FORMAL SERIES, FINITENESS PROPERTIES AND DECISION PROBLEMS

BY

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Abstract

Some finiteness properties of formal series over several non-commutative variables are investigated. Some unsolvability results for those properties are proved. A simple example of a context sensitive language over a single letter alphabet which is not probabilistic is exhibited.

Introduction [1]

An interesting connection between some classical results in analysis dating back to Cauchy and Jacobi and some quite new theories such as System Theory, Automata Theory and Computability has been established in the past few years. The classical results are concerned with a special type on infinite matrices, called Hankel matrices, connected with the Routh-Hurwitz problem (see Cantmacher (1959)), with formal series expansion of rational and alg braic functions (see Hurwitz (1889)) and approximations of formal series by an expansion of a rational function — Pade' approximation (see Frobenius (1881)). The formal approach has been rejuvenated recently by Schutzenberger (1961) followed by several others (e.g. Shamir (1967)), Nivat (1968), Flies (1969) who introduced noncommutative variables and studied formal series in connection with the theory of formal languages and automata. Kalman and Ho and others (see Kalman Arbib and Falb (1969)) have applied a generalized form of Hankel matrices to System Theory and solved the realization problem of input-output linear maps. Nasu and Honda, emerging from probabilistic automata theory, have established an interesting finiteness property of a formal series generated by a probablistic automaton, a property which was used by them to prove some undecidebility theorems (see Nasu and Honda (1969)). Finally Carlyle and Paz (1970) have found a connection between a generalized form of Hankel matrices and formal series which are generated by pseudo-probabilistic automata.

The first three sections of this paper are concerned with four finiteness properties of formal series over several noncommutative variables. Those

properties are related to both classical results and modern ones as mentioned above. While the four properties above where known to be equivalent when a single variable is involved, it is shown that this is no longer true over several noncommutative variables, and the relationship between these properties is investigated in general.

In the next two sections some nonsolvability results connected to the properties described in the first sections are proved, including some open problems left over by Nasu and Honda (1969).

The last section deals with formal languages over a single letter alphabet defined by formal series satisfying finiteness conditions as described in the first sections. A simple example of a context sensitive language which is not probabilistic is presented and it is shown that the problem whether an arbitrary probabilistic language over a single letter alphabet is regular or context-free is not solvable.

A. Word-functions

Let $X = \langle x , y \rangle$ be a finite alphabet (for the sake of simplicity we shall assume that X contains at most two elements unless otherwise stated), let X^* be the free semigroup over X (the set of all »words», or finite sequences of »symbols» or elements, from X including the empty word to be denoted by Λ) and let f be a function $f : \Sigma^* \to A$ where A stands for some division ring (in particular A may be equal to R — the set of real numbers which is a field). Such functions will be called »word-functions».

Consider the following properties of a given word-function:

(I) For every word $v \varepsilon X^*$ there exists an integer $n \leq n_0$, where n_0 is a fixed integer which does not depend on v , and elements $C_{n-1} , \ldots , C_0 \varepsilon A$ such that for any two words $u , w \varepsilon X^*$ the following equation holds true

(1) $$f(uv^n w) = C_{n-1} f(uv^{n-1} w) + C_{n-2} f(uv^{n-2} w) \ldots + C_0 f(uw)$$

where v^n is the word $\underbrace{vvv \ldots v}_{n}$ and uv is the catenation of the words u and v (i.e. if $u = x_1 , \ldots , x_k , v = y_1 , \ldots , y_j$ then $uv = x_1 \ldots x_k y_1 \ldots y_j$) .

(II) There exists an integer n , $n \times n$ matrices $A(x) , A(y)$ over A , and n-dimensional vectors over A , π and η , π a row vector and η a column vector such that for any $w = x_1 \ldots x_k \varepsilon X^*$

(2) $$f(w) = \pi A(x_1) \ldots A(x_k)\eta = \pi A(w)\eta$$

(III) Let $\triangle(f)$ be the infinite matrix $\triangle(f) = \| a_{ij}(f)\|$ with $a_{ij}(f) = f(u_i u_j)$ where $u_1 u_2 u_3 \ldots$ is a fixed preassigned lexicographical ennumeration of the elements of X^*

(3) $$\triangle(f) \text{ is of finite rank}$$

(IV) There exists two polynomials $P(x,y)$ and $Q(x,y)$ with coefficients in A and (noncommutative) variables in X such that:

(4) $$Q(x,y)\left(\sum_{v \varepsilon X^*} f(v)v^{-1}\right) = P(x,y) \text{ or } \left(\sum_{v \varepsilon X^*} f(v)v^{-1}\right) Q(x,y) = P(x,y)$$

where the product in the left-hand side is defined in the ordinary way.

