ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

497

ON THE BEHAVIOUR OF THE SOLUTIONS OF SOME FIRST ORDER DIFFERENTIAL EQUATIONS

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HELSINKI 1971
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doi:10.5186/aasfm.1971.497

Communicated 14 May 1971 by LAURI MYRBERG

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§ 1. INTRODUCTION

1.1. Summary. This paper contains some generalizations of the results of Wittich ([10], [12] and [13]) on the behaviour of the solutions of Riccati differential equations. Wittich considered Riccati equations with rational and specially with polynomial coefficients. We merely assume that the growth of a given solution is large compared to the growth of the coefficients. Such solutions to a certain extent take the role shared by transcendental solutions among all solutions of the equations with rational coefficients.

Foremost we are concentrating our attention on the following two types of differential equations in the complex domain:

$$\frac{dw}{dz} = a_0(z) + a_1(z)w + a_2(z)w^2 \tag{1.1}$$

and

$$\left(\frac{dw}{dz}\right)^n = \sum_{i=0}^{n+k} a_i(z)w^i, \text{ where } 1 \leqq k \leqq n\,. \tag{1.2}$$

1.2. Notations. We suppose that the reader is familiar with the concepts of the value distribution theory of Nevanlinna. Our notations will follow those of Wittich [12] and Hayman [5].

Throughout this paper we assume that $a_2(z) \not\equiv 0$ in the equation (1.1) and $a_{n+k}(z) \not\equiv 0$ in (1.2). We also assume all coefficients and solutions to be meromorphic in the whole plane.

Important to our estimations are some classes $S(r, w)$ and $S^0(r, w)$ of real functions, whose members will be denoted with the same symbols.

Definition 1. *If $w = w(z)$ is a given transcendental meromorphic function, then a real-valued function defined in the positive real axis belongs to $S(r, w)$, if*

$$\limsup_{r \to \infty} \frac{S(r, w)}{T(r, w)} = 0 \tag{1.3}$$

outside of a possible exceptional set E with a finite linear measure: $\int_E dr < \infty$.

A function $S(r, w)$ *specially belongs to* $S^0(r, w)$, *if* (1.3) *is valid without an exceptional set.*

Let $\sum_{i=0}^{s} a_i(z)w^i$ be a polynomial in w. The smallness of the growth of the coefficients $a_i(z)$ compared to the growth of $w = w(z)$ is defined as follows:

Definition 2. *A transcendental meromorphic function* $w = w(z)$ *is admissible* (*with respect to the coefficients* $a_i(z)$), *if*

$$\sum_{i=0}^{s} T(r, a_i(z)) = S(r, w) .$$

If specially

$$\sum_{i=0}^{s} T(r, a_i(z)) = S^0(r, w) ,$$

then $w = w(z)$ *is said to be strongly admissible.*

Remark. If $w = w(z)$ is admissible, then trivially

$$T(r, a_i(z)) = S(r, w)$$

for every function $a_i(z)$, $0 \leqq i \leqq s$. A similar conclusion naturally holds, if $w(z)$ is strongly admissible.

1.3. Auxiliary results. Many considerations in this paper utilize the following lemma which goes back to Valiron (Bieberbach [1], p. 99):

Lemma 1. *Let* $R(w, z)$ *be a rational function of the variables* w *and* z. *If* $R(w, z)$ *is of degree* q *in* w *and if* $w = w(z)$ *is any transcendental meromorphic function, then*

$$T(r, R(w(z), z)) = q\, T(r, w(z)) + O(\log r) .$$

Equally important is the following lemma originating from Clunie ([2], p. 20):

Lemma 2. *Let* $P = P(z) = P(w, w', \ldots, w^{(p)}, z)$ *and* $Q = Q(z) = Q(w, w', \ldots, w^{(q)}, z)$ *be polynomials in* w *and its derivatives. If* $w = w(z)$ *is an admissible function with respect to the coefficients of* P *and* Q, *if the degree of* Q *counted with respect to the arguments* $w, w', \ldots, w^{(q)}$ *is at most* n *and if* $w(z)^n P(z) \equiv Q(z)$ *after the substitution* $w = w(z)$, *then*

$$m(r, P(z)) = S(r, w).$$

Examining that proof of lemma 2 presented in [5], p. 68—69, we verify

Lemma 3. *In the situation of lemma 2*

$$m(r, P(z)) = S^0(r, w),$$

if $w = w(z)$ *is strongly admissible and of finite order.*

§ 2. THE EQUATION $\frac{dw}{dz} = a_0 + a_1 w + a_2 w^2$

2.1. Global solutions. The equations we consider in this paper are almost all of the general form

$$\left(\frac{dw}{dz}\right)^n = R(w, z) , \tag{2.1}$$

where $R(w, z)$ is a rational function in w. The meromorphicity of all coefficients and of the solutions we are treating is preassumed. If either the solution $w(z)$ or at least one of the coefficients of $R(w, z)$ induce a pole to the equation (2.1), then the equality of both sides is to be understood in the formal sense.

2.2. The theorem of Malmquist. Malmquist proved that the equations of type (1.1) with rational coefficients form a distinguished class among the equations

$$\frac{dw}{dz} = R(w, z) , \tag{2.2}$$

where $R(w, z)$ is rational in both variables ([7], p. 311). We generalize this result by proving

Theorem 1. *Let*

$$R(w, z) = \frac{P(w, z)}{Q(w, z)} = \frac{\sum_{i=0}^{p} a_i(z) w^i}{\sum_{i=0}^{q} b_i(z) w^i}$$

be an irreducible rational function in w. *If* $w = w(z)$ *is an admissible solution* (*with respect to the coefficients of* $P(w, z)$ *and of* $Q(w, z)$) *of the equation*

$$\left(\frac{dw}{dz}\right)^n = R(w, z) ,$$

then $R(w, z)$ *is in* w *a polynomial whose degree is* $\leqq 2n$.

