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490

ON ESTIMATING OF A FOURTH ORDER FUNCTIONAL FOR BOUNDED UNIVALENT FUNCTIONS

BY

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Preface

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1. Introduction and statement of results

The authors are concerned with the class $S(b_1)$ of univalent functions

$$f(z) = b_1 z + b_2 z^2 + \ldots , 0 < b_1 \leqq 1 ,$$

which map the closed unit disc into itself. This class has often been studied in the equivalent from of functions

$$F = (1/b_1) f ,$$

whose coefficients we shall denote by a_n. Both classes have been developed quite extensively since 1950 (cf. e.g. [14], [4], and [9]). In this paper the authors investigate the problem of finding the sharp estimate for the functional of the fourth order

$$B = |a_4 - p a_2 a_3 + q a_2^3| , p, q \text{ real},$$

in dependence on p, q, and b_1, where f ranges over $S(b_1)$.

Given real p, q, and b_1, $0 < b_1 \leqq 1$, let

$$\begin{aligned} M(x) = & -3(1 - b_1)[1 - 5p - p^2 + 12q - (11 - 9p - p^2 + 12q)b_1] \\ & - \tfrac{3}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]x \\ & + \tfrac{1}{4}(7 - 3p + 3p^2 - 12q)x^2 + \tfrac{3}{2}(4 - x)(\{\tfrac{1}{3}(2 - p)^2 \\ & + \tfrac{1}{4}[5 - 9p + 12q - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]^2\}^{\frac{1}{2}} \\ & - \tfrac{1}{2}[5 - 9p + 12q - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]) , \end{aligned}$$

where x is supposed to be real. Further, let D denote the set of pairs (p, q) for which there is a $b(p, q)$ such that if $0 < b_1 \leqq b(p, q)$, then the corresponding M satisfies the condition $M(x) \leqq 0$ for $0 < x \leqq 2(1 - b_1)$.

Table 1

Table of values of the functions b^* estimated from below

$6p$ \ $6q$	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12
−12	.000	.018	.108	.172	.189	.205	.220	.234	.248	.260	.272	.283	.294	.304	.314	.323	.332
−11	.000	.009	.103	.166	.185	.202	.217	.232	.246	.259	.272	.283	.295	.305	.315	.325	.334
−10	.000	.000	.096	.160	.179	.197	.214	.230	.245	.259	.272	.284	.295	.306	.317	.327	.336
−9	.000	.000	.085	.152	.173	.193	.211	.227	.243	.258	.271	.284	.296	.308	.319	.329	.339
−8	.000	.000	.071	.144	.167	.188	.207	.225	.241	.257	.271	.285	.297	.309	.321	.331	.342
−7	.000	.000	.051	.134	.159	.182	.202	.221	.239	.255	.271	.285	.298	.311	.323	.334	.345
−6	.000	.000	.025	.123	.150	.175	.197	.218	.237	.254	.270	.285	.300	.313	.325	.337	.348
−5	.000	.000	.000	.109	.140	.167	.192	.214	.234	.253	.270	.286	.301	.315	.328	.340	.352
−4	.000	.000	.000	.086	.128	.158	.185	.209	.231	.251	.270	.287	.302	.317	.331	.344	.356
−3	.000	.000	.000	.048	.113	.147	.177	.204	.228	.249	.269	.287	.304	.320	.334	.348	.360
−2	.000	.000	.000	.000	.096	.134	.168	.197	.224	.247	.269	.288	.306	.323	.338	.352	.366
−1	.000	.000	.000	.000	.074	.119	.157	.190	.219	.245	.268	.289	.308	.326	.342	.357	.371
0	.000	.000	.000	.000	.039	.099	.143	.180	.213	.242	.267	.290	.311	.330	.347	.363	.378
1	.000	.000	.000	.000	.000	.074	.125	.169	.206	.238	.266	.291	.314	.334	.353	.370	.386
2	.000	.000	.000	.000	.000	.041	.103	.154	.197	.233	.265	.293	.318	.340	.360	.378	.395
3	.000	.000	.000	.000	.000	.000	.073	.135	.186	.228	.264	.295	.322	.346	.368	.388	.406
4	.000	.000	.000	.000	.000	.000	.032	.109	.171	.220	.262	.297	.328	.354	.378	.399	.419
5	.000	.000	.000	.000	.000	.000	.000	.073	.150	.210	.259	.300	.335	.365	.391	.414	.435
6	.000	.000	.000	.000	.000	.000	.000	.016	.119	.196	.256	.304	.344	.378	.407	.432	.455
7	.000	.000	.000	.000	.000	.000	.000	.000	.071	.174	.250	.309	.357	.396	.429	.457	.481
8	.000	.000	.000	.000	.000	.000	.000	.000	.000	.138	.241	.317	.375	.421	.459	.491	.518
9	.000	.000	.000	.000	.000	.000	.000	.000	.000	.067	.224	.329	.404	.461	.506	.542	.572
10	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.183	.346	.450	.523	.577	.619	.652
11	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.051	.351	.502	.594	.656	.701	.735
12	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.250	.500	.625	.700	.750	.785

In all cases listed above, where $b^*(p, q) \approx .000$, we have $b^*(p, q) = 0$. For $-12 \leqq 6p \leqq 12$, $-12 \leqq 6q \leqq -4$ we also have $b^*(p, q) = 0$.

Given a (p, q) in D, let $b^*(p, q)$ denote the least upper bound of $b(p, q)$, while let $b^*(p, q) = 0$ otherwise. Hence D is the set of (p, q) such that $b^*(p, q) > 0$. The table enclosed gives the values of b^* for various p and q. For more clarity this is also given in the form of a map in the (p, q)-plane with level-lines corresponding to fixed values of b^*.

The results obtained in this paper may be formulated as follows.