The following theorems are either known or can be easily derived from known theorems.

Theorem 1. If X contains a single letter then the four properties (I), (II), (III), (IV) are equivalent for a given word-function.

Proof. That (I) is equivalent to (II) follows as a particular case from a more general theorem proved by Ho (see Kalman Arbib and Falb (1969)). That (I) is equivalent to (III) and (IV) is a classical result which can be found in many books, e.g. Gantmacher (1959) pp. 243—247.

Theorem 2. For the general case (X contains two or more letters), the properties (II) and (III) are equivalent and both of them imply property (I) for a given word function f.

Proof. That (II) is equivalent to (III) follows as a particular case from a more general theorem proved by Carlyle and Paz (1970). That (II) implies (I) has been proved by Nasu and Honda (1968).

B. Some negative results

Theorem 3. If in property (I) $n \geq 3$ for every $v\varepsilon\Sigma^*$ and X contains two (or more) *noncommutative* letters (variables) then (I) does not imply (III).

Proof. We shall show how to construct a word function f such that f satisfies (I) but it does not satisfy (III). Let $X = \langle 0, 1 \rangle$ and consider the following infinite sequence

$$U = 0110100110010110 \ldots = x_1 x_2 x_3 \ldots$$

constructed as follows: for $x_i = 0$ or $x_i = 1$ let $\bar{x}_i = 1 - x_i$. The first letter in the sequence is 0. If the first 2^k leters (for $k = 0, 1, 2, \ldots$) in the sequence have already been fixed and are equal to $x_1 x_2 \ldots x_{2^k}$ then the next 2^k letters are $\bar{x}_1 \bar{x}_2 \ldots \bar{x}_{2^k}$. Let v, w denote finite subsequences of U of the form $v = x_i x_{i+1} \ldots x_{i+m}$, $w = x_j x_{j+1} \ldots x_{j+n}$. It is known that the

infinite sequence U , constructed as above, contains no finite subsequence v such that $v = www = w^3$ (see Morse and Hedlund (1938)). Consider now the following two sequences of words over X^*.

$$u_0 = \lambda \, , u_1 = x_1 \, , u_2 = x_1 x_2 \, , \ldots u_i = x_1 x_2 \ldots x_i$$
$$v_0 = \lambda \, , v_1 = x_2 \, , v_2 = x_3 x_4 \, , \ldots v_i = x_{i+1} \ldots x_{2i}$$

and construct a word function $f , f : X^* \to R$ as follows: f will satisfy a relation of the form (1) with $n \geq 3$ and in addition the following determinants

$$\begin{vmatrix} f(u_0 v_0) \ldots f(u_0 v_i) \\ \vdots \\ f(u_i v_0) \ldots f(u_i v_i) \end{vmatrix} = \triangle_i$$

will all have nonzero value for $i = 1 , 2 , 3 , 4 , \ldots$ We prove by induction that the construction of such a function is possible. For $i = 1$ we have

$$\triangle_1 = \begin{vmatrix} f(\lambda) & f(x_2) \\ f(x_1) & f(x_1 x_2) \end{vmatrix}$$

and the values of f appearing in this determinant can be choosen at will and in a way such that $\triangle_1 \neq 0$ without affecting the relation (1) to be satisfied by f; for the arguments of f in the entries of $\triangle_1$ are words of length ≤ 2 (the »length» of a word $w = x_1 \ldots x_k$, to be denoted $|w|$ is the number of letters in it, i.e. $|w| = k$) . Assume now that all the values of f have been choosen for words w with $|w| \leq 2i$ such that f satisfies the relation (1) for such words and $\triangle_i \neq 0$. $\triangle_{i+1}$ has the form

$$\triangle_{i+1} = \begin{vmatrix} \triangle_i & f(u_0 v_{i+1}) \\ & \vdots \\ f(u_{i+1} v_0) \ldots f(u_{i+1} v_{i+1}) \end{vmatrix}$$