Proof. Let us pick out a finite complex value α such that $\Theta(w, \alpha) = 0$ and that $\lim_{r \to \infty} \frac{m(r, \alpha)}{T(r, w)} = 0$. Substituting $w = \alpha + \frac{1}{u}$ we get a modified equation

$$\left(\frac{du}{dz}\right)^n = \frac{P_\alpha(u, z)}{Q_\alpha(u, z)} = \frac{\sum_{i=0}^{p_\alpha} a_i^\alpha(z) u^i}{\sum_{i=0}^{q_\alpha} b_i^\alpha(z) u^i} . \tag{2.3}$$

Since the new coefficients are obtained from the old ones by applying only rational operations a finite number of times, $u(z)$ is an admissible solution of (2.3). Thus

$$\sum_{i=0}^{p_\alpha} \bar{N}(r, a_i^\alpha) + \sum_{i=0}^{p_\alpha} \bar{N}\left(r, \frac{1}{a_i^\alpha}\right) + \sum_{i=0}^{q_\alpha} \bar{N}(r, b_i^\alpha) + \sum_{i=0}^{q_\alpha} \bar{N}\left(r, \frac{1}{b_i^\alpha}\right) \leqq$$

$$\leqq 2 \sum_{i=0}^{p_\alpha} T(r, a_i^\alpha) + 2 \sum_{i=0}^{q_\alpha} T(r, b_i^\alpha) + O(\log r) < \frac{1}{2} T(r, u) ,$$

if r is sufficiently large and outside of a possible exceptional set of finite linear measure. Denoting now by $N_1(r, u)$ that counting function induced by simple poles of $u(z)$ we have

$$\bar{N}(r, u) \leqq \frac{1}{2} (N_1(r, u) + N(r, u)) .$$

Since $\Theta(u, \infty) = \Theta(w, \infty) = \delta(u, \infty) = \delta(w, \infty) = 0$, then

$$1 = \limsup_{r \to \infty} \frac{\bar{N}(r, u)}{T(r, u)} \leqq \frac{1}{2} \limsup_{r \to \infty} \frac{N_1(r, u) + N(r, u)}{T(r, u)}$$

$$\leqq \frac{1}{2} \limsup_{r \to \infty} \frac{N_1(r, u)}{T(r, u)} + \frac{1}{2} .$$

Thus $\limsup_{r \to \infty} \frac{N_1(r, u)}{T(r, u)} = 1$ and so $\sup_{r \geqq \varrho} \frac{N_1(r, u)}{T(r, u)} > \frac{1}{2}$ for a sufficiently large ϱ. Thus there exists a value r_0 such that

$$\sum_{i=0}^{p_\alpha} \bar{N}(r_0, a_i^\alpha) + \sum_{i=0}^{p_\alpha} \bar{N}\left(r_0, \frac{1}{a_i^\alpha}\right) + \sum_{i=0}^{q_\alpha} \bar{N}(r_0, b_i^\alpha) + \sum_{i=0}^{q_\alpha} \bar{N}\left(r_0, \frac{1}{b_i^\alpha}\right) < N_1(r_0, u) .$$

This implies the existence of a point z_0, where u has a simple pole and where all the coefficients a_i^α and b_i^α have finite, non-zero values. Utilizing the Laurent expansion of $u(z)$ at z_0 we now get $p_\alpha = q_\alpha + 2n$, thus

$$\left(\frac{du}{dz}\right)^n = \frac{P_\alpha(u, z)}{Q_\alpha(u, z)} = \frac{\sum_{i=0}^{q_\alpha + 2n} a_i^\alpha(z) u^i}{\sum_{i=0}^{q_\alpha} b_i^\alpha(z) u^i} . \tag{2.4}$$

We further modify (2.4) to the form

$$\left(\frac{du}{dz}\right)^n = A_{2n}(z) u^{2n} + \ldots + A_1(z) u + A_0(z) + \frac{P_0(u, z)}{Q_\alpha(u, z)} , \tag{2.5}$$

where the degree of the polynomial P_0 in u is at most $q_\alpha - 1$. Of course, the admissibility of u is not destroyed. Let us suppose now that there exists at least one non-zero coefficient of $P_0(u, z)$. Since $\lim_{r\to\infty} \dfrac{m(r, u)}{T(r, u)} = \lim_{r\to\infty} \dfrac{m(r, \alpha)}{T(r, w)} = 0$, we immediately have

$$m(r, u) = S(r, u) .$$

To estimate $N(r, u)$ we proceed as follows. If $u(z)$ has a pole of order k, then

$$F(z) = \frac{P_0(u(z), z)}{Q_\alpha(u(z), z)}$$

has a zero of order $\geqq k$ at this point save that at least one coefficient of P_0 has a pole or at least one coefficient of Q_α has a zero at the same time. Thus

$$N(r, u) \leqq N\left(r, \frac{1}{F}\right) + S(r, u) .$$

Since

$$F(z) = \left(\frac{du(z)}{dz}\right)^n - \sum_{i=0}^{2n} A_i(z)u(z)^i ,$$

a regular point of $u(z)$ can have a contribution to the number of the poles of $F(z)$ only if at least one of the coefficients $A_i(z)$ has a pole at that point. Hence

$$\begin{aligned} T(r, F) &= m(r, F) + N(r, F) = m(r, F) + S(r, u) \\ &= m\left(r, \left(\frac{du}{dz}\right)^n - \sum_{i=0}^{2n} A_i\, u^i\right) + S(r, u) \\ &= O(m(r, u)) + S(r, u) = S(r, u) . \end{aligned}$$

Since

$$N(r, u) \leqq N\left(r, \frac{1}{F}\right) + S(r, u) \leqq T(r, F) + S(r, u) = S(r, u),$$

we have a contradiction

$$T(r, u) = m(r, u) + N(r, u) = S(r, u) .$$

Hence $P_0(u, z) \equiv 0$ and the modified equation (2.5) is of the form

$$\left(\frac{du}{dz}\right)^n = A_{2n}(z)u^{2n} + \ldots + A_1(z)u + A_0(z) . \tag{2.6}$$

Utilizing the inverse substitution $u = \dfrac{1}{w - \alpha}$ we see that the original equation has the asserted form. The theorem follows.

Corollary 1. *Let*

$$R(w, z) = \frac{P(w, z)}{Q(w, z)} = \frac{\sum\limits_{i=0}^{p} a_i(z) w^i}{\sum\limits_{i=0}^{q} b_i(z) w^i}$$

be an irreducible rational function in w. *If* $w = w(z)$ *is an admissible solution (with respect to the coefficients of* $P(w, z)$ *and of* $Q(w, z)$) *of the equation*

$$\frac{dw}{dz} = R(w, z), \tag{2.7}$$

then (2.7) *reduces to the Riccati differential equation* (1.1).

Corollary 2. *Let* $w = w(z)$ *be an admissible solution of the equation*

$$\frac{dw}{dz} = \sum_{i=0}^{p} a_i(z) w^i . \tag{2.8}$$

Then $p \leqq 2$.

Remark. Every transcendental meromorphic solution is admissible, if all coefficients of (2.1) are rational functions. In this case our corollary 1 generalizes the theorem of Malmquist. Theorem 1, on the other hand, is a direct generalization of a theorem presented by Yosida ([14], p. 255).

