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ON THE COMPLETE INTEGRABILITY OF THE FIRST ORDER TOTAL DIFFERENTIAL EQUATION

BY

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Preface

I am deeply indebted to Professor Heikki Haahti, my teacher, for suggesting this subject and pushing me to start. His fruitful advice during the numerous discussions and his tireless and stimulating interest in the progress of the study helped me essentially to carry out this work.

I wish to express my sincere thanks to Professor Rolf Nevanlinna for his valuable criticism of the manuscript, yielding many improvements in the text. I am also much indebted to Professor Tapio Klemola for his reading the manuscript and for his observations, and to Professor Yrjö Kilpi, head of the Department of Mathematics of the University of Oulu, for his encouragement during this work.

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INDRODUCTION

1. Let X be a normed linear space over the reals $\mathbf{R}$ with dimension ≥ 2, Y a Banach space over $\mathbf{R}$, and let A and B be open subsets of X and Y, respectively. Denote by $L(X\,;Y)$ the Banach space of all bounded linear maps L from X to Y, with the norm $||L|| = \sup\limits_{|x|\leq 1} |Lx|$. Given a mapping F from $A\times B$ to $L(X\,;Y)$, consider the *total differential equation*

$$y'(x) = F(x\,, y(x)) \tag{1}$$

where y' denotes the Fréchet derivative of the mapping y. The differential equation (1) is said to be *completely integrable* in $A\times B$, if it has for each point $(x_0\,, y_0)$ of $A\times B$ a unique solution y in a neighbourhood of x_0, satisfying the initial condition $y(x_0) = y_0$.

2. If F is continuously differentiable and if F_1' and F_2' denote the partial derivatives of F, the *theorem of Frobenius* (Dieudonné [2]) states that the vanishing

$$R(x\,, y) = 0 \tag{2}$$

of the bilinear alternating mapping $R(x\,, y)$ from $X\times X$ to Y, given by

$$R(x\,, y)hk = \wedge\, \{F_1'(x\,, y)hk + F_2'(x\,, y)\,[F(x\,, y)h]k\}^{1)} \tag{3}$$

$(h\,, k \in X)$, for all $(x\,, y) \in A\times B$ is a necessary and sufficient condition for the complete integrability of the total differential equation (1) in $A\times B$. The necessity of this condition is a direct consequence of the symmetry

$$y''(x)hk = y''(x)kh$$

of the second derivative y''. Various methods can be used to prove the sufficiency (see e.g. Nevanlinna [8], Dieudonné [2], Keller [4], Louhivaara [5], Tienari [11], Scriba [10], Penot [9]).

3. The complete integrability of (1) is studied in Nevanlinna [6, 7, 8] when F is continuous in x and linear in y, and in Bächli [1] when F is continuous and satisfies a *Lipschitz condition*

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1) For a bilinear mapping D we denote by $\wedge D$ the alternating part of D, i.e. $\wedge Dhk = 1/2\,(Dhk - Dkh)$.

$$||F(x, y) - F(x, \bar{y})|| \le K|y - \bar{y}| \qquad (K > 0) \tag{4}$$

in $A \times B$. For a given initial value $y_0 \in B$ the differential equation (1) can be integrated in both these cases along sufficiently short oriented piecewise smooth paths l in A. More precisely, if l is such a path with the total length $|l|$ small enough and, if arc length is chosen as a parameter in a representation $s \mapsto l(s)$ of l, then the integral equation

$$u(t) = y_0 + \int_0^t F(l(s), u(s)) l'(s) ds \tag{5}$$

or equivalently, the initial value problem

$$u'(t) = F(l(t), u(t)) \, l'(t), \qquad u(0) = y_0, \tag{6}$$

has a continuous solution u on the closed interval $[0, |l|]$. Moreover, this solution is uniquely determined by the integration path l and the initial value y_0, so that the equations

$$T(l, y_0) = y_0 + U(l, y_0) = u(|l|) \tag{7}$$

define two operators T and U of the pair (l, y_0) with values in Y.

Suppose now (1) to be completely integrable in $A \times B$ and let y be a solution of (1) with $y(x_0) = y_0$. Then

$$T(l, y_0) = y(x) = y_0 + \int_l F(z, y(z)) dz$$ [2]

for each piecewise smooth path l from x_0 to x in the domain of y, since $u(t) = y(l(t))$ is the solution of (5). Thus $T(l, y_0)$ depends for fixed $y_0 \in B$ only on the end points of l, or equivalently, $U(l, y_0) = T(l, y_0) - y_0$ vanishes for closed paths l. Particularly, the condition: for each $y_0 \in B$

$$U(\delta, y_0) = 0 \tag{8}$$

whenever $\delta = \delta\sigma$ is an oriented boundary of sufficiently small simplex $\sigma \subset A$[3]), is necessary for the complete integrability of (1). In the cited cases this condition is shown to be also sufficient. Using Goursat's idea to

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[2]) $\int_l G(z) dz$ denotes the integral $\int_0^{|l|} G(l(s)) \, l'(s) ds$.

[3]) By a simplex we mean here a non-degenerate triangle, i.e. if x_0, x_1 and x_2 are its vertices, then $x_1 - x_0$ and $x_2 - x_0$ are linearly independent. Area Δ of such simplex is defined by $\Delta = |D(x_1 - x_0, x_2 - x_0)|$ where D is a nontrivial bilinear alternating real form of vectors in the subspace of X generated by $x_1 - x_0$ and $x_2 - x_0$.

estimate the norm of $U(\delta , y_0)$ the condition (8) is then reduced to a local integrability condition, equivalent to the Frobenius condition (3) when F is also differentiable.

4. In this paper we shall study the complete integrability of (1) under more general assumptions. Denoting by $\delta \to \bar{x}$ the regular convergence of δ to a point $\bar{x}$ of A in all two-dimensional planes E of X containing $\bar{x}$[4], our main result, which is derived by the above described method due to R. Nevanlinna, can be stated as follows:

Suppose that F is continuous and satisfies locally in $A \times B$ an Osgood condition

$$\|F(x , y) - F(x , \bar{y})\| \leq \varphi(|y - \bar{y}|) \tag{9}$$

where φ is a continuous and increasing function on the set $\mathbf{R}^+$ of non-negative reals such that

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dr}{\varphi(r)} = \infty \tag{$\varphi 1$}$$

and

$$\overline{\lim_{r \to 0+}} \frac{\varphi\,(r/2)}{\varphi\,(r)} < 2^{-1/2} . \tag{$\varphi 2$}$$

Then the condition

$$\lim_{\delta \to \bar{x},\, |y - \bar{y}| \leq C|\delta|} \sup \frac{|U(\delta , y)|}{|\delta|^2} = 0 \tag{10}$$

for all $(\bar{x} , \bar{y}) \in A \times B$ and for all $C > 0$, is necessary and sufficient for the complete integrability of the total differential equation (1) *in* $A \times B$.

In particular, when $\varphi(r) = Kr$ in (9), we get the basic result of Bächli [1] as a corollary. There are some inaccuracies in [1] which are corrected here (see p. 27). Other special cases are obtained by choosing

$$\varphi(r) = Kr \ \log \frac{1}{r} \ldots \log_n \frac{1}{r}$$

where $\log_n$ denotes n times iterated logarithm.

The theorem of Frobenius follows as a corollary if F is supposed also to be differentiable, but not necessarily continuously differentiable.

The hypothesis $(\varphi 2)$, which is not generally included in the Osgood condition, is added to show the sufficiency of the condition (10) for the complete integrability of (1). Actually, our proof fails if this hypothesis is replaced by

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4) $\delta = \delta\sigma$ is said to converge regularly to $\bar{x}$ in E if $\bar{x} \in \sigma \subset E$ and if $|\delta|^2/\Delta$ remains bounded for $|\delta| \to 0$.

$$(\varphi 2)' \qquad \overline{\lim_{r \to 0+}} \frac{\varphi(r/2)}{\varphi(r)} \leq 2^{-1/2} ,$$

as we shall show by a counter-example (p. 27).

The domain of the solution of (1) is also estimated (Lemma 5 p. 12), and finally we shall study the possibility to generalize further the hypotheses of the mapping F.

1. Preliminaries

For simplicity we shall suppose in this chapter that A and B are the open balls

$$(1.1) \qquad A = \{x \in X \mid |x| < \varrho\} , \quad B = \{y \in Y \mid |y| < \varrho'\}$$

and that the mapping $F: A \times B \to L(X ; Y)$ has the following properties:

1° *F is bounded and continuous in $A \times B$,*

2° *F satisfies in $A \times B$ the Osgood condition* (9) *where φ is a bounded continuous and increasing function on $\mathbf{R}^+$ satisfying the hypothesis* (φ 1).

