(\bar{\alpha}(t)) \neq 0$ for $t \in [-1,1)$. By assumption, there is $\delta, 0 \leqq \delta < 1$, such that $[\delta, 1) \cap T_0 = \emptyset$. We choose an increasing sequence of points $t_j \in T \cap [\delta, 1)$ such that (1) $t_j \to 1$, (2) $|\beta(t)| < r_j = |\beta(t_j)|$ for $t \in (t_j, 1)$, (3) $|\bar{\beta}(t)| > r_{j+1}$ for $t \in [-1, t_j]$. As above, we set $D_j = D(t_j)$. There are two subcases: (a) $D_j \subset D_{j+1}$ for all j, (b) $\bar{D}_j \cap \bar{D}_{j+1} = \emptyset$ for some j.

Suppose that (a) is true. For $j > 1$ let A_j be the spherical ring $B^n(r_1) \setminus \bar{B}^n(r_j)$. As in the case 1 we conclude that there is a component A_j^* of $g^{-1}A_j$ such that f maps A_j^* homeomorphically onto A_j and such that $\partial D_1 \subset \partial A_j^*$. Since $\alpha(t_1, 1) \subset \complement \bar{D}_1$, $A_j^* \subset D_j \setminus \bar{D}_1$. Proceeding as before we obtain a section $s: B^n(r_1) \setminus \{0\} \to U_0 \setminus B_g$ of f. Now the cluster set $C(s, 0)$ is a non-degenerate continuum in $\bar{U}_0$. This is in contradiction with 3.10.

Suppose next that (b) is true. We first observe that $\alpha(t_j, 1) \subset \complement \bar{D}_j$. Set $u_{j+1} = \sup\{t \mid \alpha(t_j, t) \subset \complement \bar{D}_{j+1}\}$. Choose a neighborhood U_{j+1} of ∂D_{j+1} such that $f \mid U_{j+1}$ is injective. Since $\beta(t_{j+1}, 1) \subset B^n(r_{j+1})$, $g(U_{j+1} \cap \complement \bar{D}_{j+1}) \subset B^n(r_{j+1})$. Hence there exists $v_1 = \max\{t \mid t_j < t < u_{j+1}, |\beta(t)| = r_{j+1}\}$. By definition $v_1 \in T$. Since $v_1 > \delta$ and since $\alpha(t_{j+1}, 1) \cap g^{-1}S^{n-1}(r_{j+1}) = \emptyset = \bar{\alpha}(-1, t_j) \cap g^{-1}S^{n-1}(r_{j+1})$, it follows that $\bar{D}(v_1) \subset \complement(\bar{D}_j \cup \bar{D}_{j+1})$. Hence $v_1' = \sup\{t \mid \alpha(t_j, t) \subset \complement \bar{D}(v_1)\} > t_j$. As above, there exists $v_2 = \max\{t \mid t_j < t < v_1', |\beta(t)| = r_{j+1}\}$ and we have $\bar{D}(v_2) \subset \complement(\bar{D}_j \cup \bar{D}_{j+1} \cup \bar{D}(v_1))$. By continuing this process we find an infinite number of components $\bar{C}^*(v_i)$ of $g^{-1}S^{n-1}(r_{j+1})$ with $v_i \in (t_j, v_{j+1})$. Hence there exists a limit point $v \in (t_j, u_{j+1})$ of the set $\{v_i \mid i = 1, 2, \ldots\}$. Every neighborhood of $\alpha(v)$ intersects infinitely many components of $g^{-1}S^{n-1}(r_{j+1})$. This is a contradiction because f is a local homeomorphism at $\alpha(v)$. The theorem is proved.

3.15. **Theorem.** *Suppose that* $n \geqq 3$ *and that* b *is an essential isolated singularity of a quasimeromorphic mapping* $f: G \to \bar{R}^n$. *Then* $\complement fG \subset \overline{fB_f}$.

Proof. Suppose that $y \in \complement fG \setminus \overline{fB_f}$. We may assume that $b = 0 = y$. Choose $r_0 > 0$ such that $\bar{B}^n(r_0) \subset G \cup \{0\}$ and set $U_0 = B^n(r_0) \setminus \{0\}$. Next choose $r' > 0$ such that $\bar{B}^n(r') \cap (fB_f \cup fS^{n-1}(r_0)) = \emptyset$. By [14, 4.4], cap $\complement fU_0 = 0$. Hence we can find a cap C of $S^{n-1}(r')$ and a set $C^* \subset U_0$ such that f maps C^* homeomorphically onto C. For each $y \in C$ let γ_y: $(0, 1] \to R^n$ be the linear path $\gamma_y(t) = ty$. Let γ_y^* be the (unique) maximal lifting of γ_y, terminating in C^*. This means that $\gamma_y^* : (r_y , 1] \to U_0$ is a path such that $0 \leqq r_y < 1$, $\gamma_y^*(1) \in C^*$, $f \circ \gamma_y^* = \gamma_y \,|(r_y , 1]$, and $\gamma_y^*(t) \to 0$ as $t \to r_y$. We show that $r_y = 0$ for almost every $y \in C$. Let $E_i = \{y \in C \mid r_y > 1/i\}$. It suffices to show that $m_{n-1}(E_i) = 0$. Set $\Gamma_i = \{\gamma_y^* \mid y \in E_i\}$. Since all paths of Γ_i converge to 0, $M(\Gamma_i) = 0$. By 2.14, this implies $M(f\Gamma_i) = 0$. On the other hand, $f\Gamma_i$ minorizes the family Δ of all segments $\alpha_y : [1/i , 1] \to R^n$, $\alpha_y(t) = ty$, $y \in E_i$. Consequently, a well-known formula [26, 7.7] yields

$$M(f\Gamma_i) \geqq M(\Delta) = r'^{1-n} m_{n-1}(E_i)(\log i)^{1-n} .$$

Thus $m_{n-1}(E_i) = 0$.

Choose $y \in C$ such that $r_y = 0$. If $t \to 0$, then $\gamma_y^*(t) \to 0$ and $f(\gamma_y^*(t)) = \gamma_y(t) \to 0$. Thus 0 is an asymptotic value of f at 0. By 3.14, this is a contradiction.

3.16. **Corollary.** *If $n \geqq 3$ and if $f : G \to R^n$ is a quasiregular mapping which has an essential isolated singularity, then fB_f is unbounded.*

3.17. **Corollary.** *If $n \geqq 3$ and if b is an essential isolated singularity of a quasiregular mapping $f : G \to R^n$, then $b \in \bar{B}_f$.*

3.18. *Remarks.* Corollary 3.17 is a special case of a theorem of Agard and Marden [1]. See also Zorič [29].

We shall indicate briefly how the proof of 3.15 can be modified so as to yield an n-dimensional version of Iversen's theorem: *If b is an essential isolated singularity of a quasimeromorphic mapping $f : G \to \bar{R}^n$, then every point in $\complement fG$ is an asymptotic value of f.* The lifting of γ_y is not necessarily unique, but we can use 3.12 to find a maximal lifting γ_y^* converging to b. The proof of the inequality $M(f\Gamma_i) \leqq KM(\Gamma_i)$ can be based on the idea of the proof of 2.14 together with [13, 7.10].

