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MODULUS AND CAPACITY INEQUALITIES FOR QUASIREGULAR MAPPINGS

BY

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1. Introduction

One of the most important tools in the theory of quasiconformal mappings is the double inequality

$$\frac{1}{K} M(\Gamma) \le M(f\Gamma) \le KM(\Gamma) , \tag{1.1}$$

valid for every K-quasiconformal mapping $f: G \to G'$ and for every path family Γ in G. The left hand inequality is not always true for K-quasiregular mappings. (For terminology, see [4].) However, by a result of Poleckii [7, Theorem 1], the right hand inequality also holds in this more general case. A related result for condensers was given in [4, 7.1]. Poleckii [7, Theorem 2] also proved that the stronger inequality

$$M(\Gamma') \le \frac{K}{m} M(\Gamma) \tag{1.2}$$

is true in the following case: D is a normal domain of f, $m = N(f, D)$ is the multiplicity of f in D, Γ' is a family of injective paths in fD, and Γ is the family of all paths γ in D such that $f \circ \gamma \in \Gamma'$. A related result for condensers was given by Martio [3, 5.1 and 5.13]. Martio and Poleckii applied their inequalities to study the local behavior of a quasiregular mapping.

The main purpose of this paper is to establish (1.2) in a more general situation. In particular, the paths of Γ need not lie in a compact part of G. We also give the corresponding result for condensers. The inequality is applied to study the behavior of a mapping at an isolated singularity.

Our terminology and notation is the same as in [4]. In particular, the notation $f: G \to R^n$ includes the assumption that G is a domain in R^n and that f is continuous.

2. Preliminary results

2.1. **Lemma.** *Suppose that $f: G \to R^n$ is discrete and open, that $E \subset G$ is compact, and that $y_0 \in fE \setminus f(B_f \cap E)$. Then there is a neighborhood V_0*

of y_0 *such that for every connected neighborhood* $V \subset V_0$ *of* y_0, *the following conditions are satisfied:*

(1) $V \cap f(B_f \cap E) = \emptyset$.

(2) *The components of* $f^{-1}V$ *which meet* E *form a finite collection* $D_1, \ldots, D_k$.

(3) f *defines homeomorphisms* $f_i : D_i \to V$.

Proof. This is essentially the same as [4, 7.15]. Thus let $U_1, \ldots, U_k$ be disjoint neighborhoods of the points in $E \cap f^{-1}(y_0)$ such that $\bar{U}_i \subset G$ and such that $f|\bar{U}_i$ is injective. Then

$$V_0 = (\bigcap_{i=1}^{k} fU_i) \setminus f(E \setminus \bigcup_{i=1}^{k} U_i)$$

is the required neighborhood of y_0.

2.2. We next consider the parametrization of a path $\alpha : I \to G$ by means of the arc length of its image $f \circ \alpha$ under a mapping $f : G \to R^n$. The interval I may be closed, half open or open. We shall use for paths the notation and terminology of [9, pp. 1—8]. Thus $l(\alpha)$ is the length of a rectifiable closed path $\alpha : [a, b] \to R^n$, $s_\alpha : [a, b] \to [0, l(\alpha)]$ is the length function of α, and $\alpha^0 : [0, l(\alpha)] \to R^n$ is the normal representation of α, satisfying $\alpha^0 \circ s_\alpha = \alpha$. If α is the restriction of a path β to a subinterval, we say that α is a subpath of β and write $\alpha \subset \beta$.

2.3. **Lemma.** *Suppose that* $f : G \to R^n$ *is a light mapping. Suppose also that* $\beta : [a, b] \to R^n$ *is a rectifiable closed path and that* $\alpha : I \to G$ *is a path such that* $f \circ \alpha \subset \beta$. *Then there is a unique path* $\alpha^* : s_\beta I \to G$ *such that* $\alpha = \alpha^* \circ (s_\beta | I)$. *Moreover,* $f \circ \alpha^* \subset \beta^0$.