By construction $u_{i+1} v_{i+1} = x_1 x_2 \ldots x_{2i+2}$ wich is an initial subsequence of U of length $2i + 2$ so that, by the property of U mentioned above, the value $f(u_{i+1} v_{i+1})$ does not depend on any other values $f(w)$ already fixed — although f is required to satisfy a relation of the form (1). But, by induction, $\triangle_i \neq 0$ and therefore, one can choose values $f(w)$ for all w with $|w| \leq 2i + 2$ not yet fixed by the relation (1) including $f(u_{i+1} v_{i+1})$ so that $\triangle_{i+1} \neq 0$. It follows that a function f satisfying a relation of the form (1) can be constructed such that $\triangle(f)$ has infinite rank, for the determinants $\triangle_i$ are subdeterminants of $\triangle(f)$ for $i = 1 , 2 , \ldots$.

Theorem 4. If X contains two or more variables then (IV) does not imply (I) in general and therefore (IV) does not imply (III) or (II) either (for (III) implies (I) by Theorem 2).

Proof. Let A be the set of real numbers and let $X = \{x, y\}$ where x and y are commutative variables over A. Consider the word function f defined as follows:

$$f(w) = \begin{cases} 1 & \text{if } w = x^m y^m \qquad m = 0, 1, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(xy - 1) \sum_{i=0}^{\infty} \frac{1}{x^i y^i} = xy$$

Thus $f(w)$ satisfies (IV). But $f(w)$ cannot satisfy (I). To prove this, assume that

$$f(u^n v) = C_{n-1} f(u^{n-1} v) + C_{n-2} f(u^{n-2} v) \dots C_0 f(v)$$

then if $u = x$ and $v = y^n$ we must have that

$$f(x^n y^n) = C_{n-1} f(x^{n-1} y^n) + \dots \dots + C_0 f(y^n)$$

which is impossible for $f(x^n y^n) = 1$ while $f(x^{n-i} y^n) = 0$ for $i = 1, \dots, n$.

Remark: Note that the variables in the example of Theorem 4 are commutative. For noncommutative variables Theorem 4 is no longer true. For noncommutative variables (IV) implies (II) (and therefore also III and (I)). This follows, as a particular case, from a Theorem of Schutzenberger (1961).

Example: Let A be the set of real numbers and let $X = \{x, y\}$ where x and y are *noncommutative* variables over A. Consider the word function f defined as follows:

$$f(w) = \begin{cases} 1 & \text{if } w = (xy)^m \quad m = 0, 1 \dots \\ 0 & \text{otherwise} \end{cases}$$

then

$$(xy - 1) \sum_{i=0}^{\infty} (xy)^{-i} = xy$$

but this function $f(w)$ satisfies also (II) as follows: Let $A(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\pi = (10)$; $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It is easily verified that $f(w) = \pi A(w) \eta$. Note that this function reduces to the function in Theorem 4 when the variables x and y are commutative.

C. Two particular cases

The previous two theorems show that the range of Theorem 2 cannot be extended to the range of Theorem 1 in general. The purpose of the next two theorems is to show that the range of Theorem 2 can be extended slightly, provided that some additional restrictions are imposed on the specified conditions (I), (II), (III).

Theorem 5. If the function $f : X^* \to A$ is such that the Variables X are commutative and A is a division ring then property (I) implies the property (III) and therefore, by Theorem 2, properties (I), (II) and (III) are equivalent.

Proof. Let f be a word function over a two letter alphabet $X = \{x , y\}$ satistiying the conditions of the theorem and satisfying a relation of the form

$$(5) \qquad f(uw^n v) = C_{n-1} f(uw^{n-1} v) + \ldots + C_0 f(uv)$$

where the constants $n , C_{n-1} \ldots C_0$ depend on w only and $n \leq n_0$, for some fixed integer n_0 not depending on w. Let η be any column in $\triangle(f)$ such that

$$\eta = (f(u_1 w) , f(u_2 w) \ldots f(u_k w) \ldots)^T$$

where $u_1 u_2 \ldots$ is the lexicographic enumeration of the words in Σ^* corresponding to nonzero rows and $|w| \geq 2n_0 - 1$. Then either $w = w' x^{n_0}$ or $w = w'' y^{n_0}$ for the variables are assumed to be commutative. Thus either $\eta = (v_i x^{n_0})^T i = 1 , 2 , \ldots$ or $\eta = (v_i' y^{n_0})^T i = 1 , 2 , \ldots$ where $v_i = u_i w'$ and $v_i' = u_i w''$ and in both cases the column η is a linear combination of of previous columns in $\triangle(f)$, because of the relation (5).