3.19. *The structure of fB_f.* Suppose that a quasimeromorphic mapping $f : G \to \bar{R}^n$ has an isolated essential singularity and that $y \in \complement fG$. By Theorem 3.15, every neighborhood U of y meets fB_f. We shall now study, for $n = 3$, the structure of the set $U \cap fB_f$. Zorič [28] has given an example of a quasiregular mapping $f : R^3 \to R^3$ which has an essential singularity at ∞ and for which $\complement fR^3 = \{0 , \infty\}$. In this example fB_f consists of four rays starting from 0. It is easy to modify this example so that fB_f consists of three rays starting from 0. We shall show that the number of the rays cannot be reduced to two.

We first give a factorization lemma, which is due to Church and Hemmingsen [4]. Let (r , φ , z) be the cylindrical coordinates in R^n. Thus

$r \geqq 0$, $\varphi \in R^1 \pmod{2\pi}$, $z \in R^{n-2}$, and $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$, $(x_3, \ldots, x_n) = z$. For every non-zero integer k, we define the *winding mapping* $g_k : R^n \to R^n$ by $g_k(r, \varphi, z) = (r, k\varphi, z)$. If $k > 0$, g_k is quasiregular with $K_I(g_k) = k$, $K_O(g_k) = k^{n-1}$.

3.20. **Lemma.** *Suppose that* $f : G \to R^n$ *is discrete and open, that* $x_0 \in B_f$, *and that* $f(x_0) = 0$. *Suppose also that* x_0 *has a neighborhood* V *such that* $f(B_f \cap V)$ *is contained in the* $(n-2)$*-dimensional subspace* $Z = \{y \in R^n \mid y_1 = y_2 = 0\}$. *Then there is a neighborhood* $U = U(x_0, f, r)$ of x_0 *and a homeomorphism* h *of* U *onto a ball* $B^n(r)$ *such that* $f \mid U = g_k \circ h$, $k = i(x_0, f)$.

Proof. Choose $r > 0$ such that $U = U(x_0, f, r)$ is a normal neighborhood of x_0 and $U \subset V$. Set $g = f \mid U$. Since f defines a k-to-one covering mapping of $U \setminus g^{-1}gB_g$ onto $B^n(r) \setminus gB_g$, $B^n(r) \setminus gB_g$ is not simply connected (cf. [4, 5.2]). Since $gB_g \subset Z$, this implies $gB_g = Z \cap B^n(r)$. The lemma follows from [4, 4.1].

3.21. *Definition.* A set $A \subset \bar{R}^n$ is said to be a *quasiconformal p-ball* if there is a neighborhood U of A and a quasiconformal mapping g of U such that $gA = B^p$. If $p = 1$, A is called a *quasiconformal arc.*

3.22. **Theorem.** *Suppose that* b *is an isolated essential singularity of a quasimeromorphic mapping* $f : G \to \bar{R}^3$ *and that* $y \in \complement fG$. *Then* $V \cap fB_f$ *is not contained in a quasiconformal arc for any neighborhood* V *of* y.

Proof. The idea of the proof is much similar to that used in Zorič [28], [29], Agard-Marden [1], and in the proofs of 2.3 and 3.14. We shall therefore omit some details. We may assume that $b = 0 = y$. Suppose that $V \cap fB_f$ is contained in a quasiconformal arc for some neighborhood V of 0. We may assume that $V \cap fB_f \subset Z = \{x \in R^3 \mid x_1 = x_2 = 0\}$. Fix $r_0 > 0$ such that $\bar{B}^3(r_0) \subset G \cup \{0\}$, and set $U_0 = B^3(r_0) \setminus \{0\}$, $g = f \mid U_0$. By 3.15, there is $r' \neq 0$ such that $r'e_3 \in gB_g$, $\bar{B}^3(|r'|) \subset V$ and $\bar{B}^3(|r'|) \cap fS^2(r_0) = \emptyset$. We may assume that $r' > 0$. Choose $x_0 \in g^{-1}(r'e_3) \cap B_g$. By 3.20, g is topologically equivalent to a winding mapping g_k in a neighborhood of x_0. Let $\beta : (0, r'] \to R^3$ be the path $\beta(t) = te_3$. Using 3.12 we choose a maximal lifting $\alpha : (\delta, r'] \to G$ of β terminating at x_0. From 3.20 it follows that for every $t \in (\delta, r']$ g is topologically equivalent to g_k in some neighborhood of $\alpha(t)$. Furthermore, it follows from 3.12 that $\alpha(t) \to \partial U_0$ as $t \to \delta$, which is possible only if $\alpha(t) \to 0$.

For $0 < r < r' - \delta$ consider the cap

$$C(r, \varphi) = \{y \in R^3 \mid |y - \delta e_3| = r, \ y_3 > \delta + r \cos \varphi\},$$

$0 < \varphi \leqq \pi$. Let $C^*(r, \varphi)$ be the component of $g^{-1}C(r, \varphi)$ which contains $\alpha(\delta + r)$. Let φ_r be the least upper bound of all $\varphi \in (0, \pi]$ such that $C^*(r, \varphi)$ is homeomorphic to $C(r, \varphi)$, and such that f defines a k-to-one covering mapping of $C^*(r, \varphi) \setminus |\alpha|$ onto $C(r, \varphi) \setminus |\beta|$. Let $E = \{r \in (0, r' - \delta) \mid 0 \in \bar{C}^*(r, \varphi_r)\}$. Using similar path family estimates as in the proof of 3.14 and applying 2.14 we see that $m_1(E) = 0$. It is not difficult to see that if $r \in (0, r' - \delta) \setminus E$, then $\varphi_r = \pi$, $\bar{C}^*(r, \varphi_r)$ is a topological 2-sphere, and the mapping $\bar{C}^*(r, \varphi_r) \to \bar{C}(r, \varphi_r) = S^2(\delta e_3, r)$ is topologically equivalent to $g_k|S^2$. Choose a sequence $r_1 > r_2 > \ldots$ such that $r_i \in (0, r' - \delta) \setminus E$ and $r_i \to 0$. If $\delta > 0$, we choose $r_1 < \delta$. Let D_i be the bounded component of $\complement\bar{C}^*(r_i, \varphi_{r_i})$. Then $\bar{D}_i \subset B^3(r_0)$, and each ∂D_i separates the points x_0 and 0 . Passing to a subsequence we may assume that one of the following cases occurs: (1) $0 \in D_{i+1} \subset D_i$ for all i or (2) $x_0 \in D_i \subset D_{i+1}$ for all i.

