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A DENSITY ESTIMATE FOR *L*-FUNCTIONS WITH A REAL CHARACTER

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A density estimate for L-functions with a real character

1. The density theorem. For any Dirichlet character χ and for real $\alpha \geqq 0$, $T > 0$, define $N(\alpha, T, \chi)$ to be the number of the zeros of the function $L(s, \chi)$ in the rectangle

$$(1.1) \qquad \alpha \leqq \sigma \leqq 1, \quad |t| \leqq T.$$

Let d be a fundamental discriminant (i. e. $d \neq 1$, and either $d \equiv 1 \pmod 4$, d square-free, or $d = 4N$, $N \equiv 2,3 \pmod 4$, N square-free), and write $\chi_d(n) = (d/n)$, where (d/n) is Kronecker's symbol. (In general, we shall write $\chi_D(n) = (D/n)$ provided (D/n) is defined). As is well known, the characters χ_d exhaust all real primitive non-principal characters, and the modulus of χ_d is $|d|$. Now, for $X \geqq 3$, consider the sums

$$N_r^*(\alpha, T, X) = \sum_{|d| \leqq X}^* N(\alpha, T, \chi_d),$$

$$N_r'(\alpha, T, X) = \sum_{|d| \leqq X}^* (X|d|^{-1})^{\frac{1}{2}} N(\alpha, T, \chi_d),$$

where Σ^* denotes a sum restricted to fundamental discriminants. It is our purpose to find estimates for these sums.

Various estimates are known for the sum

$$N^*(\alpha, T, X) = \sum_{q \leq X} \sum_{\chi_q^* \bmod q} N(\alpha, T, \chi_q^*),$$

where χ_q^* runs over all primitive characters to the modulus q. (For an excellent account of this subject, see Montgomery [2], Ch. 12). Now, wanting to obtain non-trivial estimates for $N_r^*(\alpha, T, X)$, we shall consider the inequalities

$$(1.2) \qquad N_r^*(\alpha, T, X) \leqq N^*(\alpha, T, X),$$

$$(1.3) \qquad N_r^*(\alpha, T, X) \leqq N_r'(\alpha, T, X).$$

In view of the known estimates for $N^*(\alpha, T, X)$, the inequality (1.2) gives sharp and non-trivial results if α is near 1, say $\alpha > \frac{4}{5}$, but near $\alpha = \frac{1}{2}$ the results become trivial. On the other hand, a recent result obtained by the author [1] on mean-values of character sums with real characters enables us to obtain for $N'_r(\alpha, T, X)$ an estimate which is sharpest near $\alpha = \frac{1}{2}$. The density theorem to be proved runs as follows:

Theorem 1. *We have uniformly for* $\frac{1}{2} \leqq \alpha \leqq 1$, $T \geqq 2$, $X \geqq 3$

$$N'_r(\alpha, T, X) \ll X^{\frac{7-6\alpha}{6-4\alpha}} T^2 \log^{68}(XT) .$$

Corollary 1. *There exists a function* $f(x, y)$, *defined for* $x > 0$, $y > 0$ *with*

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

such that for any numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *and for* $X \geqq X(\varepsilon_1, \varepsilon_2)$, *the region*

$$\sigma \geqq \tfrac{1}{2} + f(\varepsilon_1, \varepsilon_2), \quad |t| \leqq X^{\varepsilon_1}$$

contains no zero of any function $L(s, \chi_d)$ *with* $|d| \leqq X$, *with a possible exception of* $X^{1-\varepsilon_2}$ *functions at most.*

Corollary 2. *We have uniformly for* $\frac{1}{2} \leqq \alpha \leqq 1$, $T \geqq 2$, $X \geqq 3$

$$N^*_r(\alpha, T, X) \ll \min\{X^{\frac{7-6\alpha}{6-4\alpha}} T^2, \ (X^2T)^{2\alpha^{-1}(1-\alpha)}\} \log^{68}(XT) . \tag{1.4}$$

Corollary 2 is a combination of our theorem 1 and a theorem of Montgomery (see [2], theorem 12.2). If T is »small» (for example, $T \ll X^{\varepsilon}$), the first estimate in (1.4) is better than the second for $\alpha \leqq 0.844$. It would be of interest to find an approach to $N^*_r(\alpha, T, X)$ which would give estimates, better than those known for $N^*(\alpha, T, X)$, also near $\alpha = 1$.