1.1 We shall first set up some properties of the operators T and U given by (7) (p. 8). PA denotes in the sequel the set of all oriented polygonal paths in A, i.e. paths of the form $l = x_n x_{n-1} \ldots x_1 x_0$ from x_0 to x_n formed by the oriented line segments $x_i x_{-1}$ from x_{i-1} to x_i, $i = 1, \ldots, n$.

Lemma 1. *The operators T and U are defined in the set*

$$(1.2) \qquad W = \left\{(l , y) \in PA \times B \,\middle|\, |l| < \frac{\varrho' - |y|}{M}\right\}$$

where

$$(1.3) \qquad M = \sup \{\|F(x , y)\| \mid (x , y) \in A \times B\} .$$

In view of the definition (7) of the operators T and U this lemma states that the continuous solution of the initial value problem (6) (p. 8) exists and is unique on the interval $[0 , |l|]$ for fixed $(l , y_0) \in W$. With the hypotheses 1° and 2° this is a well-known result of the theory of the ordinary differential equations (see Nevanlinna [8], p. 153). By this theory we get also

Lemma 2. *If l_1 and l_2 are paths of PA such that the product path $l_2 l_1$ is defined* (i.e. the final point of l_1 agrees with the initial point of l_2), *then*

$$(\mathrm{T1}) \qquad T(l_2 l_1 , y) = T(l_2 , T(l_1 , y)) ,$$

both sides being defined whenever one side is.

Furthermore,

$$T(l^{-1}, T(l, y)) = y \tag{T2}$$

for all (l, y) *in the domain of* T, l^{-1} *being the inverse path of* l.

The hypotheses given for φ ensure that the integral equation

$$v(t, r) = r + \int_0^t \varphi(v(s, r))ds \tag{1.4}$$

has a unique solution v in $\mathbf{R}^+ \times \mathbf{R}^+$. Since v is for $r > 0$ also the solution of the integral equation

$$\int_r^{v(t,r)} \frac{d\alpha}{\varphi(\alpha)} = t, \tag{1.5}$$

we see that v is increasing in its both arguments. Moreover $v(t, 0) \equiv 0$. In the proof of our main theorem we shall also need the following inequality, which will be proved in the last section (p. 31):

Lemma 3. *For all* $(l, y_j) \in W$, $j = 1, 2$

$$|T(l, y_1) - T(l, y_2)| \leq v(|l|, |y_1 - y_2|). \tag{1.6}$$

1.2. For the sake of completeness we shall prove the following result (cf. Bächli [1] Prop. 2):

Lemma 4. *Given a point* (x_0, y_0) *of* $A \times B$, *the initial value problem*

$$y'(x) = F(x, y(x)), \qquad y(x_0) = y_0 \tag{1.7}$$

has a solution in an open star-shaped neighbourhood V *of* x_0 *if and only if*

$$U(\delta, y_0) = 0 \tag{8}$$

whenever δ *is a path of the form* $x_0 z x x_0$ *in* V. *If a solution exists, it is uniquely determined by*

$$y(x) = T(xx_0, y_0) \tag{1.8}$$

where xx_0 *denotes the oriented line segment from* x_0 *to* x.

Proof. Suppose first that the initial value problem (1.7) has a solution y in V. Then the restriction of y to any polygonal path l of V starting from x_0 defines a continuous solution of the initial value problem (6) (p. 8). By the hypotheses 1° and 2° this solution is unique (see Nevanlinna [8] p. 147), so that by the definition (7) of T and U we get the representation (1.8) for the solution y, and

$$U(\delta\,,y_0) = T(\delta\,,y_0) - y_0 = y(x_0) - y_0 = 0$$

for each $\delta = x_0 z x x_0$ in V.

Conversely, suppose that $U(\delta\,,y_0)$ exists and vanishes whenever $\delta = x_0 z x x_0$ is in V. To show that (1.8) defines the solution of the initial value problem (1.7), choose an arbitrary point x from V, and such a neighbourhood N of x that for each $z \in N$ the path $x_0 z x x_0$ lies in V. By the hypothesis $T(x_0 z x x_0\,,y_0) = y_0 + U(x_0 z x x_0\,,y_0) = y_0$ for all $z \in N$. Since $x x_0$ is a subpath of $x_0 z x x_0 = (x_0 z)(z x)(x x_0)$, then $T(x x_0\,,y_0)$ is defined by Lemma 2. By the arbitrary choice of x from V it follows that $T(z x_0\,,y_0)$ is defined for all $z \in N$. Applying Lemma 2 and writing for convenience $T(l)y$ instead of $T(l\,,y)$, we have

$$\begin{aligned} y(z) &= T(z x_0\,,y_0) = T(z x_0)\,T(x_0 z x x_0) y_0 \\ &= T(z x_0)\,T(x_0 z)\,T(z x)\,T(x x_0) y_0 = T(z x)\,y(x) \end{aligned}$$

for all $z \in N$, so that

$$y(z) - y(x) = U(z x\,,y(x)) = \int_{zx} F(\xi\,,y(\xi)) d\xi\,.$$

Thus y is continuous at x by the boundedness of F and, for all $z \in N$, $z \neq x$,

$$y(z) - y(x) = F(x\,,y(x))\,(z - x) + |z - x|\,\langle z - x\rangle$$

where the expression

$$\langle z - x\rangle = \frac{1}{|\,z - x\,|}\int_{zx} (F(\xi\,,y(\xi)) - F(x\,,y(x))) d\xi$$

tends to 0 as $z \to x$ by the continuity of F and y. This shows the Fréchet-differentiability of y at $x \in V$, the derivative being

$$y'(x) = F(x\,,y(x))\,.$$

The initial condition $y(x_0) = y_0$ is by (1.8) obviously satisfied, whence the lemma is proved.

As an application of this result we shall prove

Lemma 5. *Suppose that the total differential equation* (1) *is completely integrable in* $A \times B$. *Then for a given* $(x_0\,,y_0) \in A \times B$ *the solution* y *of* (1) *which satisfies the initial condition* $y(x_0) = y_0$ *is defined and agrees with the mapping*

$$y(x) = T(x x_0\,,y_0) \tag{1.8}$$

in the domain of this mapping, particularly in the ball

$$(1.9) \qquad N(x_0, y_0) = \left\{ x \in X \mid |x - x_0| < \min \left[\varrho - |x_0| , \frac{\varrho' - |y_0|}{M} \right] \right\} .$$

Proof. Let V denote the domain of the mapping (1.8). By Lemma 1, $N(x_0, y_0)$ is contained in V and by Lemma 4 (1.8) is the necessary expression of the solution y of (1) with $y(x_0) = y_0$. This solution is defined by the hypothesis in a neighbourhood of x_0, and we have to show that V is contained in this neighbourhood.

Choose $\bar{x} \in V$. The complete integrability of (1) implies that, for each point z of the line segment $\bar{x}x_0$, the initial value problem

$$(1.10) \qquad y'(x) = F(x, y(x)), \quad y(z) = T(zx_0, y_0)$$

has a uniquely determined solution in a ball $N_z \subset A$ with center z. The segment $\bar{x}x_0$ is compact, whence we can select a finite open covering $\{N_{z_i}\}_{i=0}^n$ of $\bar{x}x_0$ such that $z_0 = x_0$ and $z_n = \bar{x}$. Since

$$d = \inf \{ |x - z| \mid x \in \bar{x}x_0 , z \in X - \bigcup_{i=0}^{n} N_{z_i} \}$$

is positive as a distance between two disjoint closed subsets of X, one of which is compact, we get the ball

$$N_d = \{ x \in X \mid |x - \bar{x}| < d \}$$

such that the line segment xx_0 is contained in $\bigcup_{i=0}^{n} N_{z_i}$ for all $x \in N_d$.

Let $\bar{y}$ be the solution of (1.10) with $z = z_n = \bar{x}$.

It suffices to show that $N_d \subset V$ and that

$$(1.11) \qquad \bar{y}(x) = T(xx_0, y_0)$$

for all $x \in N_d$, since by (1.8) and (1.11) we then have

$$y'(\bar{x}) = \bar{y}'(\bar{x}) = F(\bar{x}, \bar{y}(\bar{x})) = F(\bar{x}, y(\bar{x})) .$$

If $x - x_0$ and $\bar{x} - x_0$ are linearly dependent, then (1.11) holds trivially by Lemma 2. For the rest, perform a triangulation of the simplex with vertices x_0, x and $\bar{x}$ as follows: Choose for each $i = 1, \ldots, n$ points

$$x_i \in \bar{x}x_0 \cap N_{z_i} \cap N_{z_{i-1}} \quad \text{and} \quad u_i \in xx_0 \cap N_{z_i} \cap N_{z_{i-1}}$$

and join each u_i to z_{i-1}, z_i and x_i by line segments (see Figure 1).