Theorem 6. Let f be a word function $f : \Sigma^* \to A$ such that $\Sigma = <x , y>$ and f satisfies the property (1) with $n_0 \leq 2$ then f satisfies (III).

Proof. Let $f(u_i)$ be the leading term of the $i-th$ column of $\triangle(f)$. As $u_0 u_1 , \ldots$ is a lexicographic ordering of the words in $\Sigma^*, i > j$ implies that $|u_i| \geq |u_j|$. If $|u_i| \geq 4$ then u_i is representable in one of the forms $u_i' x^2 u_i''$ or $u_i' y^2 u_i''$ or $u_i' (xy)^2 u_i''$ or $u_i' (yx)^2 u_i''$. This follows from the fact that any word of length 4 which is not equal to $(xy)^2$ or $(yx)^2$ must have a subword of the form x^2 or a subword of the form y^2 (this can be checked by straight forward inspection). It follows from the above observation and from the conditions of the theorem that any column in $\triangle(f)$ whose leading term is $f(u_i)$ with $|u_i| \geq 4$ is a linear combination of two columns with leading terms $f(u_j)$ with $|u_j| < |u_i|$ and this concludes the proof.

D. Solvability problems

This and the following section assume basic knowledge of Automata Theory and Recursive Functions Theory.

Let f_1, f_2 be word functions over the same alphabet X and define the following operations on them:

$$f_3(u) = (f_1 \lor f_2)(u) = max(f_1(u), f_2(u))$$
$$f_4(u) = (f_1 \land f_2)(u) = min(f_1(u), f_2(u))$$

Nasu and Honda (1969) considered the following problem: Is it recursively solvable to ascertain for arbitrary word functions f_1 and f_2 satisfying the property (II) whether $f_1 \lor f_2$ and $f_1 \land f_2$ satisfy that same property?

A negative answer to the above question was given by the above authors using an example as follows: A set of languages $\{L\}$ (L is a set of words $L \subset \Sigma^*$) and two sets of word functions $\{f_1\}\{f_2\}$ over an alphabet with two letters are given such that:

(1) It is recursively unsolvable for an arbitrary element L in $\{L\}$ to ascertain whether $L = \emptyset$ ($\emptyset$ is the empty set of words).

(2) $L = \emptyset$ implies that $f_1 \lor f_2$ satisfies the property (II) where $f_1 \varepsilon \{f_1\}$ $f_2 \varepsilon \{f_2\}$ and $f_1 f_2$ are determined by L.

(3) $L \neq \emptyset$ implies that $f_1 \lor f_2$ does not satisfy the property (I), $f_1 f_2$ are as in (2). As (II) implies (I) we have that

(4) $L \neq \emptyset$ if and only if $f_1 \lor f_2$ satisfies (II) and therfore it is recursively unsolvable to ascertain for arbitrary $f_1 \lor f_2, f_1 \varepsilon \{f_1\}, f_2 \varepsilon \{f_2\}$ whether $f_1 \lor f_2$ satisfies (II) (a similar argument do for $f_1 \land f_2$). This is Theorem 24 in Nasu and Honda (1969). There are however some additional consequences to be drawn from the above example and our work here. Those consequences are given in the following corollaries.

Corollary 1. For two arbitrary functions f_1 and f_2 which satisfy both property (I) or (III), it is recursively unsolvable to decide whether $f_1 \lor f_2$ and $f_1 \land f_2$ satisfy (I) or (III), respectively, provided that the alphabet X contains two or more letters.

Proof: (2) and (3) above imply not only (4) but also that:

(5) $L = \emptyset$ if and only if $f_1 \lor f_2$ satisfy (I) (this following from the fact that (II) implies (I) and if f_1 and f_2 satisfy (II) they also satisfy (I)). The statement regarding property (III) follows from Theorem 2.

Remark: As properties (I) and (II) are not equivalent (Theorem 2 and Theorem 3 here) Corollary 1 is not equivalent to the theorem of Nasu and Honda cited above.

Corollary 2. It is recursively unsolvable to decide for an arbitrary word function f whether f satisfies property (I) or property (II) or property (III) provided the alphabet X contains two or more letters.