Proof. The function $s_\beta : [a, b] \to [0, l(\beta)]$ is continuous and increasing. If $s_\beta(t_1) = s_\beta(t_2)$ for $t_1, t_2 \in I$, then β is constant on $[t_1, t_2]$. Since $f \circ \alpha \subset \beta$ and since f is light, also α is constant on $[t_1, t_2]$. Hence there is a unique mapping α^* of $I^* = s_\beta I$ into G such that $\alpha = \alpha^* \circ (s_\beta | I)$. The continuity of α^* follows easily from the continuity of α. If $t \in I$, then $f(\alpha^*(s_\beta(t))) = f(\alpha(t)) = \beta(t) = \beta^0(s_\beta(t))$. Hence $f \circ \alpha^* = \beta^0 | I^*$.

2.4. *Definition.* Suppose that $f : G \to R^n$ is a light mapping and that $\alpha : I \to G$ is a closed path. We say that f is *absolutely precontinuous* on α if $\beta = f \circ \alpha$ is rectifiable and if the path $\alpha^* : [0, l(\beta)] \to G$, given by 2.3, is absolutely continuous. If the path α is open or half open, we say that f is absolutely precontinuous on α if it is absolutely precontinuous on every closed subpath of α.

2.5. *Remarks.* (1) If f is a homeomorphism onto a domain G', then f is absolutely precontinuous on α if and only if f^{-1} is (locally) absolutely continuous on $f \circ \alpha$.

(2) We shall make use of the following elementary observation: In the situation of Lemma 2.3, the path α^* depends on the path β, but only up to a translation of the parameter. More precisely, if $f \circ \alpha \subset \beta_1$ and $f \circ \alpha \subset \beta_2$, and if α_1^*, α_2^* are the corresponding paths given by 2.3, then $\alpha_1^*(t) = \alpha_2^*(t + t_0)$ for some constant t_0. In particular, α^* is (locally) absolutely continuous if and only if f is absolutely precontinuous on α.

2.6. **Lemma.** *Suppose that $f : G \to R^n$ is quasiregular. Let Γ_0 be the family of all paths β in R^n such that either β is non-rectifiable or there is a path α in G such that $f \circ \alpha \subset \beta$ and f is not absolutely precontinuous on α. Then $M(\Gamma_0) = 0$.*

Proof. This lemma is a slight extension of an important result of Poleckii [7, Lemma 6]. The topological part of Poleckii's proof has been simplified by Rickman [8]. Choose an exhaustion (G_i) of G. This means that $G_1, G_2, \ldots$ is a sequence of domains such that $\bar{G}_i \subset G_{i+1}$ and $G = \bigcup \{G_i \mid i \in N\}$. Let Γ_i be the family of all closed paths in G_i on which f is not absolutely precontinuous. By the aforementioned result of Poleckii, $M(f\Gamma_i) = 0$. Furthermore, the family Γ_{non} of all non-rectifiable paths is of modulus zero. Since Γ_0 is minorized by the union of Γ_{non} and all $f\Gamma_i$, $M(\Gamma_0) = 0$.

3. Modulus and capacity inequalities

3.1. **Theorem.** *Suppose that $f : G \to R^n$ is a non-constant quasiregular mapping, that Γ is a path family in G, that Γ' is a path family in R^n, and that m is a positive integer such that the following condition is satisfied:*

There is a set $E_0 \subset G$ of measure zero such that for every path $\beta : I \to R^n$ in Γ' there are paths $\alpha_1, \ldots, \alpha_m$ in Γ such that $f \circ \alpha_i \subset \beta$ for all i and such that for every $x \in G \setminus E_0$ and $t \in I$, $\alpha_i(t) = x$ for at most one i.

Then

$$M(\Gamma') \leq \frac{K_I(f)}{m} M(\Gamma). \tag{3.2}$$

3.3. *Remarks.* It is not required that $\Gamma' = f\Gamma$. We shall later apply the theorem with $E_0 = B_f$. If $\Gamma' = f\Gamma$, the condition is trivially true for $m = 1$, and we obtain Poleckii's inequality $M(f\Gamma) \leq K_I(f) M(\Gamma)$. If D is a normal domain of f, if Γ' is a family of paths in fD, and if Γ is the family of all paths α in D such that $f \circ \alpha \in \Gamma'$, then the condition is satisfied for $m = N(f, D)$ by Rickman [8]. Hence we obtain Poleckii's second inequality (1.2).