The exponent of X in theorem 1 tends to $\frac{1}{2}$ as $\alpha \to 1$. A better limit would be a very deep result since it is possible that, for some d, there exists an exceptional zero, giving to $N'_r(\alpha, T, X)$ a term $\gg (X|d|^{-1})^{\frac{1}{2}}$ which tends to $X^{\frac{1}{2}}$ as $|d|$ decreases.

In the proof of theorem 1 we shall need some mean-value results for L-functions and Dirichlet polynomials (theorem 2 and lemmas 1, 2, 4 below) from which the density estimate is deduced by the classical arguments of Littlewood and Ingham.

2. The mean-value theorem. To formulate the mean-value theorem, mentioned above, define $\mathcal{D}$ to be the set of the integers D satisfying the following conditions:

(i) D is not a square,
(ii) $D \equiv 1 \pmod 4$, or $D = 4N$, $N \equiv 1, 2, 3 \pmod 4$.

In [1] it is proved that uniformly for $X \geqq 3$, $Y \geqq 1$

$$\sideset{}{'}\sum_{|D| \leq X} \left| \sum_{1 \leqq n \leqq Y} \left(\frac{D}{n}\right) \right|^2 \ll XY \log^8 X , \tag{2.1}$$

where Σ' denotes a summation restricted to the set $\mathcal{D}$.

Using (2.1) we shall prove that following mean-value result:

Theorem 2. *We have uniformly for* $X \geqq 3$ *and for all real* t

$$\sum_{|D| \leqq X}' |L(\tfrac{1}{2} + it, \chi_D)|^2 \ll X(|t| + 1)^2 \log^{10}(X(|t| + 1)) .$$

Proof. If $|D| \leqq X$ then by partial summation and the Polya-Vinogradov estimate we have

$$L(\tfrac{1}{2} + it, \chi_D) \ll (|t| + 1) \sum_{1 \leqq n \leqq N} |s_D(n)| n^{-\frac{3}{2}} + 1 ,$$

where $N = X(|t| + 1)^2 \log^2 X$,

$$s_D(n) = \sum_{1 \leqq m \leqq n} \left(\frac{D}{m}\right) .$$

Hence by Schwarz's inequality and (2.1)

$$\sum_{|D| \leqq X}' |L(\tfrac{1}{2} + it, \chi_D)|^2 \ll (|t| + 1)^2 \sum_{1 \leqq m \leqq N} m^{-1} \sum_{1 \leqq n \leqq N} n^{-2} \sum_{|D| \leqq X}' |s_D(n)|^2 + X$$
$$\ll X(|t| + 1)^2 \log^{10}(X(|t| + 1)) .$$

3. Preliminaries for the proof of the density theorem. First note that if $D \in \mathfrak{D}$ then D is of the form $D = da^2$, where d is a fundamental discriminant. The non-trivial zeros of the functions $L(s, \chi_D)$ and $L(s, \chi_d)$ coincide, whence a zero of $L(s, \chi_d)$, lying in the rectangle (1.1), occurs in the sum

$$\sum_{|D| \leqq X}' N(\alpha, T, \chi_D) \tag{3.1}$$

$\gg (X|d|^{-1})^{\frac{1}{2}}$ times. Hence

$$N_r'(\alpha, T, X) \ll \sum_{|D| \leqq X}' N(\alpha, T, \chi_D) ,$$

and so it will be sufficient to find an estimate for the sum (3.1).

Let J_1 be the value of the sum (3.1), and let $D_1, \ldots, D_j$ be the numbers D in (3.1) for which $N(\alpha, T, \chi_D) > 0$. Obviously $j \leqq J_1$.