2.3. Deficiencies of admissible solutions of (1.1). Admissible solutions of a Riccati differential equation have deficiency properties, which to a certain extent coincide with those of the transcendental solutions of Riccati equations with polynomial coefficients (see e.g. Wittich [12], p. 78—80). We first prove

Lemma 4. *Let* $w = w(z)$ *be an admissible function with respect to the coefficients of* $P(w, z) = a_0(z) + a_1(z)w + \ldots + a_p(z)w^p$. *Then*

$$N(r, P(w(z), z)) = p\, N(r, w) + S(r, w) ,$$

or, more definitely,

$$p\, N(r, w) - S(r, w) \leqq N(r, P(w(z), z)) \leqq p\, N(r, w) + S(r, w) ,$$

where the quantities $S(r, w)$ *are non-negative.*

Proof. Let us denote by $\nu(f, \alpha)$ the multiplicity of f at a given α-point. Considering a pole of $P(w(z), z)$ we immediately have

$$\nu(P, \infty) \leqq p\nu(w, \infty) + \sum_{i=0}^{p} \nu(a_i, \infty) .$$

On the other hand, if $a_p w^p$ is the only maximal term in P, then

$$\nu(P, \infty) \geqq p\nu(w, \infty) - \nu(a_p, 0) .$$

If the only maximal term is $a_k w^k$ with $k \neq p$ or if there exist at least two maximal terms, then we easily verify that

$$\nu(w, \infty) \leqq \sum_{i=0}^{p} \nu(a_i, \infty) + \sum_{i=0}^{p} \nu(a_i, 0) .$$

Thus in any case

$$\nu(P, \infty) \geqq p\nu(w, \infty) - \nu(a_p, 0) - p\sum_{i=0}^{p} \nu(a_i, \infty) - p\sum_{i=0}^{p} \nu(a_i, 0) .$$

Summation over all poles of $P(w(z), z)$ and integration now gives

$$pN(r, w) - N\left(r, \frac{1}{a_p}\right) - p\sum_{i=0}^{p} N(r, a_i) - p\sum_{i=0}^{p} N\left(r, \frac{1}{a_i}\right) \leqq N(r, P)$$

and

$$N(r, P) \leqq pN(r, w) + \sum_{i=0}^{p} N(r, a_i) .$$

The assertion follows by admissibility of $w(z)$.

Corollary. *In the situation of lemma* 4

$$p\, N(r, w) - S^0(r, w) \leqq N(r, P(w(z), z)) \leqq p\, N(r, w) + S^0(r, w),$$

if $w = w(z)$ *is strongly admissible.*

Theorem 2. *If* $w = w(z)$ *is an admissible solution of the equation* (1.1), *then* $\Theta(w, \infty) = 0$.

Proof. Since $w(a_2 w) = w' - a_1 w - a_0$, we have

$$m(r, w) \leqq m(r, a_2 w) + m\left(r, \frac{1}{a_2}\right) + O(1) = S(r, w)$$

by lemma 2. Thus

$$T(r, w) = N(r, w) + S(r, w) .$$

From (1.1) we deduce by lemma 4

$$N(r, w') = N(r, w) + \bar{N}(r, w) \geqq 2\, N(r, w) - S(r, w) ,$$

hence

$$\bar{N}(r, w) \geqq N(r, w) - S(r, w)$$

and so

$$T(r, w) \leqq \bar{N}(r, w) + S(r, w) .$$

Since $S(r, w)$ is non-negative, we have

$$1 \geqq \limsup_{r\to\infty} \frac{\bar{N}(r, w)}{T(r, w)} \geqq \limsup_{r\to\infty} \left(1 - \frac{S(r, w)}{T(r, w)}\right) = 1 - \liminf_{r\to\infty} \frac{S(r, w)}{T(r, w)} = 1.$$

The theorem follows.

Corollary 1. *If* $w = w(z)$ *is an admissible solution of the equation* (1.1), *then* $\delta(w, \infty) = 0$. *Thus* (1.1) *does not admit an admissible integral solution.*

By lemma 3 we also have

Corollary 2. *If* $w = w(z)$ *is a strongly admissible solution of finite order of the equation* (1.1), *then also the Valiron deficiency vanishes:* $\Delta(w, \infty) = 0$.

Theorem 3. *If* $w = w(z)$ *is an admissible solution of the equation* (1.1), *where the coefficients are supposed to be integral functions, then for every* $\alpha \neq \infty$ *we have*

$$\begin{cases} a_0(z) + \alpha a_1(z) + \alpha^2 a_2(z) \equiv 0 \Rightarrow \Theta(w, \alpha) = \delta(w, \alpha) = \Delta(w, \alpha) = 1 \\ a_0(z) + \alpha a_1(z) + \alpha^2 a_2(z) \not\equiv 0 \Rightarrow \Theta(w, \alpha) = \delta(w, \alpha) = 0. \end{cases}$$

Proof. Substituting $w = \alpha + \dfrac{1}{u}$ we get

$$u' = - a_2 - (a_1 + 2\alpha a_2)u - (a_0 + \alpha a_1 + \alpha^2 a_2)u^2 .$$

Now $a_0 + \alpha a_1 + \alpha^2 a_2 \not\equiv 0$ implies $0 = \Theta(u, \infty) = \Theta(w, \alpha) \geqq \delta(w, \alpha) = 0$ by theorem 2.

If, on the other hand, $a_0 + \alpha a_1 + \alpha^2 a_2 \equiv 0$, then the equation (1.1) has the special form

$$\frac{dw}{dz} = (w - \alpha)(b + a_2 w) , \tag{2.9}$$

where $b = a_1 + \alpha a_2$. The uniqueness theorem for the solution of the first order differential equtions ([3], p. 34) implies that α is a Picard value of $w(z)$. The theorem follows.

Corollary. *In the normal case, where* $\delta(w, \alpha) = 0$, *we even have* $\Delta(w, \alpha) = 0$, *if* $w = w(z)$ *is strongly admissible and of finite order.*

We will more closely consider the admissible solutions $w = w(z)$ with at least one deficient value.

The equation (2.9) has two distinct deficient values α and β if and only if it has the form

$$\frac{dw}{dz} = a_2(w - \alpha)(w - \beta) . \tag{2.10}$$

By theorem 2 and theorem 3 $\sum_{\omega} \Theta(w, \omega) = 2$ in this case. Thus by Singh and Gopalakrishna ([9], p. 129) it is possible to omit the assumption of strong admissibility in corollary 2 to theorem 2 and in corollary to theorem 3.