3.4. *Proof of Theorem* 3.1. Let Γ_0 be the family of Lemma 2.6. Setting $\Gamma_1 = \Gamma' \setminus \Gamma_0$ we have $M(\Gamma_1) = M(\Gamma')$. Hence it suffices to prove that

$$M(\Gamma_1) \leq \frac{K_I(f)}{m} M(\Gamma) .$$

We may assume that E_0 is a Borel set. By [4, 8.2], we may also assume that at all points $x \in G \setminus E_0$, f is differentiable and $J(x, f) > 0$. Thus $B_f \subset E_0$. Let $\varrho \in F(\Gamma)$. Define $\sigma : G \to \dot{R}^1$ by

$$\sigma(x) = \varrho(x)/l(f'(x)) \quad \text{for} \quad x \in G \setminus E_0 .$$
$$\sigma(x) = \infty \quad \text{for} \quad x \in E_0 .$$

Then σ is a Borel function. Next define $\varrho' : R^n \to \dot{R}^1$ by

$$\varrho'(y) = m^{-1} \sup_B \sum_{x \in B} \sigma(x) ,$$

where B runs through all subsets of $f^{-1}(y)$ such that $\operatorname{card} B \leq m$. For $y \in \complement fG$ put $\varrho'(y) = 0$. Then $\varrho'(y) = \infty$ for $y \in fE_0$.

We shall prove that $\varrho' \in F(\Gamma_1)$. To show that ϱ' is a Borel function, we choose an exhaustion of G with domains G_i, $\bar{G}_i \subset G_{i+1}$. Denoting by χ_A the characteristic function of a set A, we set

$$\varrho_i = \varrho \chi_{\bar{G}_i} , \ \sigma_i = \sigma \chi_{\bar{G}_i} , \ \varrho_i'(y) = m^{-1} \sup_B \sum_{x \in B} \sigma_i(x) . \tag{3.5}$$

As $i \to \infty$, $\varrho_i(x) \to \varrho(x)$ and $\varrho_i'(y) \to \varrho'(y)$ for all $x \in G$ and $y \in R^n$. Furthermore, $\varrho_i'(y) = 0$ for $y \in \complement f\bar{G}_i$, and $\varrho_i'(y) = \infty$ for $y \in f(\bar{G}_i \cap B_f)$. Hence it suffices to show that every $y_0 \in f\bar{G}_i \setminus f(\bar{G}_i \cap B_f)$ has a neighborhood in which f is a Borel function. Apply Lemma 2.1 with $E = \bar{G}_i$, and let V be the corresponding neighborhood of y_0. Setting $g_j = f_j^{-1}$, we have

$$\varrho_i'(y) = m^{-1} \sup_J \sum_{j \in J} \sigma_i(g_j(y))$$

for $y \in V$, where J runs through all subsets of $\{1, \ldots, k\}$ such that $\operatorname{card} J \leq m$. Since every $\sigma_i \circ g_j$ is a Borel function, $\varrho_i' | V$ is a Borel function.

Next let β be a member of Γ_1. We must show that

$$\int_\beta \varrho' ds \geq 1 . \tag{3.6}$$

Assume first that $\beta : [a, b] \to R^n$ is a closed path. By the hypothesis, there are paths $\alpha_1, \ldots, \alpha_m$ in Γ such that $f \circ \alpha_i \subset \beta$ and such that $\operatorname{card} \{i | \alpha_i(t) = x\} \leq 1$ for all $x \in G \setminus E_0$ and $t \in [a, b]$. Set $c = l(\beta)$,

$\gamma = \beta^0$, and let $\gamma_i : I_i \to G$ be the path α^* given by Lemma 2.3 for $\alpha = \alpha_i$. Thus $\alpha_i(t) = \gamma_i(s_\beta(t))$ and $f \circ \gamma_i \subset \gamma$.