Lemma 2 and Lemma 4 imply that, for each $i = 0, \ldots, n$,

$$T(uz_i, y_i) = T(uvz_i, y_i) \quad \text{whenever} \quad u, v \in N_{z_i} \quad \text{and} \quad y_i = T(z_i x_0, y_0).$$

Applying this and Lemma 2, and denoting $x = u_{n+1}$ and $T(l, y) = T(l)y$, we get for each $i = 1, \ldots, n$

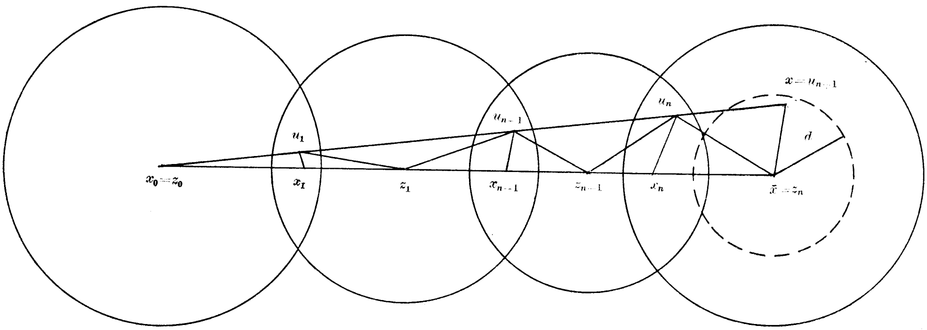

Figure 1.

$$T(u_{i+1}z_i)y_i = T(u_{i+1}u_iz_i)y_i = T(u_{i+1}u_ix_iz_i)y_i$$
$$= T(u_{i+1}u_ix_iz_{i-1})y_{i-1} = T(u_{i+1}u_iz_{i-1})y_{i-1} ,$$

so that

$$T(u_{i+1}z_i)y_i = T(u_{i+1}u_iz_{i-1})y_{i-1} \tag{1.12}$$

for all $i = 1, \ldots, n$. By Lemma 4 we also have

$$\bar{y}(x) = T(xz_n)\, T(z_nx_0)y_0 ,$$

which by the notations $x = u_{n+1}$ and $y_n = T(z_nx_0)y_0$ can be written as

$$\bar{y}(x) = T(u_{n+1}z_n)y_n . \tag{1.13}$$

From (1.13) we finally get by repeated application of (1.12)

$$\bar{y}(x) = T(u_{n+1}z_n)y_n = T(u_{n+1}u_nz_{n-1})y_{n-1}$$
$$= T(u_{n+1}u_nu_{n-1}z_{n-2})y_{n-2} = \ldots$$
$$= T(u_{n+1}u_n \ldots u_1z_0)y_0 = T(u_{n+1}x_0)y_0$$
$$= T(xx_0)y_0 = T(xx_0 , y_0) ,$$

as desired.

Remark. Lemma 4, the first part of Lemma 5, and their proofs remain unchanged, if A and B are open subsets of X and Y, respectively.

2. The main theorem

2.1. Theorem. *Let F be a bounded and continuous mapping from $A \times B$[5]) to $L(X ; Y)$ satisfying an Osgood condition*

[5]) A and B are the balls given by (1.1) (p. 10).

$$(9) \qquad \|F(x, y) - F(x, \bar{y})\| \leq \varphi(|y - \bar{y}|)$$

for all $(x, y), (x, \bar{y}) \in A \times B$, *where* φ *is a bounded, continuous and increasing function on* $\mathbf{R}^+$ *such that*

$$(\varphi 1) \qquad \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dr}{\varphi(r)} = \infty$$

and

$$(\varphi 2) \qquad \overline{\lim_{r \to 0+}} \frac{\varphi(r/2)}{\varphi(r)} < 2^{-1/2} .$$

Then the total differential equation

$$(1) \qquad y'(x) = F(x, y(x))$$

is completely integrable in $A \times B$ *if and only if*

$$(10) \qquad \lim_{\delta \to \bar{x}, |y - \bar{y}| \leq C|\delta|} \sup \frac{|U(\delta, y)|}{|\delta|^2} = 0$$

for all $(\bar{x}, \bar{y}) \in A \times B$ *and for all* $C > 0$.

To show that the complete integrability of (1) implies the condition (10), choose an arbitrary point $(\bar{x}, \bar{y})$ of $A \times B$ and a simplex σ containing the point $\bar{x}$. If x_0 denotes the initial point of $\delta = \delta\sigma$ and if y_0 is such a point of B that $|y_0 - \bar{y}| \leq C|\delta|$ for fixed $C > 0$, then, for $|\delta|$ small enough, δ is contained in the ball $N(x_0, y_0)$ given by (1.9) (p. 13). Thus $U(\delta, y_0) = 0$ by Lemma 4 and Lemma 5, which implies particularly that the condition (10) holds.

2.2. To prove *the sufficiency of the condition* (10), let (x_0, y_0) be a point of $A \times B$ and let $\delta_0 = x_0 x_0^2 x_0^1 x_0$ be an oriented boundary of a simplex σ_0 in the ball

$$(2.1) \qquad V(x_0, y_0) = \left\{ x \in X \mid |x - x_0| < r = \min \left[\varrho - |x_0|, \frac{\varrho' - |y_0|}{40M} \right] \right\} .$$

(The radius r of this ball is so chosen that the terms appearing in the following construction are defined.)

By Lemma 4 it suffices to show that $U(\delta_0, y_0) = 0$, which will be done in two steps.

Denote by x_1 the midpoint of the line segment $x_0^2 x_0^1$ and set (see Figure 2)

$$l_0 = x_1 x_0 , \quad l_1 = x_1 x_0^1 x_0 x_1 ,$$
$$l_2 = x_1 x_0 x_0^2 x_1 , \text{ and } l_3 = x_0 x_1 .$$

Denoting

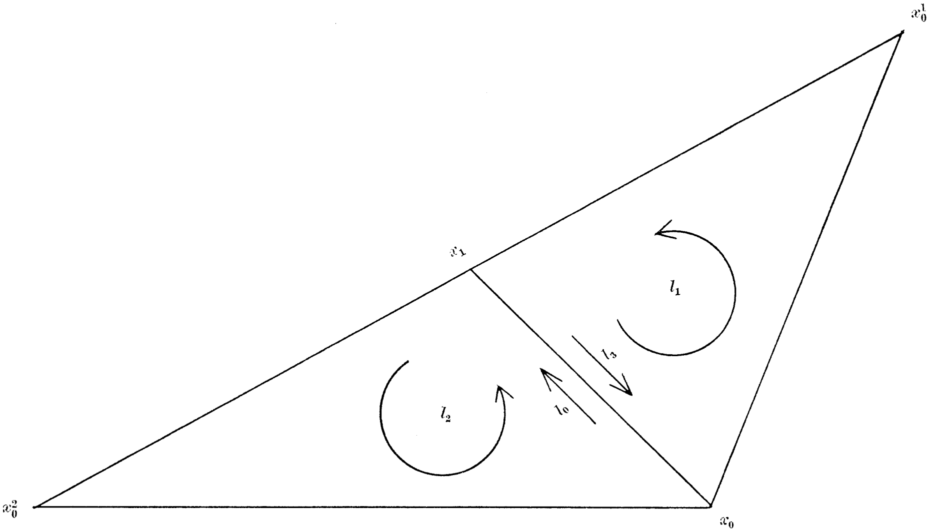

Figure 2.

(2.2) $$z_1 = T(l_0\,,y_0) \quad \text{and} \quad z_{j+1} = T(l_j\,,z_j)$$

for $j = 1\,,2\,,$ and applying Lemma 2 we get

$$U(\delta_0\,,y_0) = T(\delta_0\,,y_0) - y_0 = T(l_3l_2l_1l_0\,,y_0) - y_0 = T(l_3\,,z_3) - y_0.$$

$l_3 = x_0x_1$ is the inverse path of $l_0 = x_1x_0$, whence the property $(T2)$ of T implies that

$$y_0 = T(l_3\,,T(l_0\,,y_0)) = T(l_3\,,z_1)\,.$$

Thus

$$U(\delta_0\,,y_0) = T(l_3\,,z_3) - T(l_3\,,z_1)\,,$$

whence

(2.3) $$|U(\delta_0\,,y_0)| \leq \sum_{j=1}^{2} |T(l_3\,,z_{j+1}) - T(l_3\,,z_j)|\,.$$