Theorem 1. *If* (p, q) *and* f *belong to* D *and* $S(b_1)$, *respectively, where* $0 < b_1 \leqq b^*(p, q)$, *then the corresponding* B *does not exceed*

$$B^* = \tfrac{2}{3}(1 - b_1^3) + 2(1 - b_1)^2[\tfrac{5}{3} - 3p + 4q - (\tfrac{20}{3} - 5p + 4q)b_1] .$$

The estimate is sharp for every p, q, *and* b_1. *All the extremal functions are given by the formula*

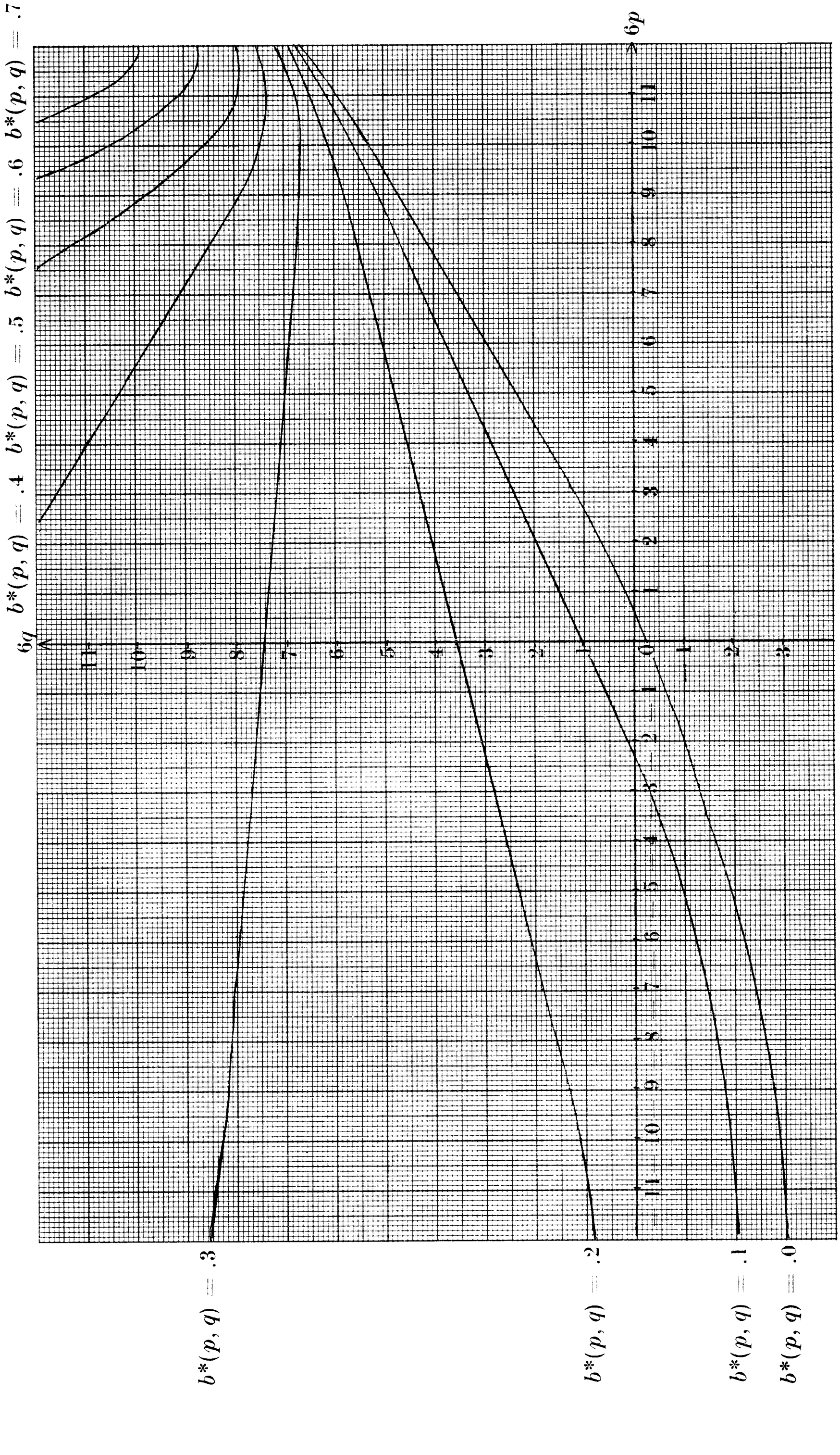

Fig. 1. Level-lines corresponding to the function b^*

$$f_c^*(z) = e^{-ic}P^{-1}(b_1 P(e^{ic}z)),\ P(z) = z/(1-z)^2,\ |z| \leqq 1,\ -\pi < c \leqq \pi .$$

Theorem 2. *If* $p \leqq \frac{5}{2}$ *and* f *belongs to* $S(b_1)$, *where*

$$\frac{(p-2)^2 + p^2 - p - 4q + \frac{7}{3}}{(p-3)^2 + p^2 - p - 4q + \frac{7}{3}} \leqq b_1 \leqq 1 \text{ for } q \leqq \tfrac{1}{4}(p^2 - p + \tfrac{7}{3}) ,$$

while

$$\frac{(p-2)^2 - 3(p^2 - p - 4q + \frac{7}{3})}{(p-3)^2 - 3(p^2 - p - 4q + \frac{7}{3})} \leqq b_1 \leqq 1 \text{ for } q \geqq \tfrac{1}{4}(p^2 - p + \tfrac{7}{3}) ,$$

then the corresponding B *does not exceed*

$$B^{**} = \tfrac{2}{3}(1 - b_1^3) .$$

The estimate is sharp for every $p, q,$ *and* b_1. *All the extremal functions are given by the formula*

$$f_c^{**}(z) = e^{-ic}\check{P}^{-1}(b_1 \check{P}(e^{ic}z)),\ \check{P}(z) = z/(1-z^3)^{\frac{2}{3}},\ |z| \leqq 1,\ -\pi < c \leqq \pi .$$

The paper is concluded by few applications and remarks. In particular, the authors obtain the sharp estimates of the third coefficients of

$$g(z) = \frac{f'(z)}{f(z)},\ h(z) = 1 + z\frac{f''(z)}{f'(z)},\ |z| \leqq 1 .$$

Actually, these classical functionals (cf. [1], [12], and [13]) gave reason to introduce the fourth order functional B of the present paper. The functional B is also an analogue of the functional $|a_3 - pa_2^2|$, p real, considered in the case where $1 \leqq p \leqq 1$ in [14], formula (37), and in the general case in [3].

2. Proof of Theorem 1

The proof is analogous to that given in [7] for $B = |a_4|$.