For almost every $t \in [0, c]$ we have $|\gamma'(t)| = 1$ by [9, 1.3.(5)]. Since f is absolutely precontinuous on each α_i, the paths γ_i are (locally) absolutely continuous (Remark 2.5.(2)). Hence the derivative $\gamma_i'(t)$ exists a.e. in I_i. It follows that for almost every $t \in I_i$, either $\gamma_i(t) \in E_0$ or

$$1 = |\gamma'(t)| = |f'(\gamma_i(t))\gamma_i'(t)| \geq l(f'(\gamma_i(t)))|\gamma_i'(t)| .$$

Since $\sigma(x) = \infty$ for $x \in E_0$, the inequality $\sigma(\gamma_i(t)) \geq \varrho(\gamma_i(t))\, |\gamma_i'(t)|$ holds a.e. in I_i. Consequently (cf. [9, 4.1]),

$$1 \leq \int_{\alpha_i} \varrho\, ds = \int_{\gamma_i} \varrho\, ds = \int_{I_i} \varrho(\gamma_i(t))\, |\gamma_i'(t)| dt \leq \int_{I_i} \sigma(\gamma_i(t))\, dt ,$$

$1 \leq i \leq m$. Set $h_i(t) = \sigma(\gamma_i(t))\, \chi_{I_i}(t)$ for $t \in [0, c]$, and let $J(t) = \{i \mid t \in I_i\}$. For every $t \in [0, c]$, either $\gamma(t) \in fE_0$, in which case $\varrho'(\gamma(t)) = \infty$, or the points $\gamma_i(t)$, $i \in J(t)$, are distinct points in $f^{-1}(\gamma(t))$. In both cases we have

$$\varrho'(\gamma(t)) \geq m^{-1} \sum_{i=1}^{m} h_i(t) ,$$

which implies

$$\int_\beta \varrho' ds = \int_0^c \varrho'(\gamma(t))\, dt \geq m^{-1} \sum_{i=1}^{m} \int_0^c h_i(t) dt = m^{-1} \sum_{i=1}^{m} \int_{I_i} \sigma(\gamma_i(t))\, dt \geq 1 .$$

If the path β is open or half open, we can apply the above proof to the closed extension [9, 3.2] of β. We have proved that $\varrho' \in F(\Gamma_1)$. Consequently,

$$M(\Gamma_1) \leq \int \varrho'^n\, dm . \tag{3.7}$$

To estimate the above integral, we again choose an exhaustion (G_i) of G and introduce the functions ϱ_i, σ_i and ϱ_i' as in (3.5). Fix i, let $y_0 \in f\bar{G}_i \setminus f(\bar{G}_i \cap B_f)$, and let V be a connected neighborhood of y_0 satisfying the conditions of Lemma 2.1 for $E = \bar{G}_i$. We have thus k homeomorphisms $g_j : V \to D_j$, $f \circ g_j = \mathrm{id}$, and $\bar{G}_i \cap f^{-1}V = \bigcup \{\bar{G}_i \cap D_j \mid 1 \leq j \leq k\}$. Put $J_0 = \{1, \ldots, k\}$, and define for each $y \in V$ a set $J(y) \subset J_0$ as follows: If $k \leq m$, then $J(y) = J_0$. If $k > m$, then $\operatorname{card} J(y) = m$, and for all $j \in J(y)$, $j' \in J_0 \setminus J(y)$, either $\sigma_i(g_j(y)) > \sigma_i(g_{j'}(y))$ or $\sigma_i(g_j(y)) = \sigma_i(g_{j'}(y))$ and $j > j'$. Then

$$\varrho_i'(y) = m^{-1} \sum_{j \in J(y)} \sigma_i(g_j(y))$$

for $y \in V$. For $J \subset J_0$, the sets $V_J = \{y \in V \mid J(y) = J\}$ are disjoint Borel sets. Using Hölder's inequelity, a transformation formula for Lebesgue integrals, and the quasiconformality of $f \mid D_j$, we obtain

$$\begin{aligned}\int_{V_J} \varrho_i'^n \, dm &\leq m^{-1} \sum_{j \in J} \int_{V_J} \sigma_i(g_j(y))^n \, dm(y) \\ &= m^{-1} \sum_{j \in J} \int_{g_j V_J} \frac{\varrho_i(x)^n}{l(f'(x))^n} J(x, f) \, dm(x) \\ &\leq \frac{K_I(f)}{m} \int_{f^{-1} V_J} \varrho_i^n \, dm .\end{aligned}$$

Summing over all $J \subset J_0$ yields

$$\int_V \varrho_i'^n \, dm \leq \frac{K_I(f)}{m} \int_{f^{-1}V} \varrho_i^n \, dm .$$

Since $\varrho_i'(y) = 0$ for $y \in \complement f\bar{G}_i$ and since $f\bar{G}_i \setminus f(\bar{G}_i \cap B_f)$ can be almost covered by a countable number of disjoint sets V as above (for example, with cubes), and since $m(fB_f) = 0$, we obtain

$$\int \varrho_i'^n \, dm \leq \frac{K_I(f)}{m} \int \varrho_i^n \, dm .$$

As $i \to \infty$, this and (3.7) yield

$$M(\Gamma_1) \leq \frac{K_I(f)}{m} \int \varrho^n \, dm .$$

Since $\varrho \in F(\Gamma)$ was arbitrary, this proves the theorem.