An application of the inequality (1.6) (p. 11) to the right hand side of (2.3) gives

$$|U(\delta_0\,,y_0)| \leq \sum_{j=1}^{2} v(|l_3|\,,|z_{j+1} - z_j|)\,,$$

which by

$$z_{j+1} - z_j = T(l_j\,,z_j) - z_j = U(l_j\,,z_j)\,,\ j = 1\,,2$$

can be rewritten as

$$|U(\delta_0\,,y_0)| \leq \sum_{j=1}^{2} v(|l_3|\,,|U(l_j\,,z_j)|)\,.$$

Denote by $U(\delta_1\,,y_1)$ that of the terms $U(l_j\,,z_j)\,,\ j=1\,,2\,,$ which has greater norm (choose $U(\delta_1\,,y_1)=U(l_1\,,z_1)$ if the norms are equal). Because $|l_3|=|x_1-x_0|$ and v is increasing in its second argument (see p. 11), it follows from the above inequality that

$$(2.4)\qquad |U(\delta_0\,,y_0)| \leq 2\,v(|x_1-x_0|\,,|U(\delta_1\,,y_1)|)\,,$$

where $(\delta_1\,;y_1)$ is one of the pairs

$$(l_1\,,z_1)=(x_1x_0^1x_0x_1\,,T(x_1x_0\,,y_0))\,,\ (l_2\,,z_2)=(x_1x_0x_0^2x_1\,,T(x_1x_0^1x_0\,,y_0))\,,$$

denoted in the sequel by

$$(\delta_1\,,y_1)=(x_1x_1^2x_1^1x_1\,,T(w_1\,,y_0))\,.$$

Apply now the above procedure to the pair $(\delta_1\,,y_1)$ instead of (δ_0,y_0), and so on. At the n:th step the initial pair being

$$(\delta_{n-1}\,,y_{n-1})=(x_{n-1}x_{n-1}^2x_{n-1}^1x_{n-1}\,,\ \ T(w_{n-1}\,,y_{n-2}))\,,$$

write $x_n=\frac{1}{2}\,(x_{n-1}^1+x_{n-1}^2)\,,$ and let

$$(2.5)\qquad (\delta_n\,,y_n)=(x_nx_n^2x_n^1x_n\,,T(w_n\,,y_{n-1}))$$

denote that of the elements

$$(x_nx_{n-1}^1x_{n-1}x_n\,,T(x_nx_{n-1}\,,y_{n-1}))\,,\ (x_nx_{n-1}x_{n-1}^2x_n\,,T(x_nx_{n-1}^1x_{n-1}\,,y_{n-1}))$$

for which the operator U has greater norm; choosing the first one if the norms are equal.

The same reasoning as in the derivation of the inequality (2.4) yields the inequality

$$(2.6)\qquad |U(\delta_{n-1}\,,y_{n-1})| \leq 2\,v(|x_n-x_{n-1}|\,,|U(\delta_n\,,y_n)|)\,.$$

By (2.5) and Lemma 2 we have

$$y_m=T(w_m\,,y_{m-1})=T(w_m\ldots w_{n+1}\,,y_n)$$

for $m=1\,,2\,,\ldots\,;\ n=0\,,1\,,\ldots\,,m-1\,,$ so that

$$(2.7)\qquad y_m-y_n=U(w_m\ldots w_{n+1}\,,y_n)\,.$$

From (5), (7) and (1.3) (pp. 8 and 10) it follows that

$$|U(l\,,y)|=\left|\int_0^{|l|} F(l(s)\,,u(s))\,l'(s)ds\right| \leq M\,|l|$$

for all $(l\,,y)\in W$. By the above construction

$$|w_m \ldots w_{n+1}| < 6|\delta_n|$$

for $m = 1, 2, \ldots;\ n = 0, 1, \ldots, m-1$ (cf. Bächli [1] p. 9). From (2.7) it follows then that

$$|y_m - y_n| < 6M|\delta_n| \tag{2.8}$$

for these values of m, n. But $|\delta_n| \to 0$ as $n \to \infty$, so that the sequence $(y_n)_{n=0}^{\infty}$ is by (2.8) a Cauchy sequence in the Banach space Y, whence $\bar{y} = \lim_{n\to\infty} y_n$ exists.

From (2.8) it follows further that

$$|\bar{y} - y_n| = \lim_{m\to\infty} |y_m - y_n| \leq 6M|\delta_n|$$

for each $n = 0, 1, \ldots$. By the choice (2.1) of the ball $V(x_0, y_0)$ we have $|\delta_0| < 4r \leq \frac{1}{10M}(\varrho' - |y_0|)$, so that

$$|\bar{y}| \leq |y_0| + |\bar{y} - y_0| \leq |y_0| + 6M|\delta_0| < \varrho',$$

whence $\bar{y} \in B$.

Let σ_n denote the simplex whose oriented boundary is δ_n, $n = 0, 1, \ldots$. By the previous construction

$$\sigma_n \subset \sigma_{n-1} \quad \text{for} \quad n = 1, 2, \ldots,$$

and the intersection of these simplexes is a well determined point $\bar{x}$ of A. From our construction it follows further that, if Δ_n denotes the area of σ_n the sequence $(|\delta_n|^2/\Delta_n)_{n=0}^{\infty}$ is bounded. Thus δ_n converges regularly to $\bar{x}$ as $n \to \infty$.

By Lemma 2 one verifies that each value of the operator T (or U) in the above construction can be written in the form $T(l, y_0)$ (or $U(l, y_0)$), where l is a path of PA whose length is by the construction less than $10|\delta_0|$ and hence less than $\frac{1}{M}(\varrho' - |y_0|)$. Thus (l, y_0) belongs to the set W where T and U are defined by Lemma 1. This justifies the construction. Summarizing its results we get

Lemma 6. *Let (x_0, y_0) be a point of $A \times B$, and let $\delta_0 = x_0 x_0^2 x_0^1 x_0$ be an oriented boundary of a simplex in the ball*

$$V(x_0, y_0) = \left\{ x \in X \,\middle|\, |x - x_0| < \min\left[\varrho - |x_0|, \frac{\varrho' - |y_0|}{40\,M}\right]\right\}. \tag{2.1}$$

Then the above construction yields a sequence $((\delta_n, y_n))_{n=0}^{\infty}$ in W and a point $(\bar{x}, \bar{y})$ of $A \times B$, such that δ_n converges regularly to $\bar{x}$ in the plane containing δ_0 and that

$$|y_n - \bar{y}| \leq 6M|\delta_n| \quad \textit{for each} \quad n = 0, 1, \ldots.$$

Furthermore, for all $n \in \mathbf{N} = \{1, 2, \ldots\}$

$$|U(\delta_{n-1}, y_{n-1})| \leq 2\, v(|x_n - x_{n-1}|, |U(\delta_n, y_n)|), \tag{2.6}$$

where v *is the solution of the integral equation* (1.4) (p. 11) *and* x_n *denotes the initial point of* δ_n.

2.3. By the condition (10) we have now

$$\lim_{n\to\infty} \frac{|U(\delta_n, y_n)|}{|\delta_n|^2} = 0 .$$

Denote

$$r_n = |U(\delta_n, y_n)| \tag{2.9}$$

for $n = 0, 1, \ldots$. Noting that by our construction the sequence $(2^n|\delta_n|^2)_{n=0}^{\infty}$ is bounded, we then get

$$\lim_{n\to\infty} 2^n r_n = 0, \tag{2.10}$$

and the inequality (2.6) can be rewritten as

$$r_{n-1} \leq 2\, v(|x_n - x_{n-1}|, r_n) \tag{2.11}$$

for all $n \in \mathbf{N}$. Our construction implies further that

$$t_0 = \sup_{n\in\mathbf{N}} \{2^{n/2}|x_n - x_{n-1}|\} \tag{2.12}$$

is a finite positive constant. Since the solution v of the integral equation (1.4) is increasing also in its first argument we get by (2.11) and (2.12)

$$r_{n-1} \leq 2\, v(t_0\, 2^{-n/2}, r_n) \tag{2.13}$$

for all $n = 1, 2, \ldots$.