We start with the estimate

$$(1)\qquad \begin{aligned} &|a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3 + x_1^2 a_2 + x_2^2(a_3 - a_2^2) + 2x_1(a_3 - \tfrac{3}{4}a_2^2)| \\ &\leqq \tfrac{2}{3}(1 - b_1^3) - \tfrac{1}{2}b_1|a_2|^2 + 2|x_1|^2(1 - b_1) \\ &+ |x_2|^2(1 - b_1^2) - 2b_1\mathrm{Re}(x_1\bar{a}_2) \end{aligned}$$

which holds for any x_1, x_2 (not necessarily real). Inequality (1) is a simple consequence of the Grunsky-Nehari inequalities (cf. [7], inequality (4)). Let us choose x_1, x_2 to be real. Then we get

$$\begin{aligned}|a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3 + 2x_1(a_3 - \tfrac{3}{4}a_2^2) + x_1^2a_2 + x_2^2(a_3 - a_2^2)| \\ \leqq \tfrac{2}{3}(1-b_1^3) - \tfrac{1}{2}b_1|a_2|^2 + 2x_1^2(1-b_1) \\ - 2x_1b_1 \operatorname{Re} a_2 + x_2^2(1-b_1^2)\,.\end{aligned}$$

Here the left-hand side is not less than

$$\begin{aligned}\operatorname{Re}[a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3 + 2x_1(a_3 - \tfrac{3}{4}a_2^2) + x_1^2a_2 + x_2^2(a_3 - a_2^2)] \\ = \operatorname{Re}[a_4 - pa_2a_3 + qa_2^3] \\ + \operatorname{Re}[(p-2)a_2a_3 - (q - \tfrac{13}{12})a_2^3 + 2x_1\lambda + x_1^2a_2 + x_2^2(\lambda - \tfrac{1}{4}a_2^2)]\,,\end{aligned}$$

where

$$\lambda = a_3 - \tfrac{3}{4}a_2^2\,.$$

Therefore

$$\begin{aligned}(2)\quad \operatorname{Re}(a_4 - pa_2a_3 + qa_2^3) \\ \leqq \tfrac{2}{3}(1-b_1^3) + (2-p)\operatorname{Re}(a_2\lambda) + (\tfrac{5}{12} - \tfrac{3}{4}p + q)\operatorname{Re} a_2^3 - \tfrac{1}{2}b_1|a_2|^2 \\ - 2x_1(\operatorname{Re}\lambda + b_1 \operatorname{Re} a_2) + 2x_1^2(1 - b_1 - \tfrac{1}{2}\operatorname{Re} a_2) \\ + x_2^2(1 - b_1^2 - \operatorname{Re}\lambda + \tfrac{1}{4}\operatorname{Re} a_2^2)\,.\end{aligned}$$

Now we notice that $|a_3 - a_2^2| \leqq 1 - b_1^2$, (cf. [15], formula (19)), whence

$$x_2^2(1 - b_1^2 - \operatorname{Re}\lambda + \tfrac{1}{4}\operatorname{Re} a_2^2) \geqq 0\,.$$

Therefore we choose

$$(3)\qquad x_2 = 0\,.$$

On the other hand, there is no loss of generality if we assume

$$(4)\qquad a_4 - pa_2a_3 + qa_2^3 > 0\,,\ \operatorname{Re} a_2 \geqq 0\,,$$

since this normalization can always be achieved by a properly chosen rotation. Consequently, from (2), (3), and (4) we infer

$$\begin{aligned}(5)\quad B = a_4 - pa_2a_3 + qa_2^3 \\ \leqq \tfrac{2}{3}(1-b_1^3) + (2-p)\operatorname{Re}(a_2\lambda) + (\tfrac{5}{12} - \tfrac{3}{4}p + q)\operatorname{Re} a_2^3 \\ - \tfrac{1}{2}b_1|a_2|^2 - 2x_1(\operatorname{Re}\lambda + b_1\operatorname{Re} a_2) + 2x_1^2(1 - b_1 - \tfrac{1}{2}\operatorname{Re} a_2)\,.\end{aligned}$$

The estimate (5) implies, in particular,

$$(6)\qquad 1 - b_1 - \tfrac{1}{2}\operatorname{Re} a_2 \geqq 0\,.$$

We shall consider, separately, two cases: $1 - b_1 - \frac{1}{2}\operatorname{Re} a_2 = 0$ and $1 - b_1 - \frac{1}{2}\operatorname{Re} a_2 > 0$.

Suppose first that $1 - b_1 - \frac{1}{2}\operatorname{Re} a_2 = 0$. It is well known (cf. [7]) that this implies $f = f_0^*$, whence

$$\frac{b_1 z + b_1 a_2 z^2 + b_1 a_3 z^3 + b_1 a_4 z^4 + \ldots}{(-1 + b_1 z + b_1 a_2 z^2 + b_1 a_3 z^3 + \ldots)^2} = \frac{b_1 z}{(1-z)^2}$$

and, consequently,

$$a_2 = 2(1 - b_1)\,,$$
$$a_3 = 3 - 8b_1 + 5b_1^2\,,$$
$$a_4 = 2(2 - 10b_1 + 15b_1^2 - 7b_1^3)\,.$$

Hence

$$B = \tfrac{2}{3}(1 - b_1^3) + 2(1 - b_1)^2[\tfrac{5}{3}(1 - 4b_1) - p(3 - 5b_1) + 4q(1 - b_1)] = B^*\,.$$

Suppose next that $1 - b_1 - \frac{1}{2}\mathrm{Re}\, a_2 > 0$. Then we can minimize the right-hand side of (5) by choosing

$$x_1 = \frac{\mathrm{Re}\,\lambda + b_1 \,\mathrm{Re}\, a_2}{2(1 - b_1) - \mathrm{Re}\, a_2}$$

and find the most favourable estimate

$$\text{(7)} \qquad B \leqq \tfrac{2}{3}(1 - b_1^3) + (2 - p)\,\mathrm{Re}(a_2 \lambda) + (\tfrac{5}{12} - \tfrac{3}{4}p + q)\mathrm{Re}\, a_2^3 - \tfrac{1}{2} b_1 |a_2|^2 - \frac{(\mathrm{Re}\,\lambda + b_1 \,\mathrm{Re}\, a_2)^2}{2(1 - b_1) - \mathrm{Re}\, a_2}\,.$$