We utilize the following notations in the sequel:

$$\Phi_e = \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{w'}\right)}{T(r, w)}$$

and

$$\overline{\Phi}_e = \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{w'}\right)}{T(r, w)} .$$

Since $\Phi_e + \sum_{\omega\neq\infty} \delta(w, \omega) \leqq 2$, we have the following trivial

Theorem 4. *If* $w = w(z)$ *is an admissible solution of* (2.10), *where the coefficients are integral functions, then*

$$\Phi_e + \sum_{\omega\neq\infty} \delta(w, \omega) = 2 .$$

Since for all meromorphic functions $2\delta(w', 0) \geqq \sum_{\omega\neq\infty} \delta(w, \omega)$ ([12], p. 22), we have a

Corollary. *If* $w = w(z)$ *is an admissible solution of* (2.10) *with integral coefficients, then*

$$\delta(w', 0) = 1 .$$

In the special case of (2.10) we are easily able to reach a small improvement to a result of Wittich who proved that any transcendental meromorphic solution of the equation (1.1) with rational coefficients has the order $\lambda_w \geqq \frac{1}{2}$ ([12], p. 76). Now we have

Theorem 5. *If* $w = w(z)$ *is an admissible solution of the equation* (2.10) *with integral coefficients, then* λ_w *is either a positive integer or infinite. Further* $w(z)$ *is of regular growth, i.e.*

$$\lambda_w = \lim_{r\to\infty} \frac{\log T(r, w)}{\log r} .$$

Proof. The function

$$g(z) = \frac{1}{w(z) - \alpha} - \frac{1}{\beta - \alpha}$$

is an integral function with $\sum_{\omega\neq\infty} \delta(g, \omega) = \delta(g, 0) = 1$. Thus

$$0 \leqq K(f) = \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{w}\right) + N(r, w)}{T(r, w)}$$

$$\leqq \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{w}\right)}{T(r, w)} + \limsup_{r\to\infty} \frac{N(r, w)}{T(r, w)} = 0.$$

Since $T(r, w) = T(r, g) + O(\log r)$, the assertion is a direct consequence of corollary 6.1 in [4], p. 298.

Remark. The order λ_w can be any positive integer or infinite. For example the equation $w' = qz^{q-1}(w - w^2)$ has a solution $w = e^{z^q}/(e^{z^q} - 1)$ of the order $\lambda_w = q$ with two deficient values: $\delta(w, 0) = \delta(w, 1) = 1$. On the other hand, $w = e^{e^z}/(e^{e^z} - 1)$ with $\lambda_w = \infty$ satisfies the equation $w' = e^z(w - w^2)$.

If now an admissible solution of (2.9) with integral coefficients has exactly one deficient value α, then either $\Delta \not\equiv 0$, where $\Delta = \frac{d}{dz}\left(\frac{b}{a_2}\right)$, or $\Delta \equiv 0$ and (2.9) has the special form

$$\frac{dw}{dz} = a_2(w - \alpha)^2. \tag{2.11}$$

Actually, it is not known, whether there exists an admissible solution of (2.11). We can assume $a_2(z)$ to be a transcendental integral function, because all solutions of (2.11) are rational, if $a_2(z)$ is a polynomial. Taking now a regular point z_0, we see that the solution $w(z)$ has the representation

$$w(z) = \alpha - \frac{1}{C + \int_{z_0}^{z} a_2(t)\, dt}$$

for a conveniently selected C. Since $T(r, w) = T\left(r, \int_{z_0}^{z} a_2(t)dt\right) + O(\log r)$, the admissibility of $w(z)$ depends on whether there exists an integral function $a_2(z)$ such that $T(r, a_2) = S\left(r, \int_{z_0}^{z} a_2(t)dt\right)$. If $a_2(z)$ is of finite order or if $\sum_{\omega \neq \infty} \Theta(a_2, \omega) > 0$, this is not possible ([6], p. 98 and [9], p. 123), but in general the question is open. If, however, an admissible solution of (2.11) exists, then a theorem analogous to theorem 4 is valid:

Theorem 6. *If* $w = w(z)$ *is an admissible solution of* (2.11) *with integral coefficients, then* $\Phi_e + \sum_{\omega \neq \infty} \delta(w, \omega) = 1.$

Proof. We immediately get

$$N\left(r, \frac{1}{w'}\right) \leqq N\left(r, \frac{1}{a_2}\right) + N\left(r, \frac{1}{(w - \alpha)^2}\right) = N\left(r, \frac{1}{a_2}\right) = S(r, w),$$

thus $\Phi_e = 0$. The theorem follows.

Without any regard to the admissibility of $w(z)$ we state that $\lambda_w = \lambda_{A_2} = \lambda_{a_2}$, where $A_2 = \int_{z_0}^{z} a_2(t)dt$, since every meromorphic function has the same order as its derivative. Thus we have

Theorem 7. *If* $w = w(z)$ *is a non-constant meromorphic solution of the equation* (2.11), *then* $\lambda_w = \lambda_{a_2}$.

If $\Delta \not\equiv 0$, the equation we are considering has the general form (2.9), where b/a_2 is not a constant. In this case the following analogies to theorem 4 have a somewhat weakened form. We first prove

Theorem 8. *If* $w = w(z)$ *is an admissible solution of* (2.9) *with integral coefficients and with* $\Delta \not\equiv 0$, *then* $\overline{\Phi}_e \geqq 1$.

Proof. The factor $F = b + a_2 w$ in the equation (2.9) satisfies a Riccati differential equation

$$\frac{dF}{dz} = c_0 + c_1 F + F^2 ,$$

where

$$\begin{cases} c_1 = \dfrac{a_2'}{a_2} - a_1 - 2\alpha a_2 \\ c_0 = a_2 \Delta . \end{cases}$$

Since $c_0 \not\equiv 0$, we have by theorem 2

$$m\left(r, \frac{1}{F}\right) = S(r, F) = S(r, w).$$

Thus

$$\begin{aligned} T(r, w) &= T\left(r, \frac{1}{a_2}(F - b)\right) \leqq T(r, F) + T\left(r, \frac{1}{a_2}\right) + T(r, b) + O(1) \\ &= m\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F}\right) + S(r, w) = N\left(r, \frac{1}{F}\right) + S(r, w) \\ &\leqq N\left(r, \frac{1}{w'}\right) + S(r, w) . \end{aligned}$$

The assertion now follows, since $S(r, w)$ is non-negative:

$$\overline{\Phi}_e \geqq \limsup_{r\to\infty} \frac{T(r, w) - S(r, w)}{T(r, w)} = 1 - \liminf_{r\to\infty} \frac{S(r, w)}{T(r, w)} = 1 .$$

Corollary 1. *If $w = w(z)$ is a strongly admissible solution of (2.9) with integral coefficients and with $\Delta \not\equiv 0$, then $\overline{\Phi}_e = 1$.*

Proof. This assertion is obtained at once, since

$$\frac{N\left(r, \dfrac{1}{w'}\right)}{T(r, w)} \leqq \frac{N\left(r, \dfrac{1}{F}\right)}{T(r, w)} \leqq \frac{T(r, b + a_2 w) + O(1)}{T(r, w)} \leqq 1 + \frac{S^0(r, w)}{T(r, w)} .$$

Corollary 2. *If $w = w(z)$ is a strongly admissible solution of finite order of (2.9) with integral coefficients and with $\Delta \not\equiv 0$, then $\Phi_e = \overline{\Phi}_e = 1$.*

Proof. Since $T(r, a_i) \leqq T(r, w), i = 1, 2,$ for a sufficiently large r, then $a_1(z)$ and $a_2(z)$ are of finite order and we have $m\left(r, \dfrac{1}{F}\right) = S^0(r, w)$ by corollary 2 to theorem 2 as in the proof of the main theorem. Thus

$$T(r, w) \leqq N\left(r, \frac{1}{w'}\right) + S^0(r, w)$$

and

$$\Phi_e \geqq \liminf_{r\to\infty} \frac{T(r, w) - S^0(r, w)}{T(r, w)} = 1 - \limsup_{r\to\infty} \frac{S^0(r, w)}{T(r, w)} = 1 .$$

Corollary 1 now implies $1 \leqq \Phi_e \leqq \overline{\Phi}_e = 1$.