To prove that $U(\delta_0, y_0) = 0$ we now make a counter-hypothesis: $r_0 = |U(\delta_0, y_0)| > 0$. Since $v(t, 0) \equiv 0$, it follows then from (2.13) that $r_n > 0$ for all $n \in \mathbf{N}$. From (1.5) (p. 11) and (2.13) we conclude further that

$$\int_{r_n}^{\frac{1}{2}r_{n-1}} \frac{dx}{\varphi(x)} \leq t_0 2^{-n/2} \tag{2.14}$$

for $n = 1, 2, \ldots$. By the hypothesis ($\varphi 2$)[6]) there exist positive numbers a and b such that

$$\frac{\varphi\left(\dfrac{r}{2}\right)}{\varphi(r)} < 2^{-\frac{1}{2}-a} \quad \text{for } 0 < r < b ,$$

[6]) This is the only time when the hypothesis ($\varphi 2$) is used.

which gives after n repeated application

(2.15) $$\frac{\varphi(r2^{-n})}{\varphi(r)} < 2^{-n/2}\, a_n \text{ for } 0 < r < b\,,\, a_n = 2^{-na}\,.$$

The condition (2.10) implies that there exists a natural number n_0 such that

$$\varepsilon_n = 2^n\, r_n < b \quad \text{for} \quad n \geq n_0\,,$$

whence for each $n > n_0$

$$\int_{\varepsilon_n}^{\varepsilon_{n-1}} \frac{dr}{\varphi(r)} < 2^{-n/2}\, a_n \int_{\varepsilon_n}^{\varepsilon_{n-1}} \frac{dr}{\varphi(r2^{-n})} = 2^{-n/2}\, a_n \int_{r_n}^{\frac{1}{2} r_{n-1}} \frac{2^n d\alpha}{\varphi(\alpha)}\,.$$

By the inequality (2.14) this gives

$$\int_{\varepsilon_n}^{\varepsilon_{n-1}} \frac{dr}{\varphi(r)} < t_0 a_n\,,$$

so that

$$\int_{\varepsilon_n}^{\varepsilon_{n_0}} \frac{dr}{\varphi(r)} = \sum_{i=n_0+1}^{n} \int_{\varepsilon_i}^{\varepsilon_{i-1}} \frac{dr}{\varphi(r)} < t_0 \sum_{i=n_0+1}^{n} a_i$$

for each $n > n_0$. Since by (2.10) $\varepsilon_n = 2^n\, r_n \to 0$ as $n \to \infty$, we thus have

$$\lim_{n\to\infty} \int_{\varepsilon_n}^{\varepsilon_{n_0}} \frac{dr}{\varphi(r)} = \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{\varepsilon_{n_0}} \frac{dr}{\varphi(r)} \leq t_0 \sum_{i=1}^{\infty} a_i\,.$$

But this contradicts with the hypothesis $(\varphi 1)$, since the series $\sum_i a_i = \sum_i 2^{-ia}$ converges. Thus $r_0 = |U(\delta_0\,, y_0)\,| = 0$, whence $U(\delta_0\,, y_0) = 0$. This completes the proof of our theorem.

3. Corollaries

From now on we shall suppose A and B to be *open subsets* of X and Y, respectively.

3.1. Since a continuous mapping between normed linear spaces is locally bounded, we get as an immediate consequence of the previous theorem

Corollary 1. *Let* $F : A \times B \to L(X\,;Y)$ *be a continuous mapping which satisfies the Osgood condition* (9) *locally in* $A \times B$, *that is, for each* $(x_0, y_0) \in A \times B$ *there exist open balls* $N \subset A$ *and* $N' \subset B$ *with centers* x_0 and y_0, *respectively, and a continuous and increasing function* φ *on* $\mathbf{R}^+$ *satisfying the conditions* ($\varphi 1$) *and* ($\varphi 2$), *such that the inequality* (9) *holds for all* (x, y), $(x, \bar{y}) \in N \times N'$.

Then the total differential equation (1) *is completely integrable in* $A \times B$ *if and only if*

$$\lim_{\delta \to \bar{x}, |y-\bar{y}| \leq C|\delta|} \sup \frac{|U(\delta, y)|}{|\delta|^2} = 0 \tag{10}$$

for all $(\bar{x}, \bar{y}) \in A \times B$ *and for all* $C > 0$.

The regular convergence takes place in planes, whence it suffices that the hypotheses of the above corollary hold for the restrictions of F to $(E \cap A) \times B$ for all two-dimensional planes E intersecting A.

3.2. An important special case of the theorem is obtained when F satisfies a Lipschitz condition (4) (p. 8). In this case we choose

$$\varphi(r) = Kr \quad (K > 0)\,. \tag{3.1}$$

Since

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dr}{Kr} = \infty \text{ and } \lim_{r \to 0+} \frac{K\frac{r}{2}}{Kr} = \tfrac{1}{2} < 2^{-\frac{1}{2}}$$

this function φ has properties ($\varphi 1$) and ($\varphi 2$) (p. 15). Thus we get by Corollary 1

Corollary 2. (cf. Bächli [1] Prop. 3) *Let* F *be a continuous mapping from* $A \times B$ *to* $L(X\,;Y)$ *satisfying the Lipschitz condition locally in* $A \times B$. *Then the total differential equation* (1) *is completely integrable in* $A \times B$ *if and only if the condition* (10) *holds for all* $(\bar{x}, \bar{y}) \in A \times B$ *and for all* $C > 0$.

For the function (3.1) the solution v of the integral equation (1.4) (p. 11) is

$$v(t, r) = r \exp(Kt)\,. \tag{3.1$'$}$$

When this expression is used, the last part in the proof of the theorem, where the conditions

$$\lim_{n \to \infty} 2^n r_n = 0 \tag{2.10}$$

and

$$(2.13) \qquad r_{n-1} \leq 2\, v(t_0\, 2^{n/2}\,, r_n)\,,\; n = 1\,, 2\,, \ldots$$

were shown to imply that $r_0 = |U(\delta_0\,, y_0)\,| = 0$ (see pp. 19—20), is trivial. In fact by (3.1)′ and (2.13) we have

$$r_0 \leq 2 r_1 \exp(K t_0\, 2^{-1/2}) \leq \ldots \leq 2^n\, r_n \exp(K t_0 \sum_{i=1}^{n} 2^{-i/2})$$

$$\leq 2^n\, r_n \exp\left(\frac{K t_0}{\sqrt{2} - 1}\right) \text{ for each } n = 1\,, 2\,, \ldots\,,$$

whence $\lim_{n\to\infty} 2^n\, r_n = 0$ implies that $r_0 = 0$.

We obtain other special cases by defining for given natural number n and positive number K

$$\varphi_n(r) = K r \log \frac{1}{r} \ldots \log_n \frac{1}{r} \text{ for } 0 < r \leq b_n = (\exp_n 1)^{-1}$$

where $\log_n$ and $\exp_n$ denote the n-fold iterated logarithm and exponential function, respectively. Defining moreover

$$\varphi_n(0) = 0 \quad \text{and} \quad \varphi_n(r) = \varphi_n(b_n) \quad \text{for} \quad r > b_n\,,$$

we get a continuous and increasing function $\varphi_n : \mathbf{R}^+ \to \mathbf{R}^+$. Since

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{b} \frac{dr}{\varphi_n(r)} = \lim_{a \to \infty} \int_{\alpha(b)}^{a} \frac{d\alpha}{\alpha} \text{ where } \alpha = \alpha(r) = \log_n \frac{1}{r}$$

when b is small enough, the integral $\int_0^1 \frac{dr}{\varphi_n(r)}$ diverges.

Furthermore,

$$\frac{\varphi_n\left(\frac{r}{2}\right)}{\varphi_n(r)} = \frac{1}{2}\, \frac{\log \frac{1}{r} + \log 2}{\log \frac{1}{r}} \cdots \frac{\log_{n-1}\left(\log \frac{1}{r} + \log 2\right)}{\log_{n-1}\left(\log \frac{1}{r}\right)}$$

for $0 < r < b_n$, from which we easily conclude that

$$\lim_{r \to 0+} \frac{\varphi_n\left(\frac{r}{2}\right)}{\varphi_n(r)} = \frac{1}{2}\,.$$

Thus φ_n satisfies the hypotheses (φ1) and (φ2) (p. 15), and we have

Corollary 3. *Let F be a continuous mapping from $A \times B$ to $L(X\,;\,Y)$ satisfying a condition*

$$(3.2) \qquad \|F(x, y) - F(x, \bar{y})\| \leq K|y - \bar{y}| \log \frac{1}{|y - \bar{y}|} \cdots \log_n \frac{1}{|y - \bar{y}|}$$

$(K > 0, n \in \mathbf{N})$ *locally in* $A \times B$. *Then the complete integrability of* (1) *is equivalent to the validity of the condition* (10) *for each* $(\bar{x}, \bar{y}) \in A \times B$ *and for each* $C > 0$.

3.3. The theorem of Frobenius. As a consequence of Corollary 2 we get

Corollary 4. (Frobenius's theorem) *Suppose that* F *is a differentiable mapping from* $A \times B$ *to* $L(X; Y)$ *such that*
(a) *the partial derivative* F_2' *is locally bounded in* $A \times B$.

Then the condition

$$(3.3) \qquad \wedge \{F_1'(\bar{x}, \bar{y})hk + F_2'(\bar{x}, \bar{y})[F(\bar{x}, \bar{y})h]k\} = 0$$

for all $(\bar{x}, \bar{y}) \in A \times B$ *and for all* $h, k \in X$ *is necessary and sufficient for the complete integrability of the total differential equation* (1) *in* $A \times B$.

Proof. The local boundedness of F_2' implies by the mean value theorem that F satisfies the Lipschitz condition locally in $A \times B$. Therefore it suffices to verify that the condition (3.3) is equivalent to the condition (10).