Suppose that (1) is true. We first show that $A_i = D_i \setminus \bar{D}_{i+1}$ is a normal domain. Suppose that this is not true. Then $A_i \cap f^{-1}f\partial A_i \neq \emptyset$. Let Q be a component of this set. From 3.7 it follows that there is a neighborhood U of ∂A_i such that $M^*(f, U) = k$. Hence $U \cap Q = \emptyset$, which implies that Q is compact. From [27, (7.5), p. 148] it follows that $fQ = S_j = S^2(\delta e_3, r_j)$ where $j = i$ or $i + 1$. Suppose first that $\delta = 0$. Let $\beta_j : (0, r_j] \to R^3$ be the path $\beta_j(t) = te_3$. By 3.12, β_j has a maximal lifting $\alpha_j : (c_j, r_j] \to G$ terminating at a point $x_1 \in Q$. Moreover, $\alpha_j(t) \to 0$ as $t \to c_j$. Hence there is $t_0 \in (c_j, r_j)$ such that $\alpha_j(t_0) \in \partial D_{i+1}$. If $j = i + 1$, this is clearly impossible. If $j = i$, then $t_0 = r_{i+1}$, and $\alpha_i(t_0)$ is the unique point in $\partial D_{i+1} \cap f^{-1}(r_{i+1}e_3)$. Hence $\alpha_i|[r_{i+1}, r_i]$ and $\alpha|[r_{i+1}, r_i]$ are both liftings of $\beta|[r_{i+1}, r_i]$ starting at $\alpha(r_{i+1})$. Since $i(\alpha(t), f)$ is constant, it follows from the last statement of 3.12 that $\alpha(t) = \alpha_i(t)$ for $t \in [r_{i+1}, r_i]$, which is impossible. Next assume that $\delta > 0$. We let $\beta_j' : (0, \delta - r_j] \to R^3$ be the path $\beta_j'(t) = te_3$. Choose a maximal lifting $\alpha_j' : (c_j', \delta - r_j] \to G$ of β_j' terminating at a point in Q . Then $\alpha_j'(t) \to 0$ as $t \to c_j'$. Hence there is $t_0' \in (c_j', \delta - r_j)$ such that $\alpha_j'(t_0') \in \partial D_{j+1}$. This is clearly impossible in both cases $j = i$ and $j = i + 1$. We have proved that A_i is a normal domain.

Let Γ_i be the family of all paths joining the boundary components of A_i in A_i . Lemma 3.9 yields $N(f, A_i) = N(f, \partial A_i) = k$. Using [13, 3.2 and 5.9] we obtain

$$\left(b_3 \frac{d(D_i)^3}{m(D_1)}\right)^{\frac{1}{2}} \leqq M(\Gamma_i) \leqq kK(f)M(f\Gamma_i) \leqq 4\pi kK(f) \left(\log \frac{r_1}{r_i}\right)^{-2} .$$

Hence $d(D_i) \to 0$ as $i \to \infty$. Thus $f(D_1 \setminus \{0\}) \subset B^3(r_1)$, which is a contradiction by [14, 4.6].

Suppose next that (2) is true. As above, we see that the ring $A_i =$

$D_i \setminus \bar{D}_1$ is a normal domain. Using similar path family estimates we obtain

$$\left(b_3 \frac{m(D_1)^3}{m(U_0)}\right)^{\frac{1}{2}} \leqq M(\Gamma_i) \leqq kK(f)M(f\Gamma_i) \leqq 4\pi kK(f)\left(\log \frac{r_1}{r_i}\right)^{-2} .$$

As $i \to \infty$, this gives a contradiction. The theorem is proved.

3.23. *Remark.* The function $f(z) = e^z$ shows that none of the results 3.14, 3.15, 3.16, 3.17, 3.22 is true for $n = 2$.

3.24. *Tamely embedded branch set image.* Now we shall study the case where fB_f is a tamely embedded $(n-2)$-manifold. From the theorem of Church and Hemmingsen (Lemma 3.20 above) it follows that B_f is also tamely embedded and that f is locally topologically equivalent to a winding mapping g_k. If f is quasiregular, we can go a bit further.

3.25. **Theorem.** *Let $f: G \to R^n$ be quasiregular and let $x_0 \in B_f$. Suppose that there is a neighborhood U of x_0 such that $f(U \cap B_f)$ is contained in a quasiconformal $(n-2)$-ball. Then there exists a neighborhood V of x_0 and quasiconformal mappings $h_1 : V \to B^n$, $h_2 : B^n \to V' \subset fU$ such that $f \mid V = h_2 \circ g_k \circ h_1$, where $k = i(x_0, f)$. In particular, $B_f \cap V$ is a quasiconformal $(n-2)$-ball.*

Proof. There is a neighborhood V' of $f(x_0)$ and a quasiconformal mapping $g: V' \to B^n$ such that $V' \subset fU$, $g(f(x_0)) = 0$, and $gf(U \cap B_f) \subset Z = \{x \in R^n \mid x_1 = x_2 = 0\}$. By 3.20, there is a neighborhood V of x_0 and a homeomorphism $k_1 : V \to B^n(r)$ such that $g \circ f \mid V = g_k \circ k_1$. Setting $h_1(x) = k_1(x)/r$ and $h_2(y) = g^{-1}(ry)$ we have $f \mid V = h_2 \circ g_k \circ h_1$. It remains to show that h_1 is quasiconformal. Every point $w \in B^n \setminus Z$ has a neighborhood W such that $\varphi = f \mid h_1^{-1} W$ is injective. Since $h_1^{-1} \mid W = \varphi^{-1} \circ h_2 \circ g_k \mid W$, h_1^{-1} is locally K-quasiconformal in $B^n \setminus Z$, where $K = K(f)K(h_2) K(g_k)$. Since $\Lambda_{n-1}(Z) = 0$, it follows from [6, p. 380] or [26, 35.1] that h_1^{-1}, and hence h_1 is quasiconformal.

3.26. We shall next study the case $n = 3$ in more detail. A bounded arc $J \subset R^n$ is said to have *bounded oscillation* [11, p. 107] if there is a constant $a < \infty$ such that $|x_1 - x_2| \leqq a\, |x_1 - x_3|$ holds for successive points x_1, x_2, x_3 of J. We shall show that if fB_f is a tamely embedded arc, then both B_f and fB_f are arcs with locally bounded oscillation. In particular, if they are piecewise smooth arcs, no zero angles can occur. For example, fB_f cannot be the set $\{x \in R^3 \mid x_3 = 0, x_1 \geqq 0, |x_2| = x_1^2\}$. We first need a modulus estimate.

3.27. **Lemma.** *Suppose that* $F \subset \bar{R}^n$ *is connected and that* $\alpha : [0, 1] \to \complement F$ *is a loop which is not homotopically trivial in* $\complement F$ *. Let* Γ *be the family of all paths which join* $|\alpha|$ *and* F *. Then* $M(\Gamma) \geqq c_n > 0$ *where* c_n *depends only on* n *.*

Proof. Choose $a \in |\alpha|$ and $b \in \bar{F}$ such that $|a-b| = d(|\alpha|, F) = s$. If $s = 0$, the spheres $S^{n-1}(a, t)$ meet both $|\alpha|$ and F for sufficiently small t, and $M(\Gamma) = \infty$ by [26, 10.12]. If $s > 0$, we set $x_0 = (a+b)/2$. Since α is not homotopically trivial in $\complement F$, $|\alpha|$ contains a point in $S^{n-1}(a, s) \setminus B^n(b, s)$. Since $\complement B^n(b, s)$ is simply connected, F contains a point in $S^{n-1}(b, s) \setminus B^n(a, s)$. Hence the sphere $S^{n-1}(x_0, t)$ meets both $|\alpha|$ and F whenever $s/2 < t < s\sqrt{3}/2$. Hence [26, 10.12] yields $M(\Gamma) \geqq b_n \log \sqrt{3}$.