Proof. Unsolvability of property (II follows from the theorem of Nasu and Honda. Unsolvability of (I) follows from Corollary 1 above and unsolvability of (III) follows from Theorem 2 and from the unsolvability of (II).

In the above cited paper of Nasu and Honda (1969) they considered the nonsolvability problems for functions over alphabeta containing at least two letters, leaving the single-letter case open. We shall prove here that the nonsolvability results proved by them are valid also for the single etter case using an example suggested by them and introduced first by this author (Paz, 1966).

E. Unsolvability for single letter alphabets

We introduce first the following three functions over a single letter alphabet $\Sigma = \{\sigma\}$.

(1) $f(\sigma^k) = \pi[A(\sigma)]^k\eta$, $k = 1 , 2 , \ldots$ with $\begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{5}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$\pi = (0\ 0\ 1) ; \eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; A(\sigma) =$$

Thus, f satisfies the condition (II) by its definition.

The following property of f can be proved using an argument similar to the one used in Paz (1966, p. 31):

For any integer n there are integers t , k_1 and k_2 such that:

$$f(\sigma^{k1}\sigma^{nt}) > \frac{4}{11} ; \qquad f(\sigma^{k2}\sigma^{nt}) \leq \frac{4}{11}$$

$$f(\sigma^{ki}\sigma^{pt}) \leq \frac{4}{11} ; i = 1 , 2 . \quad p = 0 , 1 . 2 , \ldots , n - 1$$

(2) Let g be the function $g(\sigma^k) = \frac{4}{11}$ for $k = 1 , 2 \ldots$. The function g satisfies the condition (II) trivially.

(3) Let $\{Z\}$ be the family of Turing machines. For every $Z \varepsilon \{Z\}$ define the word function h_Z as follows: $h_Z\{\sigma^n\} = 1$ if Z is still moving after n steps, beginning from blank tape and $h_Z(\sigma^n) = 0$ otherwise.

Then h_Z is recursive and total assuming one of the two following forms:

$$\text{Either} \quad h_Z(\sigma^n) = 1 \quad \text{for} \quad n = 1, 2, \ldots$$

$$\text{or} \quad h_Z(\sigma^n) = 0 \quad \text{for} \quad n > n_0$$

(if Z halts after n_0 steps).

Although it is undecidable to ascertain for arbitrary h_Z which of the two forms is the actual form of h_Z still in both cases h_Z satisfies the condition (II) for any Z as one verifies easily.

Proposition 1. For arbitrary h_Z as above the functions fh_Z and gh_Z satisfy the condition (II).

Proof. It was proved in Paz (1966) that the class of functions satisfying the condition (II) with the additional restriction that the matrices $A(\sigma)$ be stochastic is closed under multiplication. The proof can be extended readily so as to remove the above additional restriction.

Proposition 2. The function $f \vee g$ does not satisfy the condition (II).

Proof: By Theorem 1 it suffices to show that $f \vee g = F$ does not satisfy condition (1). Assume the contrary, then there is some integer n such that for any integer t the function F satisfies a relation of the form

$$F(\sigma^k \sigma^{nt}) = \sum_{i=1}^{n} C_{n-1} F(\sigma^k \sigma^{(n-i)t})) \quad k = 0, 1, \ldots$$

Evaluating this relation first for t and k_1 as in the remark after the definition of the function f one finds that:

$$\frac{4}{11} < \max\left(\frac{4}{11}, f(\sigma^{k1}\sigma^{nt})\right) = F(\sigma^{k1}\sigma^{nt}) =$$

$$= \sum_{i=1}^{n} C_{n-1} F(\sigma^{k1}\sigma^{(n-i)t}) = \sum_{i=1}^{n} C_{n-1} \max\left(f(\sigma^{k1}\sigma^{(n-i)t}) \cdot \frac{4}{11}\right) =$$

$$= \frac{4}{11} \sum_{i=1}^{n} C_{n-1}$$

$$\text{or} \quad \sum_{i=1}^{n} C_{n-1} > \frac{4}{11}$$

On the other hand, evaluating the relation for t and k_2 one finds in the same way that $\sum_{i=1}^{n} C_{n-1} = \frac{4}{11}$ which is a contradiction.