Let $(\bar{x}, \bar{y})$ be a point of $A \times B$, and let $h, k \in X$, $(x, y) \in A \times B$ be so chosen that $U(\delta, y)$ is defined for $\delta = x(x+k)(x+h)x$. If we denote for each $z \in \delta$ by $\delta(z)$ the subpath of δ from x to z and

$$u(z) = T(\delta(z), y) = y + U(\delta(z), y),$$

we have

$$(3.4) \qquad U(\delta, y) = \int_\delta F(z, u(z))dz.$$

The differentiability hypothesis implies that

$$(3.4)' \qquad F(z, u(z)) = F(\bar{x}, \bar{y}) + F_1'(\bar{x}, \bar{y})(z - \bar{x}) + F_2'(\bar{x}, \bar{y})(u(z) - \bar{y}) + |\varepsilon(z)| \langle \varepsilon(z) \rangle$$

where

$$\varepsilon(z) = (z, u(z)) - (\bar{x}, \bar{y}) \in X \times Y$$

and $\langle \varepsilon(z) \rangle \to 0$ in $L(X; Y)$ as $|\varepsilon(z)| = |z - \bar{x}| + |u(z) - \bar{y}| \to 0$. Since $\int_\delta F(\bar{x}, \bar{y})dz = 0$ and since

$$\int_\delta F_1'(\bar{x}, \bar{y})(z - \bar{x})dz = \frac{1}{2}[F_1'(\bar{x}, \bar{y})hk - F_1'(\bar{x}, \bar{y})kh] = \wedge F_1'(\bar{x}, \bar{y})hk,$$

we get by (3.4) and (3.4)'

$$(3.4)'' \qquad U(\delta\,,y) = \wedge\, F_1'(\bar{x}\,,\bar{y})hk + \int_\delta F_2'(\bar{x}\,,\bar{y})\,(u(z)-\bar{y})dz$$
$$+ \int_\delta |\varepsilon(z)\,|\,\langle\,\varepsilon(z)\,\rangle\,dz\,.$$

Moreover,

$$\int_\delta F_2'(\bar{x}\,,\bar{y})\,(u(z)-\bar{y})dz = \int_\delta F_2'(\bar{x}\,,\bar{y})\,(u(z)-y)dz + \int_\delta F_2'\,(\bar{x}\,,\bar{y})\,(y-\bar{y})dz$$
$$= \int_\delta F_2'\,(\bar{x}\,,\bar{y})\,(u(z)-y)dz\,.$$

By the choice of the mapping u we have

$$u(z) - y = U(\delta(z)\,,y) = \int_{\delta(z)} F(\xi\,,u(\xi))d\xi$$
$$= F(\bar{x}\,,\bar{y})\,(z-x) + \int_{\delta(z)} (F(\xi\,,u(\xi)) - F(\bar{x}\,,\bar{y}))d\xi\,.$$

Noting also that

$$\int_\delta F_2'(\bar{x}\,,\bar{y})\,[F(\bar{x}\,,\bar{y})\,(z-x)]dz = \wedge\ F_2'(\bar{x}\,,\bar{y})\,[F[\bar{x}\,,\bar{y})h]k$$

we then have

$$\int_\delta F_2'(\bar{x}\,,\bar{y})\,(u(z)-\bar{y})\,dz = \wedge\ F_2'(\bar{x}\,,\bar{y})\,[F(\bar{x}\,,\bar{y})h]k$$
$$+ \int_\delta F_2'(\bar{x}\,,\bar{y}) \left[\int_{\delta(z)} (F(\xi\,,u(\xi)) - F(\bar{x}\,,\bar{y}))d\xi\right] dz\,.$$

The formula (3.4)″ can thus be rewritten as

$$(3.5) \qquad U(\delta\,,y) = \wedge\,\{F_1'(\bar{x}\,,\bar{y})hk + F_2'(\bar{x}\,,\bar{y})\,[F(\bar{x}\,,\bar{y})h]k\}$$
$$+ |\delta|^2\,(\langle\,\delta\,,y\,\rangle_1 + \langle\,\delta\,,y\,\rangle_2)$$

where

$$(3.5)' \qquad \langle\,\delta\,,y\,\rangle_1 = \frac{1}{|\delta|^2}\int_\delta |\varepsilon(z)\,|\langle\,\varepsilon(z)\,\rangle\,dz$$

and

$$(3.5)'' \qquad \langle \delta , y \rangle_2 = \frac{1}{|\delta|^2} \int_\delta F_2'(\bar{x} , \bar{y}) \left[\int_{\delta(z)} (F(\xi , u(\xi)) - F(\bar{x} , \bar{y}))d\xi \right] dz .$$

The proof is complete if

$$(3.5)''' \qquad \lim_{\delta \to \bar{x}} \sup_{|y-\bar{y}| \leq C|\delta|} |\langle \delta , y \rangle_1 + \langle \delta , y \rangle_2| = 0 \text{ for all } C > 0 ,$$

since the equation (3.5) implies then the equivalence of the conditions (3.3) and (10). To show this convergence note first that for each $z \in \delta$

$$\begin{aligned} |\varepsilon(z)| &= | (z , u(z)) - (\bar{x} , \bar{y}) | = |z - \bar{x}| + |u(z) - \bar{y}| \\ &\leq |z - \bar{x}| + |y - \bar{y}| + |u(z) - y| \\ &< |\delta| + |y - \bar{y}| + \left| \int_{\delta(z)} F(\xi , u(\xi))d\xi \right| . \end{aligned}$$

As differentiable mapping F is locally bounded, so that $\|F(x , y)\| \leq M$ where $M = \|F(\bar{x} , \bar{y})\| + 1$, in a neighbourhood of $(\bar{x} , \bar{y})$. Hence, if δ is close enough to $\bar{x}$ in the sense of regular convergence, and if $|y - \bar{y}| \leq C|\delta|$ for fixed $C > 0$, then

$$|\varepsilon(z)| < |\delta| + C|\delta| + M|\delta|$$

for all $z \in \delta$, whence

$$(3.6) \qquad \lim_{\delta \to \bar{x}} \frac{1}{|\delta|} \sup\{ |\varepsilon(z)| \mid z \in \delta , |y - \bar{y}| \leq C |\delta| \} \leq 1 + C + M .$$

From (3.5)′ and (3.5)″ we conclude that

$$(3.6)' \qquad |\langle \delta , y \rangle_1| \leq \frac{1}{|\delta|} \sup_{z \in \delta} \{ |\varepsilon(z)| \, \| \langle \varepsilon(z) \rangle \| \}$$

and

$$(3.6)'' \qquad |\langle \delta , y \rangle_2| \leq \| F_2'(\bar{x} , \bar{y}) \| \sup_{z \in \delta} \|F(z , u(z)) - F(\bar{x} , \bar{y})\| .$$

Noting that F is continuous at $(\bar{x} , \bar{y})$, and that $\varepsilon(z) = (z , u(z)) - (\bar{x} , \bar{y})$, we get by (3.6), (3.6)′ and (3.6)″

$$\lim_{\delta \to \bar{x}} \sup_{|y-\bar{y}| \leq C|\delta|} |\langle \delta , y \rangle_1 + \langle \delta , y \rangle_2| = 0$$

for all $C > 0$, thus completing the proof of the corollary.

Remark 1. The hypotheses of the above corollary are valid particularly when F is continuously differentiable in $A \times B$. On the other hand, the differentiability of F can be replaced by the differentiability of F's restriction to $(E \cap A) \times B$ for each two-dimensional plane E of X that intersects A (cf. p. 21). Moreover, the hypothesis (a) of this corollary can be replaced by

(b) *Each* $(x_0, y_0) \in A \times B$ *has such a convex neighbourhood* $N \times N'$ *that for all* (x, y), $(x, y+z) \in N \times N'$, $z \neq 0$, *and for all* $r \in (0, |z|)$[7].

$$(9)' \qquad \|F_2'(x, y + re)e\| \leq \varphi'(r), \quad e = z/|z|$$

where φ *is a continuous function on* $\mathbf{R}^+$ *which satisfies the conditions* (φ1) *and* (φ2) (p. 15) *and which has non-negative derivative* $\varphi'(r)$ *for* $0 < r <$ diam N'[7]).

To see this, denote $f(r) = F(x, y + re)$ for $0 \leq r \leq |z|$. Then (9)′ is equivalent to $\|f'(r)\| \leq \varphi'(r)$, whence by the mean value theorem (Dieudonne [2] p. 153)

$$\|f(|z|) - f(0)\| \leq \varphi(|z|) - \varphi(0)$$

so that

$$\|F(x, y+z) - F(x, y)\| \leq \varphi(|z|) .$$

Thus the hypotheses of Corollary 1 are valid for F.