The right-hand side of (7) may be rewritten in the form

$$\tfrac{2}{3}(1 - b_1^3) - (\tfrac{5}{2} - p) b_1 \,\mathrm{Re}^2 a_2 + (\tfrac{5}{12} - \tfrac{3}{4}p + q)\,\mathrm{Re}\, a_2^3 - (2 - p)\mathrm{Im}\, a_2 \,\mathrm{Im}\,\lambda - \tfrac{1}{2} b_1 \,\mathrm{Im}^2 a_2 + (2 - p)\mathrm{Re}\, a_2\, (\mathrm{Re}\,\lambda + b_1 \mathrm{Re}\, a_2) - \frac{(\mathrm{Re}\,\lambda + b_1 \,\mathrm{Re}\, a_2)^2}{2(1 - b_1) - \mathrm{Re}\, a_2}$$

and it attains its maximum with respect to $\mathrm{Re}\,\lambda + b_1 \,\mathrm{Re}\, a_2$ treated as the only variable for

$$\mathrm{Re}\,\lambda + b_1 \,\mathrm{Re}\, a_2 = (2 - p)\mathrm{Re}\, a_2 (1 - b_1 - \tfrac{1}{2}\mathrm{Re}\, a_2)\,.$$

Hence

$$B \leqq \tfrac{2}{3}(1 - b_1^3) - (\tfrac{5}{2} - p) b_1 \mathrm{Re}^2 a_2 + (\tfrac{5}{12} - \tfrac{3}{4}p + q)\,\mathrm{Re}\, a_2^3 - \tfrac{1}{2} b_1 \,\mathrm{Im}^2 a_2 - (2 - p)\,\mathrm{Im}\, a_2 \,\mathrm{Im}\,\lambda + \tfrac{1}{2}(2 - p)^2 \mathrm{Re}^2 a_2 (1 - b_1 - \tfrac{1}{2}\mathrm{Re}\, a_2)$$

or, after a rearrangement,

$$B \leqq \tfrac{2}{3}(1 - b_1^3) + \tfrac{1}{2}[(2 - p)^2 - (3 - p)^2 b_1]\mathrm{Re}^2 a_2 - (\tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q)\mathrm{Re}^3 a_2 - (\tfrac{5}{4} - \tfrac{9}{4}p + 3q)\mathrm{Re}\, a_2 \mathrm{Im}^2 a_2 - (2 - p)\mathrm{Im}\, a_2 \,\mathrm{Im}\,\lambda - \tfrac{1}{2} b_1 \mathrm{Im}^2 a_2\,.$$

Thus we have removed the parameter $\operatorname{Re}\lambda$. In order to remove $\operatorname{Im}\lambda$ we introduce the quantities

$$\operatorname{Re} a_2 = 2(1 - b_1) - x\,,\ 0 < x \leqq 2(1 - b_1)\,,$$
$$\operatorname{Im} a_2 = y\,,\ \operatorname{Im}\lambda = \eta - b_1 \operatorname{Im} a_2\,.$$

Under these notation we have

$$\begin{aligned} B \leqq\ & \tfrac{2}{3}(1 - b_1^3) + \tfrac{1}{2}[(2-p)^2 - (3-p)^2 b_1]\,[2(1-b_1) - x]^2 \\ & - (\tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q)\,[2(1-b_1) - x]^3 \\ & - (\tfrac{5}{4} - \tfrac{9}{4}p + 3q)\,[2(1-b_1) - x]y^2 - (2-p)y\eta + (\tfrac{3}{2} - p)b_1 y^2 \\ =\ & c_0 + c_1 x + c_2 x^2 + c_3 x^3 - \tfrac{1}{2}[5 - 9p + 12q \\ & - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]y^2 - (2-p)y\eta\,, \end{aligned}$$

where c_0, c_1, c_2, c_3 do not depend on x. Direct calculation gives

$$\begin{aligned} c_0 =\ & \tfrac{2}{3}(1 - b_1^3) + 2[(2-p)^2 - (3-p^2)b_1]\,(1-b_1)^2 \\ & - 8(\tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q)\,(1-b_1)^3 \\ =\ & \tfrac{2}{3}(1 - b_1^3) + 2(1-b_1)^2[\tfrac{5}{3} - 3p + 4q - (\tfrac{20}{3} - 5p + 4q)b_1] \\ =\ & B^*\,, \\ c_1 =\ & -2[(2-p)^2 - (3-p)^2 b_1]\,(1-b_1) \\ & + 12(\tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q)\,(1-b_1)^2 \\ =\ & -(1-b_1)\,[1 - 5p - p^2 + 12q - (11 - 9p - p^2 + 12q)b_1]\,, \\ c_2 =\ & \tfrac{1}{2}[(2-p)^2 - (3-p)^2 b_1] - 6(\tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q)\,(1-b_1) \\ =\ & -\tfrac{1}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]\,, \\ c_3 =\ & \tfrac{7}{12} - \tfrac{1}{4}p + \tfrac{1}{4}p^2 - q \\ =\ & \tfrac{1}{12}(7 - 3p + 3p^2 - 12q)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \text{(8)}\quad B - B^* \leqq\ & -(1-b_1)[1-5p-p^2+12q-(11-9p-p^2+12q)b_1]x \\ & - \tfrac{1}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]x^2 \\ & + \tfrac{1}{12}(7 - 3p + 3p^2 - 12q)x^3 \\ & - \tfrac{1}{2}[5 - 9p + 12q - (8 - 11p + 12q)b_1 \\ & - \tfrac{1}{2}(5 - 9p + 12q)x]y^2 - (2-p)y\eta\,. \end{aligned}$$

Now we apply the identity

$$-(2-p)y\eta = |1 - \tfrac{1}{2}p|(\alpha y^2 + \tfrac{1}{\alpha}\eta^2) - |1 - \tfrac{1}{2}p|\alpha\left(y + \frac{1 - \tfrac{1}{2}p}{\alpha|1 - \tfrac{1}{2}p|}\,\eta\right)^2$$

with a free parameter $\alpha > 0$. Clearly,

$$- |1 - \tfrac{1}{2}p|\alpha\left(y + \frac{1 - \frac{1}{2}p}{\alpha|1 - \frac{1}{2}p|}\,\eta\right)^2 \leqq 0\,,$$

whence

$$- (2 - p)y\eta \leqq |1 - \tfrac{1}{2}p|(\alpha y^2 + \alpha^{-1}\eta^2)\,.$$

We utilize further the estimate (cf. [7], inequality (27′))

$$\eta^2 \leqq \tfrac{4}{3}x - \tfrac{1}{3}(x^2 + y^2)\,.$$

Therefore

$$(9) \qquad - (2 - p)y\eta \leqq |1 - \tfrac{1}{2}p|\,[\tfrac{1}{3}(4x - x^2 - y^2)\alpha^{-1} + y^2\alpha]\,.$$