Theorem 7. It is recursively unsolvable to determine for arbitrary word functions f, g satisfying both the condition (II) and over a single-letter alphabet $\Sigma = \{\sigma\}$ whether $f \vee g$ or $f \wedge g$ satisfy the condition (II).

Proof: We shall prove the Theorem for $f \vee g$, the proof of the other statement being similar.

Consider the functions fh_Z and gh_Z for arbitrary h_Z as exhibited at the beginning of this section. Both of them satisfy the condition (II) (Pro-

position 1). But the function $fh_z \vee gh_z = (f \vee g)h_z$ does not satisfy this condition if and only if $h(\sigma^n) = 1$ for all n (for if $f(\sigma^n) = 0$ for $n > n_0$, which is the only alternative, then $(f \vee g)h_z$ will assume only finitely many nonzero values), which is undecidable by the choice of the set $\{h_z\}$.

Corollary 3. It is recursively unsolvable to determine for an arbitrary given word function over a single letter alphabet whether this function satisfies either of the conditions (I), (II), (III) or (IV).

Proof: Property (II) is undecidable for the function exhibited in Theorem 7 and the other properties are equivalent to property (II) by Theorem 1.

F. Languages over a single letter alphabet

Given a word function f we define the languages:

$$L(f) = \{x : x\varepsilon\Sigma^* , f(x) \neq 0\}$$

$$L(f , \lambda) = \{x : x\varepsilon\Sigma^* , f(x) > \lambda\}$$

where λ is a real number.

Languages of the first form have been introduced by Shutzenberger (1961) and languages of the second form have been introduced by Rabin (1963). Let us assume now that the word function f in the definition of the languages above satisfies the property (II). The following properties are well known or easily proved for this case:

(1) There are nonregular languages which can be represented in the form $L(f)$ (Turakainen, 1968).

(2) Every language representable in the form $L(f)$ is representable also in the form $L(f , \lambda)$ (f in both cases satisfies (II)) but the converse is not true.

For the first part see Turakainen (1968). For the second part consider the language $L = \{x^n y^n : n = 1 , 2 , \ldots\}$. It has been proved by Turakainen (1969) that it can be represented in the form $L(f , \lambda)$ with f satisfying (II). That it cannot be represented in the form $L(f)$ can be proved using an argument identical to the argument used in the proof of Theorem 4: If $L(f) = \{x^n y^n : n = 1 , 2 , \ldots\} = \{w : f(w) \neq 0\}$ then f cannot satisfy property (I), but f is assumed to satisfy property II which implies property (I).

(3) The cardinality of the set of languages representable in the form $L(f , \lambda)$ with f satisfying property (II) is equal to the cardinality of the continuum (Rabin (1963)) which is also the cardinality of the set of all languages (Σ being finite, Σ^* is countable and the set of all subsets of Σ^* has the cardinality of the continuum).

The following problem is now natural. Are there languages not represent-
able in the form $L(f, \lambda)$ with f satisfying (II)? Bukharaev (1965) has
given an example of such a language over a single letter alphabet. Unfortu-
nately, he did not prove his example (which is quite complicated, being
defined on the basis of four arbitrary primitive recursive functions). We
have not been able to prove Bukharaev's result, not have we succeeded in
establishing contact with him (directly or indirectly) and, to the best of
our knowledge nobody else in the west knows how to prove that result (see,
e.g. Salomaa who needed that results in order to establish some relation
between time variant events and events defined as above). Note that, when
looking for a language as above, one must take into consideration that
there may be different word functions defining the same language; in
addition, if a single letter alphabet is assumed then, the required language
if existent must be at least context sensitive, for context free languages
over a single letter alphabet are regular and therefore also representable
in the form $L(f, \lambda)$ with f satisfying (II) (see Paz (1966)).

We show now an example of a nonrepresentable language over a single
letter alphabet.

Let $X = \{x\}$ and let $\triangle = \{a, b\}$. Let $u_1 u_2 \ldots$ be a lexicographical
enummeration of all nonempty words over $\triangle$ and let U be the infinite
sequence of letters from $\triangle$ resulting from the concatenation of the Words
$u_1 u_2 \ldots$ in their given order. Let $U(i)$ denote the i-th letter in U and
define the language $L_1 = \{x^k : U(k) = a\}$. L is not representable in the
form $L(f, \lambda)$ with f satisfying property (II).