3.8. *Examples.* Let us consider the complex analytic function $f : R^2 \to R^2$, $f(z) = e^z$. Let m be a positive integer, and let Q be the rectangle $0 < \operatorname{Re} z < 1$, $0 \leq \operatorname{Im} z < 2\pi m$. Then fQ is the annulus $B^2(e) \setminus \bar{B}^2$. Let Γ be the family of all horizontal segments of line joining the vertical sides of Q. It is easy to see that the condition of Theorem 3.1 is satisfied for Γ and $\Gamma' = f\Gamma$. Hence $M(\Gamma') \leq M(\Gamma)/m$. On the other hand, it is well known that $M(\Gamma') = 2\pi = M(\Gamma)/m$. Hence the inequality (3.2) is sharp in this case. This also follows from the inequality [4, 3.2] $M(\Gamma) \leq N(f, Q)\, K_O(f)\, M(f\Gamma)$.

Next let Γ be the family of all vertical segments joining the horizontal sides of Q. Now we have $M(f\Gamma) = 1/2\pi m^2 = M(\Gamma)/m$. Hence (3.2) is also true in this case. However, the condition of Theorem 3.1 is not satisfied. We shall give a result which applies to situations like this.

If $\alpha : [a, b] \to G$ is a closed path, we say that f *winds* α m *times around itself* if $f \circ \alpha = \beta$ is rectifiable and if the following condition is satisfied: Let $\beta^0 : [0, c] \to R^n$ be the normal representation of β, let $\alpha^* : [0, c] \to G$ be the path given by 2.3, and let $h = c/m$. Then $\beta^0(t+jh) = \beta^0(t)$ and $\alpha^*(t+jh) \neq \alpha^*(t)$ whenever $0 \leq t < t + jh < c$ and $j \in \{1, \ldots, m-1\}$.

3.9. **Theorem.** *Suppose that* $f : G \to R^n$ *is a non-constant quasiregular mapping, that* Γ *is a path family in* G, *that* m *is a positive integer, and that* f *winds every path of* Γ m *times around itself. Then*

$$M(f\Gamma) \leq \frac{K_I(f)}{m} M(\Gamma) .$$

Proof. The proof is closely similar to the proof of Theorem 3.1. The only difference is the proof of the inequality

$$\int_\beta \varrho' \, ds \geq 1 \tag{3.6}$$

for $\beta = f \circ \alpha \in f\Gamma$. Now

$$\int_\beta \varrho' \, ds = m \int_0^h \varrho'(\beta^0(t)) \, dt .$$

If $0 < t < h$, then $\alpha^*(t), \alpha^*(t+h), \ldots, \alpha^*(t+(m-1)h)$ are distinct points in $f^{-1}(\beta^0(t))$. Hence

$$\varrho'(\beta^0(t)) \geq m^{-1} \sum_{j=0}^{m-1} \sigma(\alpha^*(t+jh))$$

for $t \in (0, h)$. As in the proof of 3.1 we obtain $\sigma(\alpha^*(t)) \geq \varrho(\alpha^*(t)) \, |\alpha^{*\prime}(t)|$ for almost every $t \in [0, c]$. Consequently,

$$\int_\beta \varrho' \, ds \geq \sum_{j=0}^{m-1} \int_0^h \varrho(\alpha^*(t+jh)) \, |\alpha^{*\prime}(t)| \, dt = \int_\alpha \varrho \, ds \geq 1 .$$

This proves (3.6).