By theorem 3 and its corollary we have

Corollary 3. *If $w = w(z)$ is a strongly admissible solution of finite order of (2.9) with integral coefficients and with $\Delta \not\equiv 0$, then $\Phi_e + \sum_{\omega\neq\infty} \delta(w, \omega) = \overline{\Phi}_e + \sum_{\omega\neq\infty} \Delta(w, \omega) = 2$.*

Corollary 4. *If $w = w(z)$ is a strongly admissible solution of finite order of (2.9) with integral coefficients and with $\Delta \not\equiv 0$, then $\delta(w', 0) = \frac{1}{2}$.*

Proof. Since $\Phi_e = 1$ by corollary 2, we can utilize the double inequality

$$\frac{1}{2} \sum_{\omega\neq\infty} \delta(w, \omega) \leqq \delta(w', 0) \leqq \frac{2 - \Theta(w, \infty) - \Phi_e}{2 - \Theta(w, \infty)}$$

of Wittich ([12], p. 22—23) to have the assertion.

Remark 1. We specially note that the previous corollaries contain those deficiency relations obtained by Wittich for transcendental solutions of the equation (2.9) with polynomial coefficients ([12], p. 78—80).

Remark 2. We are not aware whether it would be possible to omit the additional assumptions, the strong admissibility and the finiteness of the order, imposed upon the solutions in the corollaries.

2.4. Non-admissible solutions. We first state that there does not exist any bound to the number of the deficient values for non-admissible solutions. For example the equation $w' = e^{-z^q}w^2$ has a solution $w = -\left(\int_0^z e^{-t^q}\,dt\right)^{-1}$ with $q+1$ deficient values for any integer q ([5], p. 46). We also state in this connection that $w(z)$ is almost admissible for sufficiently large values of q in the sense that

$$\lim_{q\to\infty}\left(\limsup_{r\to\infty}\frac{T(r, e^{-z^q})}{T(r, w)}\right) = \lim_{q\to\infty}\frac{1}{q} = 0\,.$$

Further, let $g(z)$ be an integral function with infinitely many deficient values. Then the equation

$$\frac{dw}{dz} = \frac{g'(z)}{g(z)^2}\,w^2$$

has solutions with infinitely many deficient values.

Examples of Riccati differential equations given in a paper of Rogai [8] contain some special types of non-admissible solutions. We can verify that the following lemmas presented in [8] retain their validity also in the complex case.

Lemma 5. *The equation* (1.1) *with* $a_1(z) \not\equiv 0$ *has a solution of the form*

$$w = (Ce^z - a_0)/a_1 \tag{2.12}$$

if and only if

$$a_1(Ce^z - a_0') - a_1'(Ce^z - a_0) = a_2(Ce^z - a_0)^2 + Ca_1^2e^z\,.$$

Lemma 6. *The equation* (1.1) *with* $a_1(z) \not\equiv 0$ *has a solution of the form*

$$w = Ce^z/a_1 \tag{2.13}$$

if and only if

$$Ca_1e^z - Ca_1'e^z = C^2a_2e^{2z} + Ca_1^2e^z + a_0a_1^2\,.$$

Lemma 7. *The equation* (1.1) *has a solution of the form*

$$w = C_1 + C_2e^z \tag{2.14}$$

if and only if

$$C_2e^z = a_2(C_1 + C_2e^z)^2 + a_1(C_1 + C_2e^z) + a_0\,.$$

Other special cases presented in [8] have in general non-meromorphic solutions in the complex case.

Theorem 9. *If a solution* $w = w(z)$ *of the equation* (1.1) *is of the type* (2.12), (2.13), *or* (2.14), *then* $w(z)$ *is non-admissible.*

Proof. Suppose contrary to the assertion that $w(z)$ were admissible. We immediately see that the classes $S(r, w)$ and $S(r, e^z)$ are identical. Thus in the first case

$$\begin{aligned} 2\,T(r, e^z) - S(r, e^z) &\leqq T(r, a_2(Ce^z - a_0)^2) \\ &= T(r, a_1(Ce^z - a_0') - a_1'(Ce^z - a_0) - Ca_1^2 e^z) \\ &= T(r, Ce^z(a_1 - a_1' - a_1^2) + a_0 a_1' - a_0' a_1) \leqq T(r, e^z) + S(r, e^z)\,, \end{aligned}$$

hence

$$2\,T(r, e^z) \leqq T(r, e^z) + S(r, e^z),$$

a contradiction.

In the second case we get a contradiction by

$$\begin{aligned} 2\,T(r, e^z) - S(r, e^z) &\leqq T(r, C^2 a_2 e^{2z}) = T(r, Ca_1 e^z - Ca_1' e^z - Ca_1^2 e^z - a_0 a_1^2) \\ &= T(r, Ce^z(a_1 - a_1' - a_1^2) - a_0 a_1^2) \leqq T(r, e^z) + S(r, e^z) \end{aligned}$$

and in the third case by

$$\begin{aligned} T(r, e^z) &= T(r, C_2 e^z) + O(1) = T(r, a_2(C_1 + C_2 e^z)^2 + a_1(C_1 + C_2 e^z) + a_0) + O(1) \\ &\geqq 2\,T(r, C_1 + C_2 e^z) - S(r, e^z) = 2\,T(r, e^z) - S(r, e^z)\,. \end{aligned}$$

Theorem 10. *If a meromorphic function* $w = w(z)$ *is a solution of the Riccati differential equation* (1.1) *and satisfies the condition*

$$N(r, w) = O\Big(\sum_{i=0}^{2} T(r, a_i)\Big) + O(\log r)$$

outside of a possible exceptional set of finite linear measure, then outside of a possible exceptional set of the same kind

$$T(r, w) = O\Big(\sum_{i=0}^{2} T(r, a_i)\Big) + O\,(\log r).$$

Proof. In the following proof all estimations are valid outside of a possible exceptional set of finite linear measure. Substituting $w = u - \dfrac{a_1}{2a_2}$ we get the following modified equation:

$$u^2 = \frac{u'}{a_2} + R = \frac{u'}{a_2} - \left[\frac{1}{2a_2^2}\,a_1' - \frac{a_1}{2a_2^3}\,a_2' + \frac{a_0}{a_2} - \frac{1}{4}\left(\frac{a_1}{a_2}\right)^2\right].$$