Besides, it is easily seen that

$$(10) \qquad \tfrac{1}{3}(4x - x^2 - y^2)\alpha^{-1} + y^2\alpha \geqq 2|y|\,[\tfrac{1}{3}(4x - x^2 - y^2)]^{\frac{1}{2}}$$

whenever $4x - x^2 - y^2 \geqq 0$, but this inequality must hold since the right-hand side in (8) is bounded from below for any $\alpha > 0$, and for the term $- (2 - p)y\eta$ in (8) we have derived the estimate (9). If $4x - x^2 - y^2 > 0$ and $y \neq 0$, in order to obtain equality in (10) we have to choose

$$\alpha = [\tfrac{1}{3}(4x - x^2 - y^2)y^{-2}]^{\frac{1}{2}}\,,$$

whence (9) becomes

$$(11) \qquad - (2 - p)y\eta \leqq |2 - p|\;|y|\,[\tfrac{1}{3}(4x - x^2 - y^2)]^{\frac{1}{2}}\,.$$

If $4x - x^2 - y^2 = 0$, (11) is an immediate consequence of (9). Finally, if $y = 0$, (11) is trivial. Combining (11) with (8) we obtain

$$\begin{aligned}(12) \quad B - B^* \leqq\; & - (1 - b_1)[1 - 5p - p^2 + 12q - (11 - 9p - p^2 + 12q)b_1]x \\ & - \tfrac{1}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]x^2 \\ & + \tfrac{1}{12}(7 - 3p + 3p^2 - 12q)x^3 \\ & - \tfrac{1}{2}[5 - 9p + 12q - (8 - 11p + 12q)b_1 \\ & - \tfrac{1}{2}(5 - 9p + 12q)x]y^2 + |2 - p|\;|y|[\tfrac{1}{3}(4x - x^2 - y^2)]^{\frac{1}{2}}\,.\end{aligned}$$

In order to find for fixed x the maximum of the right-hand side in (12) with respect to y we consider the function

$$(13) \quad Q(y) = - Uy^2 + |2 - p|y(V - \tfrac{1}{4}y)^{\frac{1}{2}}, \; -(3V)^{\frac{1}{2}} \leqq y \leqq (3V)^{\frac{1}{2}}\,,$$

where

$$U = \tfrac{1}{2}[5 - 9p + 12q - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]\,,$$
$$V = \tfrac{1}{3}x(4 - x)\,.$$

Since Q is even in y, we may, without loss of generality, assume that $y \geqq 0$. Straightforward differentiation leads to the following condition for the value of y at internal extrema:

$$(14)\qquad |1 - \tfrac{1}{2}p|(V - \tfrac{2}{3}y^2)/(V - \tfrac{1}{3}y^2)^{\frac{1}{2}} = Uy\,.$$

Solving this equation for y, we obtain two possibilities:

$$y_1^2 = \tfrac{3}{2}V(1 - \{U^2/[\tfrac{1}{3}(2-p)^2 + U^2]\}^{\frac{1}{2}})\,,$$
$$y_2^2 = \tfrac{3}{2}V(1 + \{U^2/[\tfrac{1}{3}(2-p)^2 + U^2]\}^{\frac{1}{2}})\,.$$

If $U > 0$, equation (14) requires $V - \frac{2}{3}y^2 > 0$ since we look for positive values of y. In this case only y^2 will be permissible. If $U < 0$, we have to demand $V - \frac{2}{3}y^2 < 0$, which leads to the only possibility y_2^2. Consequently the internal extremal point for Q satisfies the equation

$$(15)\qquad y^2 = \tfrac{3}{2}V\{1 - U/[\tfrac{1}{3}(2-p)^2 + U^2]^{\frac{1}{2}}\}\,.$$

Since

$$Q'(0) = 0\,,\ Q'(y) \to -\infty \text{ as } y \to (3V)^{\frac{1}{2}}\,,$$

relation (15) leads to a maximum of Q, and we find from (13) and (14)

$$\max Q(y) = \tfrac{3}{2}V\{[\tfrac{1}{3}(2-p)^2 + U^2]^{\frac{1}{2}} - U\}\,.$$

Thus from (12) we obtain

$$\begin{aligned}
B - B^* \leqq{} & -(1-b_1)[1 - 5p - p^2 + 12q + (2 - 3p - 2p^2 + 12q)b_1]x \\
& - \tfrac{1}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]x^2 \\
& + \tfrac{1}{12}(7 - 3p + 3p^2 - 12q)x^3 + \tfrac{3}{2}V\{[\tfrac{1}{3}(2-p)^2 + U^2]^{\frac{1}{2}} - U\} \\
={} & -(1-b_1)[1 - 5p - p^2 + 12q - (11 - 9p - p^2 + 12q)b_1]x \\
& - \tfrac{1}{2}[3 + p + 2p^2 - 12q + (2 - 3p - 2p^2 + 12q)b_1]x^2 \\
& + \tfrac{1}{12}(7 - 3p + 3p^2 - 12q)x^3 + \tfrac{1}{2}x(4-x)\,(\{\tfrac{1}{3}(2-p)^2 \\
& + \tfrac{1}{4}[5 - 9p + 12q - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]^2\}^{\frac{1}{2}} \\
& - \tfrac{1}{2}[5 - 9p + 12q - (8 - 11p + 12q)b_1 - \tfrac{1}{2}(5 - 9p + 12q)x]) \\
={} & \tfrac{1}{3}xM(x)\,.
\end{aligned}$$

If (p, q) belongs to D and $0 < b_1 \leqq b^*(p, q)$, then, by the definition of M, we have $M(x) \leqq 0$ for $0 < x \leqq 2(1 - b_1)$. Since x has been restricted to this interval, $B - B^* \leqq 0$, as desired.

Finally we notice that, by (5), where we have assumed (4), in order to demonstrate that $B = B^*$ can only hold for $f = f_c^*$ it is sufficient to show that equality in (6) can only hold for $f = f_0^*$, but this is a well known result (cf. [7]). On the other hand, for any f_c^*, $-\pi < c \leqq \pi$, we have $B = B^*$, as it was verified before by direct calculation. Thus the proof is completed.