3.28. *The linking number.* We also need the notion of a linking number, see e.g. Spanier [22, p. 361]. If the orientation of $\bar{R}^3$ is fixed, the linking number $\mathrm{lk}(y, z)$ is an integer, defined for 1-dimensional singular cycles y, z in $\bar{R}^3$ such that $|y| \cap |z| = \emptyset$. We recall some properties of the linking number.

(1) $\mathrm{lk}(y, z) = \mathrm{lk}(z, y)$.

(2) $\mathrm{lk}(y, z) + \mathrm{lk}(y', z) = \mathrm{lk}(y + y', z)$.

(3) If $y \sim y'$ in $\complement|z|$, then $\mathrm{lk}(y, z) = \mathrm{lk}(y', z)$. In particular, $\mathrm{lk}(y, z) = 0$ if y bounds in $\complement|z|$.

(4) If y represents a line in $\bar{R}^3$ and if z represents an orthogonal circle with center in $|y|$, then $\mathrm{lk}(y, z) = \pm 1$.

(5) The linking number is invariant under sense-preserving homeomorphisms.

3.29. **Lemma.** *Let* $k \geqq 2$, *let* $g_k : R^3 \to R^3$ *be the winding mapping* $g_k(r, \varphi, z) = (r, k\varphi, z)$, *and let* Z *be the line* $\{te_3 \mid t \in R^1\}$. *Suppose that* $-1 < a < b < c < 1$, *and that* E, F *are open arcs such that* E *joins* be_3 *and a point* $P \in S^2 \setminus Z$ *in* $B^3 \setminus Z$, *and such that* F *joins* ae_3 *and* ce_3 *in* $B^3 \setminus Z \setminus E$. *Then* $g_k^{-1}\bar{F}$ *contains a loop which is not homotopically trivial in* $B^3 \setminus g_k^{-1}\bar{E}$.

Proof. Let $h : \bar{R}^3 \to \bar{R}^3$ be the rotation $h(r, \varphi, z) = (r, \varphi + 2\pi/k, z)$. Fix a component E_0 of $g_k^{-1}E$ and a component F_0 of $g_k^{-1}F$. Let (r_0, φ_0, z_0) be the unique point in $\bar{E}_0 \cap S^2$. We choose injective singular 1-simplexes $\sigma_0, \tau_0, \sigma, \tau$ such that $|\sigma_0| = \bar{E}_0$, $|\tau_0| = \bar{F}_0$, $|\sigma| = \{(r_0, \varphi, z_0) \mid \varphi_0 \leqq \varphi \leqq \varphi_0 + 2\pi/k\}$, $|\tau| = (Z \setminus \{te_3 \mid a < t < c\}) \cup \{\infty\}$, and choose the orientations so that the chains $y_0 = \sigma_0 + \sigma - h\sigma_0$ and $w_0 = \tau_0 + \tau$ are cycles.

For $0 \leqq j \leqq k-1$ we define the cycles $y_j = hy_{j-1}$ and $w_j = hw_{j-1}$. Then $y = \Sigma y_j$ represents the circle $r = r_0, z = z_0$. Since w_0 is in the obvious sense homotopic to $Z \cup \{\infty\}$ in $\mathbf{C}|y|$, it follows from (3) and (4) in 3.28 that $\operatorname{lk}(y, w_0) = \pm 1$. On the other hand, 3.28 (5) implies $\operatorname{lk}(y_0, w_j) = \operatorname{lk}(y_{k-j}, w_0)$ for $1 \leqq j \leqq k-1$. Hence

$$\sum_{j=0}^{k-1} \operatorname{lk}(y_0, w_j) = \sum_{j=0}^{k-1} \operatorname{lk}(y_j, w_0) = \operatorname{lk}(y, w_0) = \pm 1 .$$

This implies

$$\sum_{j=0}^{k-1} \operatorname{lk}(y_0, w_j - w_0) = \pm 1 - k \operatorname{lk}(y_0, w_0) \neq 0 .$$

Hence there is j such that $\operatorname{lk}(y_0, w_j - w_0) \neq 0$. On the other hand, $w_j - w_0 = h^j\tau_0 - \tau_0$ represents a Jordan curve $J \subset g_k^{-1}\bar{F}$. By 3.28(3), $w_j - w_0 \not\sim 0$ in $\mathbf{C}|y_0| \supset B^3 \setminus g_k^{-1}\bar{E}$. Hence J is not homotopically trivial in $B^3 \setminus g_k^{-1}\bar{E}$.

3.30. **Theorem.** *Let $f: G \to R^3$ be a quasiregular mapping, let $x \in B_f$, and suppose that there exist a neighborhood U of x and a homeomorphism $g: fU \to B^3$ such that $g(f(x)) = 0$ and $gf(B_f \cap U) = B^1$. Then there is a neighborhood V of x such that $V \cap B_f$ and $f(V \cap B_f)$ are Jordan arcs with bounded oscillation.*

Proof. We first show that $f(W \cap B_f)$ has bounded oscillation for some neighborhood W of x. By 3.20, there exist $r_1 > 0$ and a homeomorphism h of $\bar{U}_1 = \bar{U}(x, g \circ f, r_1)$ onto $\bar{B}^3(r_1)$ such that $g \circ f \mid \bar{U}_1 = g_k \circ h$, where $k = i(x, f)$. Choose $t > 0$ such that $B^3(f(x), 2t) \subset fU_1$. Next choose $r > 0$ such that $U(f(x), g, r) \subset B^3(f(x), t)$. Set $W = U(x, g \circ f, r)$ and $J = B_f \cap W$. We claim that $J' = fJ$ has bounded oscillation. Suppose that this is not the case. Then there is for every $m > 1$ a triple y_1, y_2, y_3 of successive points of J' such that $|y_1 - y_2| > m\,|y_1 - y_3|$. Choose an open arc $E \subset f(U_1 \setminus B_f) \setminus B^3(y_1, |y_1 - y_2|)$ which joins y_2 and a point $P \in \partial fU_1$. Next choose an open arc $F \subset B^3(f(x), t) \cap B^3(y_1, |y_1 - y_3|) \setminus f(B_f \cap U_1)$ which joins y_1 and y_3. Let Γ be the family of all paths joining $U_1 \cap f^{-1}\bar{F}$ and $\partial U_1 \cup f^{-1}\bar{E}$ in U_1. Since $f \mid \bar{U}_1$ is topologically equivalent to g_k, it follows from Lemma 3.29 that there is a loop α in $U_1 \cap f^{-1}\bar{F}$ which is not homotopically trivial in $U_1 \setminus f^{-1}\bar{E}$. Since Γ minorizes the family of all paths joining $|\alpha|$ and $\partial U_1 \cup (U_1 \cap f^{-1}\bar{E})$, 3.27 implies $M(\Gamma) \geqq c_3 > 0$. On the other hand, $M(f\Gamma) \leqq 4\pi/(\log m)^2$. By [13, 3.2], we have $M(\Gamma) \leqq kK_O(f)M(f\Gamma)$. Thus $c_3(\log m)^2 \leqq 4\pi k K_O(f)$. Letting $m \to \infty$ yields a contradiction. Thus J' has bounded oscillation.