Let $z = X^{\frac{1}{2}}$,

$$M_D(s) = \sum_{n<z} \mu(n) \chi_D(n) n^{-s} ,$$

$$F_D(s) = L(s, \chi_D) M_D(s) ,$$

$$H(s) = \prod_{i=1}^{j} F_{D_i}(s) ,$$

and let J be the number of the zeros of $H(s)$ in the rectangle (1.1). Then $J_1 \leqq J$, so that an estimate for J will suffice.

Since for Re $s \geqq \frac{1}{2}$

$$L(s\,,\chi_D) = \sum_{n \leqq M} \chi_D(n) n^{-s} + O(X^{-1})\,,$$

where $M = X^3\,(\,|t| + 1)^2 \log^2 X$, we have

$$F_D(s) = 1 + \sum_{z \leqq n \leqq Mz} a_n \chi_D\,(n) n^{-s} + O(X^{-3/4}) \tag{3.2}$$
$$= 1 + f_D(s) + O(X^{-3/4})\,,$$

say. The coefficients a_n are bounded by $|a_n| \leqq \tau(n)$.

For the proof of theorem 1 we shall need some mean-value results for $F_D(s)$ and $f_D(s)$ which will be proved in the next section.

4. Mean-value lemmas. We begin with a mean-value estimate for $|F_D(s)\,|$ on the critical line.

Lemma 1.

$$\sum_{|D| \leqq X}' |F_D(\tfrac{1}{2} + it)\,| \ll X(\,|t| + 1) \log^7 (X\,(|t| + 1))\,. \tag{4.1}$$

Proof. By Schwarz's inequality

$$(\sum_{|D| \leqq X}' |F_D(\tfrac{1}{2} + it)\,|)^2 \ll \sum_{|D| \leqq X}' |L(\tfrac{1}{2} + it\,,\,\chi_D)\,|^2 \sum_{|D| \leqq X}' |M_D(\tfrac{1}{2} + it)\,|^2\,. \tag{4.2}$$

Now

$$\sum_{D \leqq |X|}' |M_D(\tfrac{1}{2} + it)\,|^2 \ll X \log^4 X \tag{4.3}$$

by a simple application of the Polya-Vinogradov inequality (along the lines of section 7 of [1]). Using (4.3) and theorem 2 in (4.2), we obtain (4.1).

Lemma 2.

$$\sum_{i=1}^{j} |f_{D_i}\,(1 + it)\,| \ll j^{1/2}\, X^{1/4} \log^{32} (X(\,|t| + 1))\,. \tag{4.4}$$

Proof. By Schwarz's inequality and (3.2) we have

$$(\sum_{i=1}^{j} |f_{D_i}\,(1 + it)\,|)^2 \ll j \sum_{|D| \leqq X}' |f_D\,(1 + it)\,|^2 \tag{4.5}$$
$$= j \sum_{z \leqq r,\, s \leqq Mz} a_r a_s r^{-1-it} s^{-1+it} \sum_{|D| \leqq X}' \chi_D(rs)$$
$$\leqq j \sum_{z^2 \leqq n \leqq M^2 z^2} b_n n^{-1} S(n)\,,$$

where

$$b_n = \sum_{rs=n} |a_r a_s| \leqq \sum_{rs=n} \tau(r)\tau(s) \leqq \tau^3(n)\,, \tag{4.6}$$

$$S(n) = \left|\ \sum_{|D| \leqq X}' \left(\frac{D}{n}\right)\right|. \tag{4.7}$$

If n is a square, then $S(n) \ll X$, so that the squares contribute to (4.5)

(4.8) $$\ll jX \sum_{n \geqq z} \tau^3(n^2) n^{-2} \ll jX \sum_{n \geqq z} \tau^6(n) n^{-2} \ll jX^{\frac{1}{2}} (\log X)^{2^6-1} .$$