3. Proof of Theorem 2

The proof is analogous to that given in [7] for $B = |a_4|$.

We again start with the estimate (1), where we put $x_1 = \check{x}_1 a_2$, $x_2^2 = \check{x}_2 a_2$, and choose $\check{x}_1, \check{x}_2$ to be real. Then we get

$$|a_4 + (2\check{x}_1 + \check{x}_2 - 2)a_2 a_3 + (\check{x}_1^2 - \tfrac{3}{2}\check{x}_1 - \check{x}_2 + \tfrac{13}{12})a_2^3|$$
$$\leqq \tfrac{3}{2}(1 - b_1^3) + |\check{x}_2|(1 - b_1^2)|a_2| + [2\check{x}_1^2(1 - b_1) - 2\check{x}_1 b_1 - \tfrac{1}{2}b_1]|a_2|^2 .$$

Here the left-hand side is not less than

$$\operatorname{Re}[a_4 + (2\check{x}_1 + \check{x}_2 - 2)a_2 a_3 + (\check{x}_1^2 - \tfrac{3}{2}\check{x}_1 - \check{x}_2 + \tfrac{13}{12})a_2^3] .$$

We choose $\check{x}_2$ so that $2\check{x}_1 + \check{x}_2 - 2 = -p$, i.e.

$$\check{x}_2 = 2 - p - 2\check{x}_1 .$$

Therefore

$$\operatorname{Re}[a_4 - pa_2 a_3 + (\check{x}_1^2 + \tfrac{1}{2}\check{x}_1 + p - \tfrac{11}{12})a_2^3]$$
$$\leqq \tfrac{2}{3}(1 - b_1^3) + |2 - p - 2\check{x}_1|\,(1 - b_1^2)|a_2|$$
$$+ [2\check{x}_1^2(1 - b_1) - 2\check{x}_1 b_1 - \tfrac{1}{2}b_1]|a_2|^2 ,$$

whence

$$\text{(16)}\qquad \operatorname{Re}(a_4 - pa_2 a_3 + qa_2^3)$$
$$\leqq \tfrac{2}{3}(1 - b_1^3) + |2 - p - 2x_1|(1 - b_1^2)|a_2|$$
$$+ [2\check{x}_1^2(1 - b_1) - 2\check{x}_1 b_1 - \tfrac{1}{2}b_1]|a_2|^2$$
$$- (\check{x}_1^2 + \tfrac{1}{2}\check{x}_1 + p - q - \tfrac{11}{12})\operatorname{Re} a_2^3 .$$

Now we notice that there is no loss of generality if we assume

$$\text{(17)}\qquad a_4 - pa_2 a_3 + qa_2^3 > 0,\ a_2 = u + iv\,,\ u \leqq 0\,,$$

since this normalization can always be achieved by a properly chosen rotation. Consequently, from (16) and (17) we infer

$$B = a_4 - pa_2 a_3 + qa_2^3$$
$$\leqq B^{**} + |2 - p - 2\check{x}_1|(1 - b_1^2)(u^2 + v^2)^{\frac{1}{2}} + [2\check{x}_1^2(1 - b_1)$$
$$- 2\check{x}_1 b_1 - \tfrac{1}{2}b_1](u^2 + v^2) - (\check{x}_1^2 + \tfrac{1}{2}\check{x}_1 + p - q - \tfrac{11}{12})(u^3 - 3uv^2) ,$$

where $B^{**} = \frac{2}{3}(1 - b_1^3)$. Since $(1 - b_1^2)(u^2 + v^2)^{\frac{1}{2}} \geqq 0$, we choose $\check{x}_1$ so that $2 - p - 2\check{x}_1 = 0$, i.e.

$$\check{x}_1 = 1 - \tfrac{1}{2}p\,.$$

Hence

$$B - B^{**} \leqq \tfrac{1}{2}[(p-2)^2 - (p-3)^2 b_1](u^2 + v^2) - \tfrac{1}{4}(p^2 - p - 4q + \tfrac{7}{3})(u^3 - 3uv^2)$$

or, after a rearrangement,

$$(18)\qquad B - B^{**} \leqq \tfrac{1}{2}u^2[(p-2)^2 - (p-3)^2 b_1 - \tfrac{1}{2}(p^2 - p - 4q + \tfrac{7}{3})u] + \tfrac{1}{2}v^2[(p-2)^2 - (p-3)^2 b_1 + \tfrac{3}{2}(p^2 - p - 4q + \tfrac{7}{3})u]\,.$$

We shall consider, separately, two cases:

$$(19)\qquad p^2 - p - 4q + \tfrac{7}{3} \geqq 0\,, p \neq 3\,,$$

and

$$(20)\qquad p^2 - p - 4q + \tfrac{7}{3} \leqq 0\,, p \neq 3\,.$$

In both cases, by (6) and (17), we have

$$(21)\qquad -2(1 - b_1) \leqq u \leqq 0\,.$$

Suppose first (19). By (21), in order to obtain

$$(22)\qquad \tfrac{1}{2}u^2[(p-2)^2 - (p-3)^2 b_1 - \tfrac{1}{2}(p^2 - p - 4q + \tfrac{7}{3})u] + \tfrac{1}{2}v^2[(p-2)^2 - (p-3)^2 b_1 + \tfrac{3}{2}(p^2 - p - 4q + \tfrac{7}{3})u] \leqq 0$$

we have to assume

$$(23)\qquad (p-2)^2 - (p-3)^2 b_1 - \tfrac{1}{2}(p^2 - p - 4q + \tfrac{7}{3})(-2)(1 - b_1) \leqq 0$$

and

$$(24)\qquad (p-2)^2 - (p-3)^2 b_1 + \tfrac{2}{3}(p^2 - p - 4q + \tfrac{7}{3}) \cdot 0 \leqq 0\,.$$

Inequalities (23) and (24) are equivalent to

$$(25)\qquad b_1 \geqq \frac{(p-2)^2 + p^2 - p - 4q + \frac{7}{3}}{(p-3)^2 + p^2 - p - 4q + \frac{7}{3}}$$

and

$$(26)\qquad b_1 \geqq (p-2)^2 / (p-3)^2\,,$$

respectively, where, by (21), we should assume

$$(27)\qquad \frac{(p-2)^2 + p^2 - p - 4q + \frac{7}{3}}{(p-3)^2 + p^2 - p - 4q + \frac{7}{3}} \leqq 1$$

and

(28) $$(p-2)^2 \,/\, (p-3)^2 \leqq 1\,.$$

On the other hand, as it can easily be verified, conditions (19) and (28) imply

$$\frac{(p-2)^2}{(p-3)^2} \leqq \frac{(p-2)^2 + p^2 - p - 4q + \frac{7}{3}}{(p-3)^2 + p^2 - p - 4q + \frac{7}{3}}$$

and (27). Consequently we have to assume (25) and (28), or, what is the same, (25) and $p \leqq \frac{5}{2}$. Under these conditions we can assert (22).