Thus

$$T(r, R) = O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r)$$

and so

$$m(r, u^2) = 2\, m(r, u) \leqq m(r, u') + O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r)$$

$$\leqq m(r, u) + O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r) + O(\log T(r, u))$$

implying

$$m(r, u) \leqq O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r) + O(\log T(r, u)) .$$

Hence

$$m(r, w) \leqq m(r, u) + O(\sum_{i=0}^{2} T(r, a_i))$$

$$\leqq O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r) + O(\log T(r, u))$$

and further

$$T(r, w) = m(r, w) + N(r, w) \leqq O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r) + O(\log T(r, u))$$

$$\leqq O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r) + O(\log\ T(r, w))$$

$$\leqq \frac{1}{2}\ T(r, w) + O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r)$$

for sufficiently large values of r. The theorem follows.

The estimations above are valid without any exceptional set, if all functions used are of finite order. Thus we have

Corollary 1. *If* $w = w(z)$ *is a meromorphic solution of finite order of the equation* (1.1), *whose coefficients are also of finite order, and if* $N(r, w) = O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r)$, *then we have* $T(r, w) = O(\sum_{i=0}^{2} T(r, a_i)) + O(\log r)$.

The following corollary 2 contains as a special case an earlier result due to Wittich ([12], p. 76).

Corollary 2. *If the equation* (1.1) *whose coefficients are of finite order, has a meromorphic solution* $w = w(z)$ *of finite order such that* $w(z)$ *has a finite number of poles, then*

$$\lambda_w \leqq \max\,(\lambda_{a_0}, \lambda_{a_1}, \lambda_{a_2}).$$

Remark. The previous main theorem does not imply any upper or lower bound for $\delta(w, \infty)$. For example the equation $w' = e^{-z^2} w^2$ has the

solution $w = -\left(\int\limits_0^z e^{-t^2} dt\right)^{-1}$ with no deficiency at $z = \infty$: $\delta(w, \infty) = 0$.

On the other hand the equation $w' = -e^{3z} + w + e^z w^2$ has the solution $w = e^z$ with $\delta(w, \infty) = 1$.

§ 3. THE EQUATION $\left(\frac{dw}{dz}\right)^n = \sum\limits_{i=0}^{n+k} a_i w^i$

3.1. Characterization by admissible solutions.

Theorem 11. *Let $w = w(z)$ be a meromorphic solution of the equation*

$$\sum_{i=0}^{p} b_i(z) \left(\frac{dw}{dz}\right)^i = \sum_{i=0}^{q} a_i(z) w^i . \tag{3.1}$$

If $w(z)$ is admissible with respect to the coefficients of both sides of (3.1), *then $q \leqq 2p$.*

Proof. Let us denote the left-hand side of (3.1) by P and the right-hand side by Q. We can assume that $p < q$. Write (3.1) in the form

$$(a_q w) w^{q-1} = \sum_{i=0}^{p} b_i \left(\frac{dw}{dz}\right)^i - \sum_{i=0}^{q-1} a_i w^i .$$

Then by lemma 2

$$m(r, w) \leqq m\left(r, \frac{1}{a_q}\right) + m(r, a_q w) + O(1) = S(r, w) .$$

Since by lemma 4

$$\begin{cases} N(r, Q) \geqq q\, N(r, w) - S(r, w) \\ N(r, P) \leqq p\, N(r, w') + S(r, w) , \end{cases}$$

we get

$$\begin{aligned} q\, T(r, w) &= q\, N(r, w) + S(r, w) \leqq N(r, Q) + S(r, w) \\ &= N(r, P) + S(r, w) \leqq p\, N(r, w') + S(r, w) \\ &\leqq 2p\, N(r, w) + S(r, w) \leqq 2p\, T(r, w) + S(r, w) . \end{aligned}$$

The theorem follows.

Corollary. *Let $w = w(z)$ be an admissible solution of the equation*

$$(3.2) \qquad \left(\frac{dw}{dz}\right)^p = \sum_{i=0}^{q} a_i(z) w^i .$$

Then $q \leqq 2p$.

3.2. Deficiencies of admissible solutions.

Theorem 12. *If* $w = w(z)$ *is an admissible solution of the equation*

$$(3.3) \qquad \left(\frac{d^p w}{dz^p}\right)^n = \sum_{i=0}^{n+k} a_i(z) w^i , \quad \textit{where} \quad 1 \leqq k \leqq n ,$$

then $\delta(w, \infty) = 0$ *and* $\Theta(w, \infty) \leqq 1 - \dfrac{k}{np}$. *If* $w = w(z)$ *is strongly admissible, then* $\Theta(w, \infty) = 1 - \dfrac{k}{np}$.

Proof. Writing (3.3) in the form

$$w^{n+k-1}(a_{n+k} w) = \left(\frac{d^p w}{dz^p}\right)^n - \sum_{i=0}^{n+k-1} a_i w^i ,$$

we get by lemma 2

$$m(r, w) \leqq m\left(r, \frac{1}{a_{n+k}}\right) + m(r, a_{n+k} w) + O(1) = S(r, w) .$$

Thus the first part of the assertion is established.

By lemma 4 we further have

$$(n + k) N(r, w) - S(r, w) \leqq N(r, \sum_{i=0}^{n+k} a_i w^i)$$

$$= N(r, (w^{(p)})^n) = n N(r, w) + n p \bar{N}(r, w) ,$$

hence

$$k T(r, w) - S(r, w) = k N(r, w) - S(r, w) \leqq n p \bar{N}(r, w)$$

and

$$n p \limsup_{r\to\infty} \frac{\bar{N}(r, w)}{T(r, w)} \geqq k - \liminf_{r\to\infty} \frac{S(r, w)}{T(r, w)} = k .$$

Thus

$$\Theta(w, \infty) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, w)}{T(r, w)} \leqq 1 - \frac{k}{np} .$$

Supposing now $w(z)$ to be strongly admissible we have by corollary to lemma 4

$$n N(r, w) + n p \bar{N}(r, w) = N(r, \sum_{i=0}^{n+k} a_i w^i) \leqq (n + k) N(r, w) + S^0(r, w) ,$$

thus

$$n\,p\,\bar{N}(r, w) \leqq k\,N(r, w) + S^0(r, w) \leqq k\,T(r, w) + S^0(r, w)$$

and

$$n\,p \limsup_{r\to\infty} \frac{\bar{N}(r, w)}{T(r, w)} \leqq \limsup_{r\to\infty} \left(k + \frac{S^0(r, w)}{T(r, w)}\right) = k$$

yielding

$$\Theta(w, \infty) \geqq 1 - \frac{k}{np}\,.$$

The theorem follows.