For a non-square integer $n \geqq 2$ we have by the quadratic reciprocity law $(D/n) = \varepsilon\, (n'/|D|)$ with $n' \in \mathfrak{D}$, $|n'| \leqq 4n$, ε depending on n and sgn D only. The same value of n' occurs for $\ll \log (X(|t| + 1))$ integers n in (4.5) at most. Hence by (2.1)

(4.9) $$\sum_{\substack{n \leqq y \\ n \neq a_2}} S^2(n) \ll Xy \log^9 (X(|t| + 1))$$

for all $y \leqq M^2 z^2$, for the sums

$$\left| \sum_{0 < D \leqq X}{}' \left(\frac{n'}{D}\right) \right| + \left| \sum_{0 > D \geqq -X}{}' \left(\frac{n'}{|D|}\right) \right|$$

are easily expressed by »complete» D-sums.

Now, using again Schwarz's inequality, we obtain from (4.5) by (4.6), (4.8) and (4.9)

$$\Big(\sum_{i=1}^{j} |f_{D_i}(1 + it)|\Big)^4 \ll j^2 X \log^{126} X + j^2 \sum_{z^2 \leqq n \leqq M^2 z^2} \tau^6(n) n^{-1} \sum_{\substack{z^2 \leqq n \leqq M^2 z^2 \\ n \neq a^2}} S^2(n) n^{-1}$$

$$\ll j^2 X \log^{126} (X(|t| + 1)) .$$

This proves lemma 2.

Lemma 3. *Let the functions* $g_1(s), \ldots, g_h(s)$ *be regular and bounded in the strip* $\sigma_1 \leqq \sigma \leqq \sigma_2$. *Let*

$$G(s) = \sum_{n=1}^{h} |g_n(s)| ,$$

$$M(\alpha) = \sup_{\sigma = \alpha} G(s) .$$

Then for $\sigma_1 \leqq \sigma \leqq \sigma_2$ *we have*

$$M(\sigma) \leqq M(\sigma_1)^{(\sigma_2 - \sigma)/(\sigma_2 - \sigma_1)} M(\sigma_2)^{(\sigma - \sigma_1)/(\sigma_2 - \sigma_1)} .$$

For a proof of this convexity result, see [3], p. 401, Satz 9.3, where the lemma is stated and proved for squares $g_n^2(s)$. However, exactly the same arguments work in the case of lemma 3, too.

Lemma 4. *Let* $\frac{1}{2} \leqq \sigma \leqq 1$, *and let*

$$\varphi(\sigma) = \max_{|t| \leq T} \sum_{i=1}^{j} |f_{D_i}(\sigma + it)| .$$

Then

$$\varphi(\sigma) \ll X^{7/4-3\sigma/2}\, T^{2-2\sigma}\, J^{\sigma-\frac{1}{2}} \log^{32}(XT)\,.$$

Proof. Use lemmas 1—3 noting that $j \leqq J$, that $|f_D(\frac{1}{2}+it)| \ll 1 + |F_D(\frac{1}{2}+it)|$, and that the boundedness condition in lemma 3 will be satisfied if we multiply each $f_{D_i}(s)$ by a suitable function tending rapidly to 0 for $T < |\mathrm{Im}\, s| \to \infty$ (see [3], p. 309).

5. Proof of the density theorem. It is sufficient to prove theorem 1 in the case $\alpha \geqq \frac{1}{2} + 2\delta$ with $\delta = (\log XT)^{-1}$, since otherwise the assertion is trivial. The number J of the zeros of the function $H(s)$ in the rectangle (1.1) will be estimated by a combination of lemma 4 with the following two well-known lemmas.

Lemma 5. *Suppose that T is not the imaginary part of a zero of $H(s)$ in the critical strip. Then*

$$J \ll \delta^{-1} \int_{-T}^{T} \{\log |H(\alpha-\delta+it)| - \log |H(5+it)|\}\, dt$$

$$+\ \delta^{-1} \int_{\alpha-\delta}^{5} \{\arg H(\sigma+iT) - \arg H(\sigma-iT)\}\, d\sigma\,,$$

where $\arg H(\sigma \mp iT)$ *is determined by a continuous variation from* $\arg H(5) = 0$ *along the lines* $\sigma = 5\,, t = \mp T$.