Choose now $s > 0$ such that $V = U(x, g \circ f, s) \subset B^3(x, u)$ and $B^3(x, 2u) \subset W$ for some u. We claim that the arc $J_0 = B_f \cap V$ has bounded oscillation. If not, there is for every $m > 1$ a triple x_1, x_2, x_3 of successive points of J_0 such that $|x_1-x_2| > m\,|x_1-x_3|$. We may assume that $|f(x_1)-f(x_2)| \leqq |f(x_2)-f(x_3)|$. Let x_0 be the end point of J such that x_0, x_3, x_2 are successive points of $\bar{J}$. Let α_1 be the open subarc of J_0 with end points x_1 and x_2, and let α_2 be the open subarc of J with end points x_0 and x_3. Let Γ be the family of all paths which join α_1 and α_2 in W. If $|x_1-x_3| < t < |x_1-x_2|$, the sphere $S^2(x_1, t)$ meets both α_1 and α_2. Hence [26, 10.12] implies

$$M(\Gamma) \geqq b_3 \log \frac{|x_1-x_2|}{|x_1-x_3|} \geqq b_3 \log m\,.$$

We next derive an upper bound for $M(f\Gamma)$. By the first part of the proof, there exists a constant a such that $|z_1-z_2| \leqq a\,|z_1-z_3|$ for successive points z_1, z_2, z_3 of J'. Set $\beta_i = f\alpha_i$, and suppose $y_i \in \beta_i$, $i = 1,2$. Then $\delta = |f(x_1)-f(x_2)| \leqq |f(x_2)-f(x_3)| \leqq |f(x_2)-y_1| + |f(x_3)-y_1| \leqq 2a\,|y_1-y_2|$. Hence $d(\beta_1, \beta_2) \geqq \delta/2a$. Define $\varrho : R^3 \to R^1$ by setting $\varrho(z) = 2a/\delta$ for $z \in \beta_1 + B^3(\delta/2a)$ and $\varrho(z) = 0$ elsewhere. Then $\varrho \in F(f\Gamma)$. Since $|y_1-f(x_1)| \leqq a\delta$, $\varrho(z) = 0$ whenever $|z-f(x_1)| \geqq a\delta+\delta/2a$. Hence

$$M(f\Gamma) \leqq \int \varrho^3 dm \leqq 8\Omega_3 a^3(a+1/2a)^3\,.$$

By [13, 3.2], we again have $M(\Gamma) \leqq kK_O(f)M(f\Gamma)$, which gives the contradiction as $m \to \infty$. The theorem is proved.

3.31. *Remark.* If a planar arc has bounded oscillation, it is a quasiconformal arc (Rickman [20], cf. also [11, p. 104]). The corresponding result in R^3 is not true, because there exist wild arcs with bounded oscillation. The authors do not know whether a tame arc with bounded oscillation must be quasiconformal.

4. Bounds for the local degree

4.1. It has been conjectured that for $n \geqq 3$ the local degree $i(x, f)$ is bounded by a constant which depends only on n and $K(f)$. This was disproved by Rickman [21] who constructed quasiregular mappings $f : R^3 \to R^3$ such that $i(0, f)$ was arbitrarily large while $K(f)$ was bounded by a universal constant. In this example, whose n-dimensional version will be given in 4.9, B_f consisted of a family of rays starting from the origin, and the local degree on B_f was 2, 3, or 4, except at the origin. In this section we show that this example is typical in two respects: (1) $i(x, f)$ cannot be large at every point of B_f. (2) If $i(x_0, f)$ is large, then B_f

must lie on »every side» of x_0. We also show that a quasiregular mapping is a local homeomorphism or constant if its dilatation is sufficiently close to 1. We close with some counterexamples.

4.2. **Theorem.** *Let* $n \geqq 3$ *and let* $f : G \to R^n$ *be a non-constant quasiregular mapping. Then either* $B_f = \emptyset$ *or*

$$\inf_{x \in B_f} i(x, f) < \left(\frac{n}{n-2}\right)^{n-1} K_I(f) \leqq 9 K_I(f) .$$

Proof. Since $\Lambda_{n-2}(fB_f) > 0$ by 3.4, the theorem follows directly from Martio [12, 6.8].

4.3. **Lemma.** *Let* $f : G \to R^n$ *be a non-constant quasiregular mapping and let* $x_0 \in G$. *Then there are* $r > 0$ *and* $C > 0$ *such that for* $x \in B^n(x_0, r)$

$$|f(x) - f(x_0)| \leqq C\, |x - x_0|^\mu$$

where $\mu = (i(x_0, f)/K_I(f))^{1/(n-1)}$.

Proof. This was proved in Martio [12, 6.1] although the formulation was less general.

4.4. **Theorem.** *Suppose that* $n \geqq 3$, *that* $f : G \to R^n$ *is* K*-quasiregular, and that* Δ *is an open cone in* $G \setminus B_f$ *with vertex at* $x_0 \in G$ *and angle* α. *Then* $i(x_0, f) \leqq C$ *where* C *depends on* n, K, and α.

Proof. Performing auxiliary similarity mappings we may assume, in view of Lemma 4.3, that (1) $x_0 = 0 = f(x_0)$, (2) $\Delta = \{x \mid |x| \cos \alpha < x_1 < 1\}$, (3) if $x \in \Delta$, then $|f(x)| \leqq |x|^\mu$ where $\mu^{n-1} = i(x_0, f)/K$. By 2.7, f is injective in the ball $B_t = B^n(te_1, t\psi(n, K) \sin \alpha)$, $0 < t \leqq 1/2$. Let J_t be the segment $\{se_1 \mid at \leqq s \leqq t\}$ where $a = 1 - \frac{1}{2}\psi(n, K) \sin \alpha$. Then $E_t = (B_t, J_t)$ is a condenser, and cap E_t depends only on n.