Remark 2. The results (3.5) and (3.5)‴, which were derived in the proof of Corollary 4 (pp. 24—25) without use of the integrability condition (3.3) or (10), show that the bilinear and alternating mapping $R(\bar{x}, \bar{y})$ from $X \times X$ to Y given by

$$(3) \qquad R(\bar{x}, \bar{y})hk = \wedge \{F_1'(\bar{x}, \bar{y})hk + F_2'(\bar{x}, \bar{y})[F(\bar{x}, \bar{y})h]k\}$$

satisfies the condition

$$(3.5) \qquad U(\delta, y) = R(\bar{x}, \bar{y})hk + |\delta|^2 \langle \delta, y \rangle, \quad \delta = x(x+k)(x+h)x$$

where $\lim\limits_{\delta \to \bar{x}, |y - \bar{y}| \leq C|\delta|} \sup |\langle \delta, y \rangle| = 0$ for all $C > 0$.

Conversely, if the previous differentiability hypotheses are replaced by the continuity of F and the Osgood condition (9), (φ1) (p. 15), and if there exists some bilinear mapping $R(\bar{x}, \bar{y})$ from $X \times X$ to Y satisfying the condition (3.5), then it can be proved to be uniquely determined and alternating. This gives us a $L(X \times X; Y)$-valued operator R, which we call a *curvature form* of F. Moreover, by this definition *the existence and vanishing of the curvature form of* F *in* $A \times B$ *is equivalent to the integrability condition* (10), thus yielding the obvious reformulations of the theorem (p. 14) and the corollaries 1, 2 and 3.

Remark 3. The 'generalized operator' A introduced in Bächli [1] p. 19

[7]) except possibly on a denumerable subset.

can be given by (3.5) with $y = \bar{y}$ and $R = A$, where $\langle \delta , \bar{y} \rangle \to 0$ as $\delta \to \bar{x}$ in the sense that $|\delta| \to 0$ for $\delta = \delta\sigma$ where σ denotes a simplex containing $\bar{x}$. The existence and vanishing of A is equivalent to

$$\lim_{\delta \to \bar{x}} \frac{U(\delta , \bar{y})}{|\delta|^2} = 0 .$$

But it is not obvious that this implies generally the condition

$$\lim_{n \to \infty} \frac{U(\delta_n , y_n)}{|\delta_n|^2} = 0$$

whenever $((\delta_n , y_n))_{n=0}^{\infty}$ is a sequence constructed by Goursat's method (see section 2.2). However, this implication is used in [1] for example to prove the sufficiency of the integrability conditions given in the summary (section 12) of [1]. Besides, $A(\bar{x} , \bar{y})$ is proved to be alternating by means of an argument $U(\delta , \bar{y}) = - U(\delta^{-1} , \bar{y})$, which is not generally valid. Replacing this argument by $U(\delta , \bar{y}) = - U(\delta^{-1} , y)$, where $y = T(\delta , \bar{y})$, the mapping $R(\bar{x} , \bar{y})$ can be proved alternating by the method used in Bächli [1].

4. On the hypotheses

4.1. A counter-examle. In proving the theorem (p. 14) the hypothesis $(\varphi 2)$ was used in section 2.3 to show that the counter-hypothesis $r_0 > 0$ leads, because of the conditions

(2.10) $$\lim_{n \to \infty} 2^n r_n = 0$$

and

(2.13) $$r_{n-1} \leq 2\, v(t_0 2^{-\frac{n}{2}} , r_n) , \quad n = 1 , 2 , \ldots ,$$

to a contradiction with the hypothesis $(\varphi\, 1)$. From this proof (p. 20) we see that $(\varphi 2)$ can be weakened to the form:

The series $\sum_n a_n$ *where*

$$a_n = 2^{\frac{n}{2}} \sup_{0<r<b} \frac{\varphi(r 2^{-n})}{\varphi(r)}$$

converges for b *small enough.*

On the other hand, we shall now show by a counter-example that this proof fails if the hypothesis $(\varphi 2)$ is replaced by

$(\varphi 2)'$ $$\overline{\lim_{r \to 0+}} \frac{\varphi\, (r/2)}{\varphi\, (r)} \leq 2^{-\frac{1}{2}} .$$

More precisely, we shall construct a bounded and increasing function[8]) $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ with

$$(\varphi 1) \qquad \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \frac{dr}{\varphi(r)} = \infty$$

and

$$(\varphi 2)'' \qquad \overline{\lim_{r \to 0+}} \frac{\varphi\,(r/2)}{\varphi\,(r)} = 2^{-\frac{1}{2}}\,,$$

and a sequence $(r_n)_{n=0}^{\infty}$ of *positive numbers,* with

$$(2.10) \qquad \lim_{n \to \infty} 2^n\, r_n = 0$$

and

$$(2.13)' \qquad \tfrac{1}{2}\, r_{n-1} = v(2^{-\frac{n}{2}-1}\,, r_n)\,, \quad n = 1\,, 2\,, \ldots,$$

where v is the solution of the integral equation

$$(1.4) \qquad v(t\,, r) = r + \int_0^t \varphi(v(s\,, r))ds\,.$$

In the construction of φ we shall use the square root to get the property $(\varphi 2)''$ and suitable 'jumps' to ensure the property $(\varphi 1)$ (see Figure 3):

Let r_0 be a positive number. Set

$$(4.1) \qquad \varphi(0) = 0 \quad \text{and} \quad \varphi(r) = r_0 \quad \text{for} \quad r > r_0\,.$$

Choose the numbers $r_1 \in (0\,, \frac{1}{2}\, r_0)$ and $r_2 \in (0\,, \frac{1}{2}\, r_1)$ so that

$$(4.2) \qquad \int_{r_1}^{\frac{1}{2} r_0} \frac{dr}{\sqrt{r_0 r}} = 2^{-3/2} \text{ and } \int_{r_2}^{\frac{1}{2} r_1} \frac{dr}{\sqrt{r_0 r}} = 2^{-2}\,,$$

and define

$$(4.1)' \qquad \varphi(r) = \sqrt{r_0 r} \quad \text{for} \quad r_2 < r \leq r_0\,.$$

Proceeding recursively, set

$$(4.3) \qquad n_0 = 0 \text{ and } n_{i+1} = n_i + 2^{\frac{1}{2} n_i + 1} \text{ for } i = 0\,, 1\,, \ldots,$$

and choose for each $i \in \mathbf{N}$ the numbers

[8]) A continuous counter-example function φ is easily obtained from the one constructed here. On the other hand, the continuity hypothesis of φ is not needed in the proof of the main theorem.

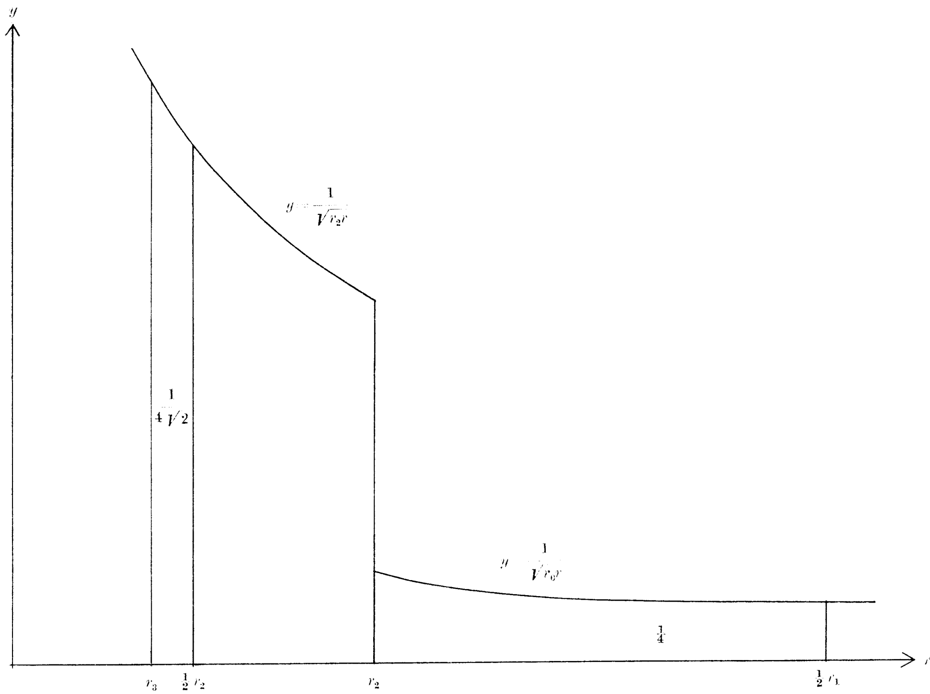

Figure 3.