This follows from a classical theorem of Littlewood (see [3], p. 397). For the following lemma, see [4], p. 180.

Lemma 6. *Let $0 \leqq \gamma < \beta < 5$, let $f(s)$ be an analytic function, real for real s, and regular for $\sigma \geqq \gamma$. Let*

$$|\mathrm{Re}\, f(5+it)| \geqq m > 0$$

and

$$|f(\sigma'+it')| \leqq M_{\sigma,t} \quad (\sigma' \geqq \sigma\,,\ |t'| \leqq t)\,.$$

Then if T is not the imaginary part of a zero of $f(s)$

$$|\arg f(\sigma+iT)| \leqq \frac{\pi}{\log\{(5-\gamma)/(5-\beta)\}}(\log M_{\gamma,T+5} + \log m^{-1}) + \frac{3\pi}{2}$$

for $5 \geqq \sigma \geqq \beta$ (with $\arg f(\sigma+iT)$ *being determined in the way indicated in Lemma 5).*

Turning to the proof of theorem 1, we apply lemma 6 with $f(s) = H(s)$. Note that $H(s)$ is real for real s, and that

$$|\mathrm{Re}\, H(5+it)| = |\mathrm{Re} \prod_{i=1}^{j} F_{D_i}(5+it)| = |\mathrm{Re}\,(1+O(X^{-3/2}))^j|$$
$$= 1 + O(X^{-\frac{1}{2}}) \geqq \tfrac{1}{2}$$

for X sufficiently large. Choose $\gamma = \alpha - 2\delta$, $\beta = \alpha - \delta$ to obtain

$$|\arg H(\sigma \mp iT)| \ll \delta^{-1} (\log M_{\alpha-2\delta, T+5} + 1) \tag{5.1}$$

for $\sigma \geqq \alpha - \delta$.

Further, by (3.2) we have for $\sigma \geqq \alpha - 2\delta$, $|t| \leqq T + 5$

$$\log |H(\sigma+it)| = \sum_{i=1}^{j} \log |F_{D_i}(\sigma+it)|$$
$$\leqq \sum_{i=1}^{j} \log (1 + |f_{D_i}(\sigma+it)| + |O(X^{-3/4})|)$$
$$\ll X^{1/4} + \sum_{i=1}^{j} |f_{D_i}(\sigma+it)|.$$

Using this in (5.1) and in lemma 5, we obtain

$$J \ll \delta^{-2} X^{1/4} T + T\delta^{-2} \sup_{\sigma \geqq \alpha - 2\delta} \ \sup_{|t| \leqq T+5} \ \sum_{i=1}^{j} |f_{D_i}(\sigma+it)|.$$

The first term can be neglected, and using lemma 4 we then have

$$J \ll T X^{7/4 - 3\alpha/2}\, T^{2-2\alpha} J^{\alpha - \frac{1}{2}} \log^{34}(XT).$$

This yields theorem 1.

6. A density conjecture. In conclusion, we state the following

Conjecture. *Uniformly for* $\frac{1}{2} \leqq \alpha \leqq 1$, $T \geqq 2$, $X \geqq 3$

$$N_r'(\alpha, T, X) \ll X^{3/2 - \alpha + \varepsilon}\, T^2,$$

the constant in $\ll$ *depending on* ε *only.*

A proof of this conjecture would require a satisfactory estimate for the sum

$$\sum_{|D| \leqq X}{}' |L(\tfrac{1}{2} + it, \chi_D)\, M_D(\tfrac{1}{2} + it)|^2.$$

For example, the validity of the generalized Lindelöf hypothesis

$$L(\tfrac{1}{2} + it, \chi_D) \ll ((|t| + 1)\, |D|)^{\varepsilon}$$

would suffice.

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