Pick $t_0 \in (0, 1/2]$ such that $f^{-1}(0) \cap B^n(2t_0) = \{0\}$. If $0 < t \leqq t_0$, then fE_t is a ringlike condenser, and $\complement fB_t$ contains 0 and ∞. Setting $g(t) = |f(te_1)|$ and using a well-known modulus estimate [26, 10.12] yield

$$\operatorname{cap} fE_t \geqq b_n \log \frac{g(t)}{g(at)}$$

where $b_n > 0$ depends only on n. Since $K \operatorname{cap} E_t \geqq \operatorname{cap} fE_t$, we obtain an inequality

$$\frac{g(t)}{g(at)} \leqq C_0$$

where C_0 depends on n and K. Since this holds for all $t \in (0, t_0]$, we obtain by iteration

$$g(t_0) \leqq C_0 g(at_0) \leqq C_0^2 g(a^2 t_0) \leqq \ldots \leqq C_0^k g(a^k t_0) .$$

On the other hand, $g(t) \leqq t^\mu$, which implies $g(t_0) \leqq C_0^k a^{k\mu} t_0^\mu$. Letting $k \to \infty$ yields $C_0 a^\mu \geqq 1$. Thus

$$i(x_0, f)/K = \mu^{n-1} \leqq \left(\frac{\log C_0}{\log (1/a)} \right)^{n-1} .$$

The theorem is proved.

4.5. **Lemma.** *Suppose that $f_j : G \to R^n$ is a sequence of sense-preserving discrete open mappings which converge to a discrete open mapping $f : G \to R^n$ uniformly on every compact part of G. Then $i(x_0, f) \geqq \limsup_{j\to\infty} i(x_0, f_j)$ for every $x_0 \in G$.*

Proof. Choose $r > 0$ such that $D = U(x_0, f, r)$ is a normal neighborhood of x_0 with respect to f. Then $i(x_0, f) = \mu(y_0, f, D)$, where $y_0 = f(x_0)$. Since $f_j \to f$ uniformly on $\bar{D}$, there is j_0 such that $|f_j(x) - f(x)| < r/2$ for $j \geqq j_0$ and $x \in \bar{D}$. By [16, Theorem 6, p. 131] this implies $\mu(f_j(x_0), f_j, D) = \mu(y_0, f, D) = i(x_0, f)$ for $j \geqq j_0$. On the other hand, $\mu(f_j(x_0), f_j, D) = \Sigma\, i(x, f_j)$ over $x \in D \cap f_j^{-1}(f_j(x_0))$. Thus $\mu(f_j(x_0), f_j, D) \geqq i(x_0, f_j)$, and the lemma follows.

4.6. **Theorem.** *For every $n \geqq 3$ there is $K > 1$ such that every non-constant K-quasiregular mapping $f : G \to R^n$ is a local homeomorphism.*

Proof. Suppose that the theorem is not true for some $n \geqq 3$. Then there exists a sequence of non-constant K_j-quasiregular mappings $f_j : G_j \to R^n$ such that $K_j \to 1$ and $B_{f_j} \neq \emptyset$ for all j. We may assume that $K_j \leqq 2$ for all j. By 4.2, there is $x_j \in B_{f_j}$ such that $i(x_j, f_j) \leqq 18$. From [13, 4.5] it follows that $H(x_j, f_j) < C$ where C depends only on n. By [13, 2.9], $U(x_j, f_j, r)$ is a normal neighborhood of x_j for sufficiently small r. Performing auxiliary similarity mappings we may assume that (i) $x_j = 0 = f(x_j)$, (ii) $U(0, f_j, r)$ is a normal neighborhood of 0 for $0 < r \leqq 1$, (iii) $l^*(0, f_j, 1) = 1$, (iv) $L(0, f_j, t) < Cl(0, f_j, t)$ for $0 < t \leqq 1$. Observe that (iii) implies $\bar{B}^n \subset G_j$.

Since $f_j B^n \subset B^n$, it follows from [14, 3.17] that the restrictions $g_j = f_j \mid B^n$ form a normal family. Passing to a subsequence, we may therefore assume that $g_j \to g$ uniformly on every compact part of B^n. By Rešetnjak [19, p. 664], $g : B^n \to R^n$ is 1-quasiregular. A 1-quasiregular mapping is either constant or a Möbius transformation. This was proved by Rešetnjak in [18], but it follows also easily from the corresponding result for quasiconformal mappings (Gehring [6, p. 388], Rešetnjak [17]) and from the condition $\dim B_f \leqq n-2$. Since $0 \in B_{f_j}$, it follows from 4.5 that g cannot be a homeomorphism. Hence g is constant. Since $g_j(0) = 0$, $g(x) = 0$ for all $x \in B^n$.

Set $l_j = l(0, f_j, 1)$. Since $L(0, f_j, 1) = 1$, (iv) implies $l_j \geqq 1/C$. Let E_j be the condenser $(U(0, f_j, l_j), \bar{U}(0, f_j, l_j/2))$. Then $f_j E_j = (B^n(l_j), \bar{B}^n(l_j/2))$, and $\operatorname{cap} f_j E_j = \omega_{n-1}(\log 2)^{1-n}$. Since $L^*(0, f_j, l_j) = 1$, we obtain the estimate $\operatorname{cap} E_j \geqq \omega_{n-1}(\log(1/l_j^*))^{1-n}$ where $l_j^* = l_j^*(0, f_j, l_j/2)$. Since E_j is a normal condenser of f_j, we have by [13, 6.2]

$$\operatorname{cap} E_j \leqq K_j N(f_j, E_j) \operatorname{cap} f E_j \leqq 36\omega_{n-1}(\log 2)^{1-n}.$$

Combining the above inequalities yields $l_j^* \leqq C_1 < 1$ where C_1 depends only on n. For each j there is $a_j \in \partial U(0, f_j, l_j/2)$ such that $|a_j| = l_j^*$. Then $|g_j(a_j)| = l_j/2 \geqq 1/2C$, which contradicts the fact that $g_j \to 0$ uniformly on $\bar{B}^n(C_1)$. The theorem is proved.

4.7. *Remark.* Theorem 4.6 does not give any explicit bound for K. We conjecture that f is a local homeomorphism if $K_I(f) < 2$. If B_f contains a rectifiable arc, this is true by Martio [12, 6.6]. However, B_f need not contain any rectifiable arc. To see this, let J be a quasiconformal curve in R^2 such that J contains no rectifiable arc [11, p. 109], and let $g : \bar{R}^2 \to \bar{R}^2$ be a quasiconformal mapping which maps J onto a line L. By Ahlfors [2], g can be extended to a quasiconformal mapping $g^* : \bar{R}^3 \to \bar{R}^3$. If $h : \bar{R}^3 \to \bar{R}^3$ is a winding mapping with $B_h = L$, then $f = h \circ g^*$ is a quasimeromorphic mapping with $B_f = J$.

4.8. *Examples.* For completeness we first construct an example of a quasiregular mapping of R^n onto R^n, $n \geqq 2$, which is in the special case $n = 3$ given in [21]. It shows that the local index has no upper bound in terms of the maximal dilatation and the dimension n.