Suppose next (20). By (21), in order to obtain (22) we have to assume

(29) $$(p-2)^2 - (p-3)^2 b_1 - \tfrac{1}{2}(p^2 - p - 4q + \tfrac{7}{3}) \cdot 0 \leqq 0$$

and

(30) $$(p-2)^2 - (p-3)^2 b_1 + \tfrac{3}{2}(p^2 - p - 4q + \tfrac{7}{3})\,(-\,2)\,(1 - b_1) \leqq 0\,.$$

Inequalities (29) and (30) are equivalent to (26) and

(31) $$b_1 \geqq \frac{(p-2)^2 - 3(p^2 - p - 4q + \frac{7}{3})}{(p-3)^2 - 3(p^2 - p - 4q + \frac{7}{3})}\,,$$

respectively, where, by (21), we should assume (28) and

(32) $$\frac{(p-2)^2 - 3(p^2 - p - 4q + \frac{7}{3})}{(p-3)^2 - 3(p^2 - p - 4q + \frac{7}{3})} \leqq 1\,.$$

On the other hand, as it can easily be verified, conditions (20) and (28) imply

$$\frac{(p-2)^2}{(p-3)^2} \leqq \frac{(p-2)^2 - 3(p^2 - p - 4q + \frac{7}{3})}{(p-3)^2 - 3(p^2 - p - 4q + \frac{7}{3})}$$

and (32). Consequently we have to assume (31) and (28), or, what is the same, (31) and $p \leqq \frac{5}{2}$. Under these conditions we can assert (22).

Inequalities (18) and (22) yield $B - B^{**} \leqq 0$, as desired.

Finally we notice that, since (17) gives no loss of generality, in order to demonstrate that $B = B^{**}$ can only hold for $f = f_c^{**}$ it is sufficient to show that $B = B^{**}$ with additional condition (17) can only hold for $f = f_0^{**}$. To this end we observe first that if

$$\frac{(p-2)^2 + p^2 - p - 4q + \frac{7}{3}}{(p-3)^2 + p^2 - p - 4q + \frac{7}{3}} < b_1 \leqq 1 \ \text{ for } q \leqq \tfrac{1}{4}(p^2 - p + \tfrac{7}{3})\,,$$

while

$$\frac{(p-2)^2 - 3(p^2 - p - 4q + \frac{7}{3})}{(p-3)^2 - 3(p^2 - p - 4q + \frac{7}{3})} < b_1 \leqq 1 \ \text{ for } q \geqq \tfrac{1}{4}(p^2 - p + \tfrac{7}{3})\,,$$

then the expressions in the square brackets in (22) are negative. Consequently (18) yields that $B - B^{**}$ can only vanish for $a_2 = u + iv = 0$. Hence we must seek the extremum functions in the subclass of $S(b_1)$ with $a_2 = 0$. Here, by (17), we assume that $a_4 - pa_2a_3 + qa_2^3 > 0$, i.e. $a_4 > 0$. For such functions we apply the estimate (1) with $x_1 = \tilde{x}_1\bar{a}_3$, $x_2^2 = \tilde{x}_2\bar{a}_3$, and choose $\tilde{x}_1, \tilde{x}_2$ to be nonnegative. Then we get

$$a_4 + \tilde{x}_2|a_3|^2 + 2\tilde{x}_1|a_3|^2$$
$$\leqq \tfrac{2}{3}(1 - b_1^3) + 2\tilde{x}_1^2(1 - b_1)|a_3|^2 + \tilde{x}_2(1 - b_1^2)|a_3| \,,$$

whence

$$B - B^{**} \leqq \tilde{x}_2(1 - b_1^2)|a_3| - 2[\tilde{x}_1 - \tilde{x}_1^2(1 - b_1) - \tfrac{1}{2}\tilde{x}_2]|a_3|^2 \,.$$

Therefore, if we put $\tilde{x}_1 = 1$ and $\tilde{x}_2 = 0$, the last inequality becomes

$$B - B^{**} \leqq -2b_1|a_3|^2 \,,$$

where equality is possible only for functions in $S(b_1)$ with $a_2 = a_3 = 0$ and $a_4 > 0$. Now we apply a known result (cf. [15], pp. 13—14) that if f belongs to $S(b_1)$ and satisfies the conditions $a_2 = a_3 = 0$ and $a_4 > 0$, then $f = f_0^{**}$. On the other hand, for any f_c^{**}, $-\pi < c \leqq \pi$, we have $B = B^{**}$, as it can easily be verified by direct calculation. Thus the proof is completed.

4. Conclusions

We begin with the applications announced in Section 1. Let

$$g(z) = B_0 + B_1z + \ldots, \; |z| \leqq 1 \,.$$

Hence, by the definition of g,

$$\frac{1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \ldots}{1 + a_2z + a_3z^2 + a_4z^3 + \ldots} = B_0 + B_1z + B_2z^2 + B_3z^3 + \ldots$$

and, consequently,

$$B_0 = 1, B_1 = a_2, B_2 = 2a_3 - a_2^2, B_3 = 3a_4 - 3a_2a_3 + a_2^3 \,.$$

Therefore, by Theorem 2,

$$|B_3| \leqq 2(1 - b_1^3) \,, \tfrac{2}{5} \leqq b_1 \leqq 1 \,.$$

Next let

$$h(z) = C_0 + C_1z + \ldots, \; |z| \leqq 1 \,.$$

Hence, by the definition of h,

$$1 + \frac{2a_2 z + 6a_3 z^2 + 12a_4 z^3 + \ldots}{1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \ldots$$

and, consequently,

$$C_0 = 1,\ C_1 = 2a_2,\ C_2 = 6a_3 - 4a_2^2,\ C_3 = 12a_4 - 18a_2a_3 + 8a_2^3 .$$

Therefore, by Theorem 2,

$$|C_3| \leqq 8(1 - b_1^3),\ \tfrac{1}{4} \leqq b_1 \leqq 1 .$$

Further we remark that in Theorems 1 and 2 the intervals for b_1 can be improved if we do not restrict the parameters x_1 and x_2 in (1) to real numbers. However in this case calculations are much more complicated. Furthermore, one may try to generalize Theorems 1 and 2 by considering complex p and q.