3.10. *Remark.* The situation of 3.9 arises in the theory of covering mappings. Suppose that f is a quasiregular covering mapping of G onto G' such that the fundamental group $\pi_1(G')$ is isomorphic to the group Z

of integers. Suppose that Δ is a path family in G' such that every member of Δ is a rectifiable loop which represents a generator of $\pi_1(G')$. Let m be a positive integer such that $m \leq N(f)$. For each $\gamma \in \Delta$, $\gamma: [a, b] \to G'$, choose a point $x \in f^{-1}(\gamma(a))$, and let α be the path in G obtained by performing m successive liftings of γ, the first one starting at x. Then f winds α m times around itself. To see this, we may assume that $\gamma : [0, h] \to G'$ is a normal representation. Then $\alpha : [0, mh] \to G$ is a path with the property $f(\alpha(t + jh)) = \gamma(t)$ for $0 \leq t \leq h$ and $1 \leq j \leq m-1$. If $\alpha(t+jh) = \alpha(t)$ for some t and $j \leq m-1$, the path $\alpha_1 = \alpha|[t, t+jh]$ is a loop such that $f \circ \alpha_1$ represents an element q in $\pi_1(G') = Z$ such that $|q| = j$. However, this is impossible, because the induced homomorphism f_* maps $\pi_1(G)$ onto $N(f)Z$ if $N(f) < \infty$, and onto $\{0\}$ if $N(f) = \infty$ [2, 15.4, p. 88].

Let Γ be the family of all liftings α. By 3.9, we have

$$M(f\Gamma) \leq \frac{K_I(f)}{m} M(\Gamma) .$$

This inequality can be written in another form. In fact, a function ϱ belongs to $F(f\Gamma)$ if and only if $m\varrho \in F(\Delta)$. Hence $M(\Delta) = m^n M(f\Gamma)$, which yields

$$M(\Delta) \leq m^{n-1} K_I(f) M(\Gamma) .$$

3.11. *Maximal liftings.* We shall need some results concerning path lifting for discrete open mappings. If $f: G \to R^n$ is a mapping, if $\beta: [a,b) \to R^n$ is a path and if $x_0 \in f^{-1}(\beta(a))$, we say that a path $\alpha : [a, c) \to G$ is a *maximal f-lifting* of β starting at x_0 if $\alpha(a) = x_0$, $f \circ \alpha \subset \beta$ and there does not exist a path $\alpha_1 : [a, c_1) \to G$ such that $\alpha \subset \alpha_1$ and $f \circ \alpha_1 \subset \beta$. See [6, p. 12]. The following result is from Rickman [8]:

3.12. **Lemma.** *Suppose that* $f: G \to R^n$ *is discrete and open, that* $\beta: [a, b) \to R^n$ *is a path and that* $x_1, \ldots, x_k$ *are points in* $f^{-1}(\beta(a))$. *Set* $m = \sum_{j=1}^{k} i(x_j, f)$. *Then there are maximal f-liftings* $\alpha_1, \ldots, \alpha_m$ *of* β *such that*

(1) $\operatorname{card}\{j \mid \alpha_j(a) = x_i\} = i(x_i, f)$ *for* $1 \leq i \leq k$.
(2) $\operatorname{card}\{j \mid \alpha_j(t) = x\} \leq i(x, f)$ *for all* $x \in G$ *and* $t \in [a, b)$.

3.13. *Condensers.* A condenser in R^n is a pair $E = (A, C)$ where $A \subset R^n$ is open and $C \subset A$ is compact. See [4, p. 24]. The capacity of a condenser $E = (A, C)$ is the number

$$\operatorname{cap} E = \inf_u \int_A |\nabla u|^n \, dm ,$$

where u runs through all C^∞-functions with compact support in A such that $u(x) \geq 1$ for $x \in C$. An alternative way to define the capacity of E is the equality

$$\operatorname{cap} E = M(\Gamma_E)\,, \tag{3.14}$$

where Γ_E is the family of all paths joining C and ∂A in A. This was proved by Ziemer [10, 3.8] for bounded condensers, and the general case can be established by a simple limiting process. For our purposes, it is most convenient to let Γ_E be the family of all half open paths $\gamma : [a\,, b) \to A$ such that $\gamma(a) \in C$ and $\gamma(t) \to \partial A$ as $t \to b$, cf. [9, 11.3]. If $f : G \to R^n$ is an open mapping and if $E = (A\,, C)$ is a condenser in G, then $fE = (fA\,, fC)$ is also a condenser. In [4, 7.1] it was proved that

$$\operatorname{cap} fE \leq K_I(f) \operatorname{cap} E \tag{3.15}$$

for non-constant quasiregular mappings f. Martio [3, 5.1] proved the inequality

$$\operatorname{cap} fE \leq \frac{K_I(f)\, N(f\,, A)^{n-1}}{M(f\,, C)^n} \operatorname{cap} E\,. \tag{3.16}$$