4.9. Let k be a positive even integer and set $Q_1 = \{x \in R^n \mid |x_i| < 1\}$. Given an integer i, $1 \leqq i \leqq n$, let J_i be the set of multi-indices $j = (j_1, \ldots, j_n)$ such that j_i is $-k$ or k and j_m is some of the numbers $-k, \ldots, -1, 0, 1, \ldots, k-1$ for $m \neq i$. For $j \in J_i$ let C_{ij} be the $(n-1)$-dimensional cube $\{x \in R^n \mid j_m/k < x_m < (j_m+1)/k$ if $m \neq i$, $x_i = j_i/k\}$. Let $\mathcal{D}_k$ be the set of sequences $D = (D^1, \ldots, D^{n-1})$ where D^{n-1} is some C_{ij} and where D^p is a p-dimensional face of the $(p+1)$-dimensional cube D^{p+1}, $p = 1, \ldots, n-2$. For $D \in \mathcal{D}_k$ let $T(D)$ be the n-dimensional simplex whose vertices are the vertices $y(D^1)$ and $z(D^1)$ of D^1, the centers $x(D^p)$ of D^p, $p = 2, \ldots, n-1$, and the origin. Denote by $T'(D)$ the $(n-1)$-dimensional simplex whose vertices are the vertices of $T(D)$ except the origin. We fix one such sequence $D_0 \in \mathcal{D}_k$ such that $y(D_0^1) = e_n$ and $z(D_0^1) = e_1/k + e_n$.

We define a function β of the vertices of the cubes C_{ij} by setting for $u = (u_1, \ldots, u_n)$

$$\beta(u) = e_n \qquad \text{if } ku_1 + \ldots + ku_n \text{ is even},$$
$$\beta(u) = e_1/k + e_n \text{ if } ku_1 + \ldots + ku_n \text{ is odd}.$$

For $D \in \mathcal{D}_k$ let $g_D : R^n \to R^n$ be the affine homeomorphism which maps $T(D)$ onto $T(D_0)$ such that $g_D(y(D^1)) = \beta(y(D^1))$, $g_D(z(D^1)) = \beta(z(D^1))$, $g_D(x(D^2)) = x(D_0^2), \ldots, g_D(x(D^{n-1})) = x(D_0^{n-1})$, $g_D(0) = 0$.

Next we define a quasiconformal mapping of the cone $E(D_0)$ onto the half space $H_+^n = \{x \in R^n \mid x_n > 0\}$, where for $D \in \mathcal{D}_k$ $E(D) = \{x \in R^n \mid x = tz, z \in T'(D), t > 0\}$. Set $Y = \{x \in R^n \mid x_n = 1\}$. Let $v \in Y$ and $r > 0$ be such that $U = B^n(v, r) \cap Y$ is the maximal $(n-1)$-dimensional ball contained in $T'(D_0)$. Let λ be the radial projection of $V = U + (e_n - v)$ into S^{n-1}, i.e. $\lambda(z) = z/|z|$. Let α, $0 < \alpha < \pi/2$, be the angle between the x_n-axis X_n and the line segments $[0, z]$, $z \in S^{n-1}(e_n, r) \cap Y$. We define a mapping w of the cone $W = \{x \in R^n \mid x = sz, z \in V, s > 0\}$ of angle α onto H_+^n as follows. For $x \in W \setminus X_n$ we set $w(x) = w(t, x', \vartheta) = (t^{\pi/2\alpha}, x', \pi\vartheta/2\alpha)$ where we have used spherical coordinates such that $t = |x|$, $x' = P_n x/|P_n x|$ where P_n is the orthogonal projection onto ∂H_+^n, and ϑ is the angle between the x_n-axis and the line segment $[0, x]$. For $x \in W \cap X_n$ we set $w(x) = x_n^{\pi/2\alpha} e_n$. The required quasiconformal mapping $h : E(D_0) \to H_+^n$ is then obtained by setting $h(x) = w(t\lambda(\psi(z) + e_n - v))$, where $x = tz$, $z \in T'(D_0)$, $t > 0$, and where ψ is the radial projection of $T'(D_0)$ onto U.

Now we define a mapping φ of the union $\cup E(D)$ over all sequences $D \in \mathcal{D}_k$ by setting $\varphi(x) = s_D(h(g_D(x)))$ for $x \in E(D)$ where s_D is the identity mapping if g_D is sense-preserving and the reflection in ∂H_+^n if g_D is sense-reversing. The mapping φ can finally be extended continuously to a mapping $f_k : R^n \to R^n$. It is left to the reader to show that f_k is quasiregular and that $K(f_k)$ has an upper bound which depends only on n and not on k. On the other hand, $i(0, f_k) \to \infty$ as $k \to \infty$. We observe here also that $f_k \partial Q_1 = \partial B^n$.

4.10. By slightly modifying the mappings f_k in 4.9 for different k we can construct a quasimeromorphic mapping f with the property $\sup i(x, f) = \infty$ as follows. For every positive integer q we set $k_q = 2 \cdot 3^{q-1}$,

$$\Theta_q(x) = 3^{q-1}x + \sum_{l=2}^{q} 4 \cdot 3^{l-2} e_n,$$

and $Q_q = \Theta_q\{x \in R^n \mid |x_i| < 1\}$. Then the cubes Q_q are disjoint and $F_q = \bar{Q}_q \cap \bar{Q}_{q+1}$ is a face of Q_q. For every q we first define mappings $\tilde{u}_q$ of $\bar{Q}_q$ by

$$\tilde{u}_q(x) = \begin{cases} f_{k_q}(\Theta_q^{-1}(x)) & \text{if } q \text{ is odd,} \\ v_1(v_2(f_{k_q}(\Theta_q^{-1}(x)))) & \text{if } q \text{ is even,} \end{cases}$$

where f_{k_q} is defined in 4.9, v_1 is the reflection in S^{n-1}, and v_2 is the reflection in ∂H^n_+. We first observe that those $(n-1)$-dimensional simplexes $\Theta_q T'(D)$ and $\Theta_{q+1}T'(\hat{D})$, $D \in \mathscr{D}_{k_q}$, $\hat{D} \in \mathscr{D}_{k_{q+1}}$ (see 4.9), that are in F_q coincide pairwise. We define a mapping $\chi_q : F_q \to F_q$ by letting $\chi_q(x)$ be the point in $\tilde{u}_{q+1}^{-1}(\tilde{u}_q(x))$ which lies in $\Theta_q\bar{T}'(D)$ if $x \in \Theta_q\bar{T}'(D)$ and if $D \in \mathscr{D}_{k_q}$. Then $\chi_q(x) = x$ if x belongs to the boundary of some simplex $\Theta_q T'(D)$, $D \in \mathscr{D}_{k_q}$. We shall modify the mappings $\tilde{u}_q$ so that the new mappings coincide pairwise in the sets F_q. To this end, we set for $q > 1$ and $x \in \bar{Q}_q$

$$u_q(x) = \begin{cases} \tilde{u}_q(t(\chi_{q-1}((x-x_q)/t+x_q)-x_q)+x_q) & \text{if } (x-x_q)/t+x_q \in F_{q-1} \text{ and } 0 \leqq t \leqq 1, \\ \tilde{u}_q(x) & \text{elsewhere,} \end{cases}$$

where $x_q = \Theta_q(0)$ is the center of Q_q. The mappings $u_q \mid Q_q$ are quasimeromorphic, and u_q and u_{q+1} coincide in F_q. Furthermore, the maximal dilatation of $u_q \mid Q_q$ has an upper bound which depends only on n. The quasimeromorphic mapping $f_0 : G_0 \to \bar{R}^n$, $G_0 = \operatorname{int}(\cup \bar{Q}_q)$, defined by $f_0(x) = u_q(x)$ for $x \in \bar{Q}_q \cap G_0$, has the required property $\sup i(x, f_0) = \infty$.