Proof: Assume $L_1 = L(f, \lambda)$ for some f satisfying property (II).
As property (II) implies property (I) (Theorem 2 here) there are constants
$C_0 \ldots C_{n-1}$ such that

(6) $f(x^{k+n}) = C_0 f(x^k) + C_1 f(x^{k+1}) = \ldots + C_{n-1} f(x^{k+n-1})$

Let $\varepsilon_0 \ldots \varepsilon_n, \delta_0 \ldots \delta_n$ be two words in $\triangle^*$ defined as follows: $\varepsilon_i = b$
if and only if $C_i > 0$, $\varepsilon_i = a$ otherwise, $\varepsilon_n = a$, $\delta_i = \triangle - \varepsilon_i$. By the
construction of U there are integers k_1 and k_2 such that:

$$U(k_1) \ldots U(k_1 + n) = \varepsilon_0 \ldots \varepsilon_n$$
$$U(k_2) \ldots U(k_2 + n) = \delta_0 \ldots \delta_n$$

It follows from the definitions that:

$$C_i > 0 \Rightarrow U(k_1 + i) = \mathrm{b}, \ U(k_2 + i) = a$$
$$\Rightarrow f(x^{k_1+i}) \leq \lambda, f(x^{k_2+i}) > \lambda.$$
$$C_i \leq 0 \Rightarrow U(k_1 + i) = a, \ U(k_2 + i) = b$$
$$\Rightarrow f(x^{k_1+i}) > \lambda, f(x^{k_2+i}) \leq \lambda.$$

$$U(k_1 + n) = a \,, \; U(k_2 + n) = b \Rightarrow f(x^{k_1+n}) > \lambda \,, f(x^{k_2+n}) \leq \lambda \,.$$

Evaluating the formula (6) first for $k = k_1$ and then for $k = k_2$ we have

$$\lambda < f(x^{k_1+n}) = \sum_{i=0}^{n-1} C_i f(x^{k_1+i}) \leq \lambda \sum_{i=0}^{n-1} C_i$$

and

$$\lambda \geq f(x^{k_2+n}) = \sum_{=0}^{n-1} C_i f(x^{k_2+i}) \geq \lambda \sum_{i=0}^{n-1} C_i$$

which is a contradiction.

Remark: It is easily seen that the language L_1 is context-sensitive for the sequence U can be generated by a linear bounded automation, and, as remarked above, no single letter and simpler language can satisfy the required properties. On the other hand by constructing the sequence U in a more complex way (but still containing as finite subsequences all words in $\triangle^*$) one can construct languages not representable in the form $L(f, \lambda)$ with f satisfying (II), but not context-sensitive either. This remark is due to S. Greibach.

We conclude this work with the following:

Theorem 8. It is recursively unsolvable to determine for an arbitrary word function f satisfying condition (II) and over a single letter alphabet, and an arbitrary number λ, $0 \leq \lambda < 1$, whether the language $L(f, \lambda)$ is regular or whether $L(f, \lambda)$ is context-free.

Proof: Consider the function fh_Z as in Proposition 1. $fh_Z = f$ if h_Z assumes only the value 1 for every argument and fh_Z has only finitely many values otherwise. It was proved in Paz (1966) that $L(f, 4/11)$ is not regular. It follows that $L(fh_Z\ 4/11)$ is not regular if and only if h_Z assumes only the value 1 which is recursively unsolvable for arbitrary h_Z. The second satement of the theorem follows from the fact that regular and single-letter languages are context-free.

Remark 1. Using the functions h_Z and $1-h_Z$ one can prove also that it is recursively unsolvable to decide for an arbitrary word function f satisfying (II) and over a single letter alphabet, and arbitrary number λ whether the language $L(f, \lambda)$ is empty or whether it is equal to Σ^*.

Remark 2. Note that the theorems 7 and 8 are existential. The functions h_Z, on which their proofs are based, are shown to satisfy (II) but their representation in the form (2) is not explicit. One may still ask therefore whether the theorems can be improved by showing how to construct the functions h_Z explicitly in the form (2), for any given Z or else by finding a subset of the set $\{Z\}$ for which this is possible and such that the halting problem from blank tape is still unsolvable for it. This remark

is specially noteworthy with regard to Theorem 8, for the problem considered in that Theorem is decidable in many cases, provided that the function is given explicitly in the form (2), as follows from the papers of Paz (1966) and Turakainen (1968).

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