Here $M(f\,, C)$ is the minimal multiplicity of f on C, defined by

$$M(f\,, C) = \inf_{y \in fC} \sum_{x \in f^{-1}(y) \cap C} i(x\,, f)\,.$$

Since $1 \leq M(f\,, C) \leq N(f\,, A)$ by [3, 3.6], the inequalities (3.15) and (3.16) are consequences of the following result:

3.17. **Theorem.** *Suppose that* $f : G \to R^n$ *is a non-constant quasiregular mapping and that* $E = (A\,, C)$ *is a condenser in* G. *Then*

$$\operatorname{cap} fE \leq \frac{K_I(f)}{M(f\,, C)} \operatorname{cap} E\,.$$

Proof. Set $\Gamma = \Gamma_E$, $\Gamma' = \Gamma_{fE}$, and $m = M(f\,, C)$. Let $\beta : [a\,, b) \to fA$ be a path in Γ'. Then $C \cap f^{-1}(\beta(a))$ contains points $x_1\,, \ldots, x_k$ such that $\sum \{i(x_j\,, f) \mid 1 \leq j \leq k\} \geq m$. By 3.12, there are maximal $(f \mid A)$-liftings $\alpha_j : [a\,, c_j) \to G$ of β, $1 \leq j \leq m$, such that $\alpha_j(a) = x_i$ for some i and such that $\operatorname{card}\{j \mid \alpha_j(t) = x\} \leq 1$ for $x \in G \setminus B_f$ and $t \in [a\,, b)$. Furthermore, it follows from [6, 3.12] that $\alpha_j(t) \to \partial A$ as $t \to c_j$. Hence $\alpha_j \in \Gamma$ for all j. The theorem follows from 3.1 and (3.14).

3.18. *Remark.* We sketch a proof of 3.17, which does not make use of path families. Let S be the set of all pairs $(x\,, k)$ such that $x \in A$ and $1 \leq k \leq i(x\,, f)$, $k \in N$. Let $P : S \to A$ be the projection $P(x\,, k) = x$.

Using the notation of [4, p. 25], we define for every $u \in W_0^\infty(E)$ a function $v : fA \to R^1$ by

$$v(y) = M(f, C)^{-1} \max_B \sum_{z \in B} u(P(z)) ,$$

where B runs through all subsets of $P^{-1}(f^{-1}(y))$ such that $\operatorname{card} B \leq M(f, C)$. Modifying the proofs of [4, 7.1] and [3, 5.1], we can show that $v \in W_0(fE)$ and that

$$\int |\nabla v|^n \, dm \leq \frac{K_I(f)}{M(f, C)} \int |\nabla u|^n \, dm .$$

3.19. *Remark.* All the results of this section can be extended without difficulties to quasimeromorphic mappings $f : G \to \bar{R}^n$ [5, 2.1].

4. Applications

4.1. Let $f : G \to R^n$ be a quasiregular mapping. An isolated boundary point b is said to be an *isolated singularity* of f. Furthermore, b is a removable singularity, a pole, or an essential singularity according as f has a finite limit, an infinite limit, or no limit at b. See [5, p. 12].

4.2. **Theorem.** *Suppose that b is an isolated singularity of a quasiregular mapping $f : G \to R^n$. Suppose also that there are finite positive constants C, p, δ such that*

$$|f(x)| \leq C|x|^{-p}$$

for $0 < |x - b| < \delta$. Then b is not an essential singularity of f.