$$r_n \in (0\,, \tfrac{1}{2} r_{n-1})\,, \quad n = n_i + 1\,, \ldots, n_{i+1}$$

by

(4.2)′
$$\int_{r_n}^{\frac{1}{2} r_{n-1}} \frac{dr}{\sqrt{r_{n_i} r}} = 2^{-\frac{n}{2}-1}\,.$$

Finally we define

(4.1)″
$$\varphi(r) = \sqrt{r_{n_i} r} \text{ for } r_{n_{i+1}} < r \leq r_{n_i}\,.$$

The equations (4.1), (4.1)′ and (4.1)″ define a bounded and increasing function φ on $\mathbf{R}^+$. Since

$$\int_{\frac{1}{2} r_{n_i}}^{r_{n_i}} \frac{dr}{\varphi(r)} = \int_{\frac{1}{2} r_{n_i}}^{r_{n_i}} \frac{dr}{\sqrt{r_{n_i} r}} = 2 - \sqrt{2}$$

for each $i \in \mathbf{N}$, then the integral $\int_0^1 \frac{dr}{\varphi(r)}$ diverges, whence φ satisfies the condition $(\varphi 1)$. The use of the square root in the previous construction implies further that

$$(\varphi 2)'' \qquad \overline{\lim_{r \to 0+}} \frac{\varphi(r/2)}{\varphi(r)} = 2^{-1/2}.$$

As the solution v of the integral equation (1.4) is also the solution of

$$(1.5) \qquad \int_r^{v(t,r)} \frac{d\alpha}{\varphi(\alpha)} = t,$$

it follows from (4.1)′, (4.1)″, (4.2) and (4.2)′ that, for the sequence $(r_n)_{n=0}^{\infty}$ constructed above,

$$(2.13)' \qquad \tfrac{1}{2} r_{n-1} = v(2^{-\frac{n}{2}-1}, r_n), \quad n = 1, 2, \ldots .$$

It remains to show that $\varepsilon_n = 2^n r_n \to 0$ as $n \to \infty$.

By the above equations (1.5) and (2.13)′ we have

$$(4.4) \qquad 2r_n < 2v(2^{-\frac{n}{2}-1}, r_n) = r_{n-1}$$

for all $n \in \mathbf{N}$, so that the sequence $(\varepsilon_n)_{n=0}^{\infty}$ is decreasing. Thus it suffices to show that the sub-sequence $(\varepsilon_{n_i})_{i=1}^{\infty}$ converges to zero. The substitution $r = 2^{-n}\alpha$ in (4.2)′ gives

$$\frac{2^{-n/2}}{\sqrt{r_{n_i}}} \int_{\varepsilon_n}^{\varepsilon_{n-1}} \frac{d\alpha}{\sqrt{\alpha}} = 2^{-\frac{n}{2}-1}$$

which implies that for any $i \in \mathbf{N}$

$$\sqrt{\varepsilon_{n-1}} - \sqrt{\varepsilon_n} = \tfrac{1}{4}\sqrt{r_{n_i}}, \quad n_i + 1 \leq n \leq n_{i+1}.$$

Adding sidewise these $n_{i+1} - n_i = 2^{(1/2)n_i+1}$ equations we obtain

$$\sqrt{\varepsilon_{n_i}} - \sqrt{\varepsilon_{n_{i+1}}} = \tfrac{1}{4}\sqrt{r_{n_i}}\, 2^{\frac{1}{2}n_i+1} = \tfrac{1}{2}\sqrt{\varepsilon_{n_i}},$$

so that

$$\varepsilon_{n_{i+1}} = \tfrac{1}{4}\varepsilon_{n_i}, \quad i = 1, 2, \ldots .$$

This implies that $\varepsilon_{n_i} \to 0$ as $i \to \infty$, which ends the proof.

Remark. The above counter-example shows that the additional hypothesis $(\varphi 2)$ cannot be dropped out from our proof of the theorem (p. 14) or even be replaced by the weaker condition $(\varphi 2)'$. However, the question whether the need of $(\varphi 2)$ is caused only by Goursat's method used in the proof remains open.

4.2. Finally we shall study whether the theorem (p. 14) can be proved if F is a bounded and continuous mapping from $A \times B$[9]) to $L(X ; Y)$ satisfying the following condition:
(F1) *For all* (x , y) , $(x , \bar{y}) \in A \times B$

$$\|F(x , y) - F(x , \bar{y})\| \leq G(x , |y - \bar{y}|) \tag{4.5}$$

where G *is a bounded and continuous function from* $A \times \mathbf{R}^+$ *to* $\mathbf{R}^+$ *such that* $G(x , r)$ *is increasing in* r *for fixed* $x \in A$ *and that for each oriented line segment* l *in* A *the integral equation*

$$v(t , r) = r + \int_0^t G(l(s) , v(s , r))ds \tag{4.6}$$

has for $r = 0$ $v(t , 0) \equiv 0$ *as the only solution.*

These hypotheses ensure that for each path l of PA the mappings

$$f(t , y) = F(l(t) , y)\, l'(t) \quad \text{and} \quad q(t , r) = G(l(t) , r)$$

satisfy the hypotheses of Bompiani's uniqueness theorem (Walter [12]). Hence, for each $(l , y_0) \in W$ (p.10) the sequence $(u_n)_{n=1}^{\infty}$ of the successive approximations defined by

$$u_{n+1}(t) = y_0 + \int_0^t F(l(s) , u_n(s))\, l'(s)ds , \quad u_1(t) \equiv y_0 \tag{4.7}$$

converges uniformly on $[0 , |l|]$ to a unique solution u of the integral equation (5) (p. 8). Thus the result of Lemma 1 (p. 10) is valid. Also lemmas 2, 4 and 5, and hence the necessity part of the theorem, are true for this F .

Now we shall prove Lemma 3 with the help of above hypotheses. Let l be an oriented line segment in A and let y_1 and y_2 be points of B such that $(l , y_j) \in W$, $j = 1 , 2$. Define the sequence $(u_n^j)_{n=1}^{\infty}$ by (4.7) with $y_0 = y_j$, $j = 1 , 2$, and let $(v_n)_{n=1}^{\infty}$ denote the sequence of successive approximations given by

$$v_{n+1}(t) = |y_1 - y_2| + \int_0^t G(l(s) , v_n(s))ds , \quad v_1(t) \equiv |y_1 - y_2| .$$

It can be shown (Edwards [3]) that the sequence $(v_n)_{n=1}^{\infty}$ converges on $[0 , |l|]$ uniformly to the *minimal* solution v of the integral equation (4.6) with $r = |y_1 - y_2|$. Moreover, it is elementary to verify by induction that

[9]) A and B are the balls given by (1.1) (p. 10).

$$|u_n^1(t) - u_n^2(t)| \leq v_n(t)$$

for all $n \in \mathbf{N}$ and $t \in [0, |l|]$. As $n \to \infty$ this implies that, for $0 \leq t \leq |l|$

$$|u_1(t) - u_2(t)| \leq v(t, |y_1 - y_2|)$$

where u_j is the solution of (5) with $y_0 = y_j$, $j = 1, 2$. For $t = |l|$ this inequality is equivalent to the inequality

$$|T(l, y_1) - T(l, y_2)| \leq v(|l|, |y_1 - y_2|) \tag{1.6}$$

which was asserted in Lemma 3.

Using successive approximations one can verify that the minimal solution $v(t, r)$ of (4.6) is increasing in r for fixed l and t. Thus also Lemma 6 (p. 18) is valid for the given F. The last step in the proof of the theorem was to verify that the following condition holds:

(F2) *If* $(x_n)_{n=0}^{\infty}$ *is a sequence in* A *such that the sequence* $(2^{n/2}|x_n - x_{n-1}|)_{n=1}^{\infty}$ *is bounded, then the conditions*

$$\lim_{n\to\infty} 2^n r_n = 0 \quad \textit{and} \quad r_{n-1} \leq 2\, v(|x_n - x_{n-1}|, r_n), \; n \in \mathbf{N}$$

are satisfied only for $r_n = 0$, $n = 0, 1, 2, \ldots$.

Hence, if we *suppose* that (F2) is valid for the minimal solutions v of the integral equation (4.6) with $l = x_n x_{n-1}$, then also the sufficiency part of the theorem is true for F.

The hypothesis (F1) is particularly valid when $G(x, r) = \varphi(r)$ where φ is a bounded, continuous and increasing function on $\mathbf{R}^+$ satisfying the condition $(\varphi 1)$. In this case the integral equation (4.6) equals to the integral equation (1.4), which has, as we saw in p. 11, a unique solution v on $\mathbf{R}^+ \times \mathbf{R}^+$. If we add the hypothesis $(\varphi 2)$ for φ, then the proof in section 2.3 shows that also the condition (F2) is valid. Thus the Osgood condition (9), $(\varphi 1)$, $(\varphi 2)$ is a special case of the hypotheses (F1), (F2).

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