Finally we remark that the same method can be applied to analogous functionals of higher orders. A counterpart of Theorem 2 for the functional of the fifth order

$$|a_5 - pa_2a_4 - qa_3^2 + ra_2^2a_3 - sa_2^4| ,\ p, q, r, s \text{ real}$$

is established in our forthcoming paper [5]. More generally, we introduce an analogous n-th order functional as follows.

Let f be a function in $S(b_1)$. Define the coefficients A_{mk} by the relation

$$\log \frac{f(z) - f(z_0)}{z - z_0} = \sum_{m,k=0} A_{mk} z_0^k z^m,\ |z| < 1,\ |z_0| < 1 .$$

Clearly, $A_{mk} = A_{km}$ for any (m, k). It is well known that the coefficients A_{mk} play an important role in the Grunsky-Nehari inequalities (cf. [2], [6], [8], and [9]). In particular (cf. [8], p. 4), we have

$$A_{00} = \log b_1 ,\ A_{02} = a_3 - \tfrac{1}{2}a_2^2 ,\ A_{03} = a_4 - a_2a_3 + \tfrac{1}{3}a_2^3 ,$$
$$A_{01} = a_2 ,\quad A_{11} = a_3 - a_2^2 ,\ A_{12} = a_4 - 2a_2a_3 + a_2^3 .$$

A polynomial of $n - 1$ variables $a_2, \ldots, a_n$, $n \geqq 2$, is said to be *related* to A_{mk}, $m + k = n$, if

(i) its coefficient at a_n is 1,

(ii) each other of its coefficients is 0 if the corresponding coefficient of A_{mk}, $m + k = n$, is 0.

It can easily be shown that if a polynomial is related to A_{mk}, $m + k = n$, then it is related to each $A_{m'k'}$, $m' + k' = n$. The modulus of a polynomial which is related to some A_{mk}, $m + k = n$, calculated at the point $(a_2, \ldots, a_n)$ determined by some function f of $S(b_1)$, is said to be an *n-th order Grunsky functional.* The coefficients of this polynomial are called the *coefficients* of the corresponding Grunsky functional.

In view of the results obtained in [10], [11], and [8] it seems to the authors natural to pose the following:

Conjecture. *Suppose that* $n, n \geqq 2,$ *is an arbitrary integer and*

$$B(a_2, \ldots, a_n; p, q, \ldots) \tag{33}$$

is an n-th order Grunsky functional with coefficients $p, q, \ldots$ *In the Euclidean* $(p, q, \ldots)$*-space there is a neighbourhood* D_n *of* $(0, 0, \ldots)$ *such that if the point* $(p, q, \ldots)$ *and a function* f *corresponding to* $(a_1, \ldots, a_n)$ *belong to* D_n *and* $S(b_1)$, *respectively, then* (33) *does not exceed*

$$B_n = \frac{2}{n-1}(1 - b_1^{n-1}) \tag{34}$$

for b_1 *in some interval*

$$b_n(p, q) \leqq b_1 \leqq 1, \, 0 < b_n(p, q) < 1 \, .$$

The estimate is sharp for every $p, q, \ldots,$ *and* b_1. *All the extremal functions are given by the formula*

$$f_{n,c}(z) = e^{-ic} P_n^{-1}(b_1 P_n(e^{ic} z)), \, P_n(z) = z/(1 - z^{n-1})^{\frac{2}{n-1}},$$
$$|z| \leqq 1, \, -\pi < c \leqq \pi \, .$$

The authors believe that the estimate of (33) by (34) can be established with help of the method applied in [8], while the totality of the extremal functions can be determined with help of the method applied in [15]. These methods, however, may fail in view of technical difficulties.

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References

[1] GOLUSIN, G. M. (Голузин, Г. М.): О коэффициентах однолистных функций. - Mat. Sb. N. S. 12 (1943), 40–47.

[2] GRUNSKY, H.: Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. - Math. Z. 45 (1939), 29–61.

[3] JAKUBOWSKI, Z. J.: Sur les coefficients des fonctions univalentes dans le cercle unité. - Ann. Polon. Math. 19 (1967), 207–233.

[4] JANOWSKI, W.: Le maximum des coefficients A_2 et A_3 des fonctions univalentes bornées. - Ibid. 2 (1955), 145–160.

[5] ŁAWRYNOWICZ, J. and TAMMI, O.: On estimating of a fifth order functional for bounded univalent functions. - Colloq. Math. [to appear].

[6] NEHARI, Z.: Some inequalities in the theory of functions. - Trans. Amer. Math. Soc. 75 (1953), 256–286.

[7] SCHIFFER, M. and TAMMI, O.: On the fourth coefficient of bounded univalent functions. - Ibid. 119 (1965), 67–78.

[8] —»— On bounded univalent functions which are close to identity. - Ann. Acad. Sci. Fenn. Ser. A I 435 (1968), 26 pp.

[9] —»— On the coefficient problem for bounded univalent functions. - Trans. Amer. Math. Soc. 140 (1969), 461–474.

[10] SIEWIERSKI, L.: The local solution of the coefficient problem for bounded schlicht functions. - Łódzkie Towarzystwo Naukowe. Soc. Sci. Lodz. Sect. III 68 (1960), 124 pp.

[11] —»— Sharp estimation of the coefficients of bounded univalent functions near the identity. - (a) Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 575–576. (b) Dissertationes Math. Rozprawy Mat. [to appear].

[12] TAMMI, O.: On the maximalization of the coefficients of schlicht and related functions. - Ann. Acad. Sci. Fenn. Ser. A I 114 (1952), 51 pp.

[13] —»— On certain combinations of the coefficients of schlicht functions. - Ibid. 140 (1952), 13 pp.

[14] —»— On the maximalization of the coefficient a_3 of bounded schlicht functions. - Ibid. 149 (1953), 14 pp.

[15] —»— Grunsky type of inequalities, and determination of the totality of the extremal functions. - Ibid. 443 (1969), 20 pp.

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