4.11. As a final example we shall exhibit a discrete and open mapping $f : G \to R^n$, $n \geqq 3$, which is not topologically equivalent to any quasiregular mapping. Given a positive integer p, we define a mapping ζ_p of the cylinder $C = \{x \in R^n \mid x_1^2+x_2^2 < 2\}$ onto itself in cylinder coordinates (see 3.19) by

$$\zeta_p(r, \varphi, z) = \begin{cases} (r, (1+4p)(\varphi-\pi/4)+\pi/4, z) & \text{if } \pi/4 \leqq \varphi < 3\pi/4, \\ (r, \varphi, z) & \text{if } 0 \leqq \varphi < \pi/4 \text{ or } 3\pi/4 \leqq \varphi < 2\pi. \end{cases}$$

Let α_p be the translation $\alpha_p(x) = x+2(p-1)e_1$. We set

$$G = \bigcup_{p=1}^{\infty} \alpha_p C$$

and define $f : G \to R^n$ by $f(x) = \alpha_p(\zeta_p(\alpha_p^{-1}(x)))$ for $x \in \alpha_p C$. Then f is a sense-preserving, discrete, and open mapping with the property $i(x, f) = p+1$ in the set $L_p = \{x \in R^n \mid x_1 = 2p, x_2 = 0\}$. Suppose that there are homeomorphisms g_1 and g_2 such that $h = g_1 \circ f \circ g_2$ is a quasiregular mapping. Then $h_p = h \mid g_2^{-1}(\alpha_p C)$ is quasiregular and $K(h_p) \leqq K(h)$. On the other hand,

$$\inf_{x \in B_{h_p}} i(x, h_p) = p+1 .$$

This gives by 4.2 a contradiction as $p \to \infty$.

In the plane the situation is different. Given a discrete open mapping $f: G \to R^2$, there exists by Stoïlow's theorem [24, p. 120] an analytic function φ and a homeomorphism g such that $f = \varphi \circ g$. The following question remains open: Let $n \geqq 3$, let $f: G \to R^n$ be a discrete open mapping, and let D be a subdomain in G such that $\bar{D} \subset G$. Is $f \mid D$ topologically equivalent to a quasiregular mapping?

5. The linear dilatations

5.1. *Upper bound for* $H^*(x, f)$. It was proved in [13] that for a non-constant quasiregular mapping $f: G \to R^n$ both $H(x, f)$ and $H^*(x, f)$ have upper bounds which depend on $K(f)$, n, and $i(x, f)$. The result for $H^*(x, f)$ will now be sharpened to the extent that the upper bound does not depend on $i(x, f)$. This does not hold for $H(x, f)$.

5.2. **Theorem.** *Let* $f: G \to R^n$ *be a non-constant* K*-quasiregular mapping and let* $x \in G$. *Then*

$$H^*(x, f) \leqq C^*(n, K)$$

where $C^*(n, K)$ *depends only on* n *and* K.

Proof. For $r > 0$ set $U = U(x, f, r)$, $l^* = l^*(x, f, r)$, $L^* = L^*(x, f, r)$, $l = l(x, f, l^*)$, $L = L(x, f, L^*)$, $U_l = U(x, f, l)$, and $U_L = U(x, f, L)$. There exists $r_0 > 0$ such that U_L is a normal neighborhood of x for $0 < r \leqq r_0$. Fix such r. We may assume that $l < r < L$. Then $(U, \bar{U}_l)$ and $(U_L, \bar{U})$ are ringlike condensers [13, 5.2]. Furthermore, ∂U_l and ∂U meet $S^{n-1}(x, l^*)$, and ∂U and ∂U_L meet $S^{n-1}(x, L^*)$. Therefore we have $\operatorname{cap}(U, \bar{U}_l)$, $\operatorname{cap}(U_L, \bar{U}) \geqq a_n$, where $a_n > 0$ depends only on n. By [13, 6.2] we have

$$\begin{aligned} &\operatorname{cap}(U, \bar{U}_l) \leqq K_O(f) i(x, f) \omega_{n-1} \left(\log \frac{r}{l}\right)^{1-n}, \\ (5.3) \qquad &\operatorname{cap}(U_L, \bar{U}) \leqq K_O(f) i(x, f) \omega_{n-1} \left(\log \frac{L}{r}\right)^{1-n}. \end{aligned}$$

On the other hand, we have by [12, 5.13, 5.15]

$$(5.4)\quad K_I(f)\,\mathrm{cap}\,(U_L, \bar{U}_l) \geqq i(x, f)\,\mathrm{cap}\,(fU_L, f\bar{U}_l) = i(x, f)\omega_{n-1}\left(\log\frac{L}{l}\right)^{1-n}.$$

We also have

$$(5.5)\qquad \mathrm{cap}\,(U_L, \bar{U}_l) \leqq \omega_{n-1}\left(\log\frac{L^*}{l^*}\right)^{1-n}.$$

From (5.3) we get

$$\left(\log\frac{L}{l}\right)^{n-1} = \left(\log\frac{L}{r} + \log\frac{r}{l}\right)^{n-1} \leqq 2^{n-1}K_O(f)i(x, f)\omega_{n-1}a_n^{-1}.$$

Then the inequalities (5.4) and (5.5) imply

$$\omega_{n-1}\left(\log\frac{L^*}{l^*}\right)^{1-n} \geqq \frac{a_n}{2^{n-1}K_I(f)K_O(f)}.$$

Hence

$$\frac{L^*}{l^*} \leqq \exp\left(2\left(\frac{\omega_{n-1}K^2}{a_n}\right)^{1/(n-1)}\right) = C^*(n, K).$$

The theorem is proved.

5.6. *Example.* Let $f_0 : G_0 \to \bar{R}^n$ be the quasimeromorphic mapping defined in 4.9. If we compose f_0 with a stretching η, $\eta(x) = x_1e_1 + \ldots + x_{n-1}e_{n-1} + Kx_ne_n$, $K > 1$, then $g = f_0 \circ \eta \mid \eta^{-1}G_0$ is a quasimeromorphic mapping with the property $\sup H(x, g) = \infty$.

Addendum

When this manuscript was completed, two papers appeared partially overlapping with our work. Goldštein [7] obtains results related to 3.16. We have not been able to follow all details of his proofs. Poleckii [15] proves the path family inequality (2.13) for arbitrary quasiregular mappings and obtains results similar to 3.3 and 4.2.

We have also received a preprint of Goldštein which contains a result similar to Theorem 4.6.

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University of Helsinki
Helsinki, Finland

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