Corollary 1. *If* $w = w(z)$ *is an admissible solution of the equation* (1.2), *then* $\delta(w, \infty) = 0$ *and* $\Theta(w, \infty) \leqq 1 - \frac{k}{n}$. *If* $w = w(z)$ *is strongly admissible or if* $k = n$, *then* $\Theta(w, \infty) = 1 - \frac{k}{n}$.

Corollary 2. *The equation* (3.3) *does not admit an admissible integral solution.*

Theorem 13. *If* $w = w(z)$ *is an admissible solution of the equation* (1.2) *with integral coefficients, then* $\delta(w, \alpha) = \Theta(w, \alpha) = 1$ *if and only if* $\alpha \neq \infty$ *is a root of multiplicity* $p \geqq n$ *of the equation*

$$a_0(z) + \alpha a_1(z) + \ldots + \alpha^{n+k}\, a_{n+k}(z) \equiv 0\,. \tag{3.4}$$

If α *is a root of multiplicity* $0 \leqq p < n$ *of the equation* (3.4), *then* $\delta(w, \alpha) = 0$ *and* $\Theta(w, \alpha) \leqq \frac{p}{n}$. *If additionally* $w = w(z)$ *is strongly admissible or if* $p = 0$, *then* $\Theta(w, \alpha) = \frac{p}{n}$.

Proof. Substituting $w = \alpha + \frac{1}{u}$ we get

$$\begin{aligned}\left(\frac{du}{dz}\right)^n &= \sum_{j=0}^{n+k} (-1)^n\, a_j(z) \left(\alpha + \frac{1}{u}\right)^j u^{2n} = \\ &= \sum_{j=0}^{n+k} (-1)^n\, a_j(z) \sum_{p=0}^{j} C_j^{(p)}\, \alpha^{j-p}\, u^{2n-p} \\ &= \sum_{p=0}^{n+k} \Big(\sum_{j=p}^{n+k} (-1)^n\, a_j(z) C_j^{(p)}\, \alpha^{j-p}\Big)\, u^{2n-p} = \sum_{p=0}^{n+k} b_{2n-p}(z) u^{2n-p}\,.\end{aligned}$$

Thus we have for $p = 0, \ldots, n-1$

$$\begin{cases} b_{2n}(z) = (-1)^n (a_0(z) + \alpha a_1(z) + \ldots + \alpha^{n+k}\, a_{n+k}(z)) \\ b_{2n-p}(z) = \displaystyle\sum_{j=p}^{n+k} \frac{(-1)^n}{p!}\, j(j-1)\ldots(j-p+1)\alpha^{j-p}\, a_j(z)\,, \end{cases} \tag{3.5}$$

where all functions $b_i(z)$ are integral functions. Since $u = u(z)$ is an admissible solution of the modified equation, we have by corollary 1 to theorem 12

$$\begin{cases} \delta(u,\ \infty) = \delta(w, \alpha) = 0 \\ \Theta(u,\ \infty) = \Theta(w, \alpha) \leqq 1 - \dfrac{n-p}{n} = \dfrac{p}{n}, \end{cases}$$

if the highest coefficient not vanishing identically is $b_{2n-p}(z)$ with $0 \leqq p < n$. Considering the right-hand sides of the equations (3.5) we observe that the functions $p!(-1)^n b_{2n-p}(z)$ for $p = 1, \ldots, n-1$ are successive derivatives of the polynomial $a_0(z) + \alpha a_1(z) + \ldots + \alpha^{n+k} a_{n+k}(z)$ in the polynomial ring $\mathcal{M}[\alpha]$, where $\mathcal{M}$ is the field of meromorphic functions. Thus $b_{2n-p}(z)$ is the highest coefficient not vanishing identically if and only if α is a root of multiplicity $0 \leqq p < n$ of the equation (3.4).

If $p = 0$ or if $w(z)$ is strongly admissible, the inequality $\Theta(w, \alpha) \leqq \dfrac{p}{n}$ evidently changes to an equality.

To complete the proof we have only to consider the case where the modified equation has a reduced form

$$\left(\frac{du}{dz}\right)^n = \sum_{i=0}^{r} b_i(z) u^i, \text{ where } r \leqq n .$$

This is the case if and only if α is a root of multiplicity $p \geqq n$ of the equation (3.4). Supposing $u(z)$ has a pole of multiplicity $\nu(u,\ \infty) \geqq 1$ at some point z we get

$$n(\nu(u,\ \infty) + 1) = \nu((u')^n,\ \infty) = \nu(\sum_{i=0}^{r} b_i\, u^i,\ \infty) \leqq r\nu(u,\ \infty) ,$$

which implies a contradiction

$$1 \leqq n \leqq (n - r)\, \nu(u,\ \infty) + n \leqq 0 .$$

Thus

$$\begin{cases} \delta(u,\ \infty) = \delta(w, \alpha) = 1 \\ \Theta(u,\ \infty) = \Theta(w, \alpha) = 1 . \end{cases}$$

3.3. Equations of type (1.2) with constant coefficients. We first note that in the case of constant coefficients any transcendental solution of (1.2) is strongly admissible, thus the equality in theorem 12 and theorem 13 is valid. Let us first consider two specific examples.

Example 1. The Weierstrass elliptic function $\wp(z)$ satisfies the equation

$$(\mathcal{P}')^2 = (\mathcal{P} - \alpha)(\mathcal{P} - \beta)(\mathcal{P} - \gamma)$$

([5], p. 44) or, equivalently,

$$(\mathcal{P}')^2 = -\alpha\beta\gamma + (\alpha\beta + \beta\gamma + \gamma\alpha)\mathcal{P} - (\alpha + \beta + \gamma)\mathcal{P}^2 + \mathcal{P}^3 .$$

The distinct complex numbers α, β and γ are simple roots of the equation

$$-\alpha\beta\gamma + \omega(\alpha\beta + \beta\gamma + \gamma\alpha) - \omega^2(\alpha + \beta + \gamma) + \omega^3 = 0 ,$$

thus by theorem 12 and theorem 13

$$\begin{cases} \Theta(\mathcal{P}, \infty) = \Theta(\mathcal{P}, \alpha) = \Theta(\mathcal{P}, \beta) = \Theta(\mathcal{P}, \gamma) = \frac{1}{2} \\ \Theta(\mathcal{P}, \omega) = 0 \text{ for } \omega \neq \infty, \alpha, \beta, \gamma \\ \delta(\mathcal{P}, \omega) = 0 \text{ for all } \omega . \end{cases}$$