Proof. Assume that b is an essential singularity. Performing a preliminary similarity transformation, we may assume that $b = 0$, that $\bar{B}^n \subset G \cup \{0\}$, and that $C = \delta = 1$. Choose $R > 0$ such that $fS^{n-1} \subset B^n(R)$. From [5, 4.6] it follows that there is y_0 in R^n such that $|y_0| > R$ and $N(y_0, f, B^n) = \infty$. Set $K = K_I(f)$, choose a positive integer m such that $m > 2Kp^{n-1}$, and choose distinct points $x_1, \ldots, x_m$ in $B^n \cap f^{-1}(y_0)$. Applying [4, 2.9], we next choose $r > 0$ such that $U_i = U(x_i, f, r)$ is a normal neighborhood of x_i for $1 \leq i \leq m$, the closures $\bar{U}_i$ are disjoint, and $\bar{U}_i \subset B^n$. Set $d = d(0, \bar{U}_1 \cup \ldots \cup \bar{U}_m)$, and let $a \in (0, d)$. Let V be the ring $B^n \setminus \bar{B}^n(a)$. Since f is open, $\partial fV \subset f\partial V = fS^{n-1} \cup fS^{n-1}(a) \subset B^n(R) \cup \bar{B}^n(a^{-p})$. Since $y_0 \in fV$, this implies $fV \subset B^n(a^{-p})$.

Consider the hemisphere $H = \{e \in S^{n-1} \mid (e \mid y_0) > 0\}$. Let Γ' be the family of all paths $\beta : [r, a^{-p}) \to R^n$, defined by $\beta(t) = y_0 + te$,

$e \in H$, and let Γ be the family of all maximal $(f|V)$-liftings of the members of Γ', starting at points of $\bar{U}_1 \cup \ldots \cup \bar{U}_m$. Then 3.1 and 3.12 imply

$$M(\Gamma') \leq \frac{K}{m} M(\Gamma) .$$

Suppose that $\alpha : [r , c) \to G$ is a member of Γ . Since $gV \subset B^n(a^{-p})$, $y_0 + a^{-p}e \in \complement fV$ for all $e \in H$. From [6, 3.12] it follows that $\alpha(t) \to \partial V$ as $t \to c$. This means that either $|\alpha(t)| \to 1$ or $|\alpha(t)| \to a$. The first case is impossible, because $\overline{|\beta|} \cap fS^{n-1} = \emptyset$ for all $\beta \in \Gamma'$. Hence $|\alpha(t)| \to a$ as $t \to c$. Hence Γ is minorized by the family Γ_1 of all paths joining the spheres $S^{n-1}(d)$ and $S^{n-1}(a)$. Consequently,

$$M(\Gamma) \leq M(\Gamma_1) = \omega_{n-1}\left(\log \frac{d}{a}\right)^{1-n} .$$

On the other hand, by [9, 7.7] we have

$$M(\Gamma') = \tfrac{1}{2}\omega_{n-1}\left(\log \frac{a^{-p}}{r}\right)^{1-n} .$$

Combining the above inequalities yields

$$a^q \geq r d^{(m/2K)^{1/(n-1)}} ,$$

where

$$q = \left(\frac{m}{2K}\right)^{1/(n-1)} - p > 0 .$$

As $a \to 0$, this gives a contradiction.

4.3. **Theorem.** *Suppose that b is an isolated singularity of a quasiregular mapping $f: G \to R^n$, and let $\alpha = K_I(f)^{1/(1-n)}$. If $\lim_{x \to b} |x - b|^\alpha |f(x)| = 0$, b is a removable singularity. The hypothesis cannot be replaced by the requirement that $|x - b|^\alpha |f(x)|$ be bounded in a neighborhood of b .*

Proof. We may assume that $b = 0$. By 4.2, b cannot be an essential singularity. Assume that b is a pole of f . Let g be a Möbius transformation of $\bar{R}^n$ such that $|g(x)| = 1/|x|$ for all $x \in \bar{R}^n$. Then $h = g \circ f$ is quasiregular in a neighborhood of 0, and $K_I(h) = K_I(f)$. From [5, 3.2] it follows that $|h(x)| \leq C|x|^\alpha$ in a neighborhood of 0 , where C is a constant. Hence $|x|^\alpha |f(x)| \geq 1/C$, which contradicts the hypothesis. Thus 0 is a removable singularity of f .

The mapping $f(x) = g(|x|^{\alpha-1}x)$, where g is as above, has a pole at the origin, $K_I(f) = \alpha^{1-n}$, and $|x|^\alpha |f(x)| = 1$ for all $x \in R^n \setminus \{0\}$.

4.4. *Remark.* In the special case $n = 2$, $K(f) = 1$, the theorems 4.2 and 4.3 are well known results for analytic functions [1, p. 124 and p. 128].

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