Example 2. Let us consider the inverse function $w = w(z)$ of

$$z = \int_0^w (t - \alpha)^{\frac{1}{m} - 1} (t - \beta)^{\frac{1}{n} - 1} (t - \gamma)^{\frac{1}{p} - 1} \, dt ,$$

where the positive integers m, n, p satisfy the condition $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1$ and where α, β, γ are distinct complex numbers ([5], p. 45). The inverse function can be continued over the whole plane as a one-valued meromorphic double-periodic function by Schwarz's reflection principle. This continued function $w = w(z)$ is an admissible solution of the differential equation

$$\left(\frac{dw}{dz}\right)^{mnp} = (w - \alpha)^{mnp - np} (w - \beta)^{mnp - mp} (w - \gamma)^{mnp - mn} = \sum_{i=0}^{2mnp} \alpha_i w^i .$$

The roots of the equation $\sum_{i=0}^{2mnp} \alpha_i \omega^i = 0$ are α, β and γ with respective multiplicities $mnp - np$, $mnp - mp$ and $mnp - mn$. Thus by theorem 13

$$\begin{cases} \Theta(w, \alpha) = \dfrac{mnp - np}{mnp} = 1 - \dfrac{1}{m} \\ \Theta(w, \beta) = \dfrac{mnp - mp}{mnp} = 1 - \dfrac{1}{n} \\ \Theta(w, \gamma) = \dfrac{mnp - mn}{mnp} = 1 - \dfrac{1}{p} \\ \Theta(w, \omega) = 0 \text{ for } \omega \neq \alpha, \beta, \gamma \\ \delta(w, \omega) = 0 \text{ for all } \omega . \end{cases}$$

Generally we have

Theorem 14. *Let* $w = w(z)$ *be an admissible solution of the equation* (1.2) *with constant coefficients and let* $n + p$ *be the highest multiplicity appearing among the distinct roots of the equation* (3.4). *Then*

$$\sum_{\alpha} \Theta(w, \alpha) = 2 - \frac{\max(0, p)}{n} .$$

Proof. When applying theorem 13 and corollary 1 to theorem 12 in this proof we naturally have the case of strong admissibility.

1°. $p > 0$. Since now all but one of the roots of (3.4) are of multiplicity $s_i < n$, we get

$$\sum_{\alpha} \Theta(w, \alpha) = \Theta(w, \infty) + \sum_{\alpha \neq \infty} \Theta(w, \alpha) = 1 - \frac{k}{n} + 1 + \sum_{i} \frac{s_i}{n}$$

$$= 2 - \frac{k}{n} + \frac{n + k - n - p}{n} = 2 - \frac{p}{n} .$$

2°. $p = 0$. If there exist two distinct roots of (3.4) with the multiplicity n, then $n = k$ and

$$\sum_{\alpha} \Theta(w, \alpha) = \sum_{\alpha \neq \infty} \Theta(w, \alpha) = 2 .$$

If, on the other hand, there exists exactly one root of (3.4) with the highest multiplicity, then all other roots have a multiplicity $s_i < n$ and so

$$\sum_{\alpha} \Theta(w, \alpha) = 1 - \frac{k}{n} + 1 + \sum_{i} \frac{s_i}{n} = 2 - \frac{k}{n} + \frac{n + k - n}{n} = 2.$$

3°. $p < 0$. Now all roots of (3.4) are at most of multiplicity $n - 1$, hence

$$\sum_{\alpha} \Theta(w, \alpha) = 1 - \frac{k}{n} + \sum_{i} \frac{s_i}{n} = 1 - \frac{k}{n} + \frac{n + k}{n} = 2 .$$

By Singh and Gopalakrishna ([9], p. 125) we get the following

Corollary. *Let* $w = w(z)$ *be an admissible solution of finite order of the equation* (1.2) *with constant coefficients and let* $s_1, \ldots, s_k$ *be the multiplicities of the distinct roots* $\alpha_1, \ldots, \alpha_k$ *of the equation* (3.4). *If* $\max(s_1, \ldots, s_k) \leq n$, *then*

$$\begin{cases} \lim\limits_{r\to\infty} \dfrac{T(r, w')}{T(r, w)} = 1 + \dfrac{k}{n} \\ \lim\limits_{r\to\infty} \dfrac{\bar{N}(r, w)}{T(r, w)} = \dfrac{k}{n} \\ \lim\limits_{r\to\infty} \dfrac{\bar{N}(r, \alpha_i)}{T(r, w)} = 1 - \dfrac{s_i}{n} \\ \lim\limits_{r\to\infty} \dfrac{\bar{N}(r, \omega)}{T(r, w)} = 1 \textit{ for } \omega \neq \alpha_i, \infty . \end{cases}$$

§ 4. CONCLUDING REMARKS

4.1. Meromorphic coefficients. In many of the previous theorems we have restricted our attention on the equations with integral coefficients. It seems to be evident that most of the results retain their validity in the general case of meromorphic coefficients, possibly in a somewhat weakened form. We omit these considerations.

4.2. Admissible integral solutions. It is obvious that the concept of the admissible solution is useful also for other types of differential equations. As an example we consider here a theorem due to Wittich ([11], p. 221).

Theorem 15. *Let $P(w, w_1, \ldots, w_n, z)$ be a polynomial in the variables $w, w_1, \ldots, w_n$ with integral coefficients $b_k(z)$. If $w = w(z)$ is an admissible integral solution of the equation*

$$P(w, w^{(1)}, \ldots, w^{(n)}, z) = f(w) , \tag{4.1}$$

where $f(w)$ is a transcendental integral function of w, then $w(z)$ is a constant.

Proof. According to Clunie ([5], p. 54) we have

$$\liminf_{r\to\infty} \frac{T(r, f(w(z)))}{T(r, w(z))} = \liminf_{r\to\infty} \frac{m(r, f(w(z)))}{m(r, w(z))} = \infty ,$$

since $f(w)$ and $w(z)$ are transcendental functions.

On the other hand, writing (4.1) in the form

$$P(w, w^{(1)}, \ldots, w^{(n)}, z) = \sum_k b_k(z) \cdot w^{k_0} \cdot (w^{(1)})^{k_1} \cdots (w^{(n)})^{k_n}$$

$$= \sum_k b_k(z) \cdot w^{k_0} \cdot \left(\frac{w^{(1)}}{w}\right)^{k_1} \cdot w^{k_1} \cdots \left(\frac{w^{(n)}}{w^{(n-1)}}\right)^{k_n} \cdots \left(\frac{w^{(1)}}{w}\right)^{k_n} \cdot w^{k_n}$$

we see that

$$m(r, P) = O(m(r, w))$$

outside of a possible exceptional set of finite linear measure. Thus

$$m(r, f(w(z)) = O(m(r, w(z)\ )\ )$$

outside of a possible exceptional set, a contradiction. The theorem follows.

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Printed September 1971