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ON A CONJECTURE OF MAHLER CONCERNING THE ALGEBRAIC INDEPENDENCE OF THE VALUES OF SOME *E*-FUNCTIONS

BY

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Preface

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Oulu, February, 1972.

Keijo Väänänen

Contents

INTRODUCTION

C. SIEGEL [7] considered the functions

$$K_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z^2/4)^n}{n!(\nu+1)\ldots(\nu+n)}, \quad \nu \neq -1, -2, \ldots,$$

satisfying the differential equation

$$w'' + \frac{2\nu+1}{z} w' + w = 0.$$

SIEGEL proved that if ν is a rational number not equal to half an odd integer and $\alpha \neq 0$ is algebraic, then $K_\nu(\alpha)$ and $K'_\nu(\alpha)$ are algebraically independent over the rational number field **Q**. He proved also some generalizations of this result. In his proof SIEGEL [7], [8] developed a method by which it is possible to establish the transcendence and algebraic independence of the values at algebraic points of certain entire functions, called E-functions (for the definition of E-functions see p. 38).

A. B. SHIDLOVSKI [4], [5] generalized SIEGEL's method and, in 1962, he [6] proved a general theorem which states that if $f_1(z), \ldots, f_m(z)$ are E-functions satisfying a system of linear differential equations with coefficients that are rational functions of z, regular at the algebraic point $\alpha \neq 0$, then the maximum number of function values $f_1(\alpha), \ldots, f_m(\alpha)$ that are algebraically independent over the rational number field, is equal to the maximum number of functions $f_1(z), \ldots, f_m(z)$ that are algebraically independent over the field of rational functions of z (see § 11). SHIDLOVSKI and his students have given many applications of this theorem (see [2]).

K. MAHLER [3] gave a further application of SHIDLOVSKI's theorem. He considered the functions

$$A_i(z) = \frac{1}{i!}\left(\frac{\partial}{\partial \nu}\right)^i K_\nu(z) \mid_{\nu=0}, \quad i = 0, 1, \ldots .$$

MAHLER proves that if $\alpha \neq 0$ is algebraic, then the elements of every one of the following four sets of six function values

$$\{A_0(\alpha), A'_0(\alpha), A_1(\alpha) \text{ or } A'_1(\alpha), A_2(\alpha), A'_2(\alpha), A_3(\alpha) \text{ or } A'_3(\alpha)\}$$

are algebraically independent over **Q**. He gives, too, some transcendental expressions involving Euler's constant γ and the constant $\zeta(3)$. The simplest involving γ is

$$\frac{\pi Y_0(2)}{2J_0(2)} - \gamma ,$$

where $J_0(z)$ and $Y_0(z)$ denote Bessel functions of the first and the second kinds of suffix 0.

In his paper MAHLER conjectures that these results could be generalized. The present paper is such a generalization.

In § 1 we consider some preliminary properties of the functions $A_i(z)$ and $B_i(z)$ (see (1)) which are solutions of the following differential equations,

$$w_0'' + \frac{1}{z} w_0' + w_0 = 0 ,$$

$$w_i'' + \frac{1}{z} w_i' + w_i + \frac{2}{z} w_{i-1}' = 0 , \quad i = 1, 2, \ldots .$$

Let $Q(n)$, $n = 0, 1, \ldots,$ denote the system of the first $2n+2$ equations of these equations, and let the functions $w_0(z), w_1(z), \ldots, w_{2n+1}(z)$ form a solution of $Q(n)$.

In § 2 we show that there are certain algebraic relations between the functions w_i and w_i', $i = 0, 1, \ldots, 2n+1$.

Next, in § 3, we present theorem 1 by which we can, under certain conditions, establish the algebraic independence of the $3n+3$ functions w_{2i}, w_{2i+1} and w_{2i}', $i = 0, 1, \ldots, n$, over the field of rational functions of z. This theorem is a generalization of MAHLER's theorem 1 ([3] p. 155). As a corollary we obtain the algebraic independence of the functions A_{2i}, A_{2i+1} and A_{2i}', $i = 0, 1, \ldots, n$.

For the proof of theorem 1 we construct, in § 4, certain functions $t_i(z)$, $i = 0, 1, \ldots, n$. By using the properties of these functions we can prove lemma 1 which states that the functions w_{2i}, w_{2i+1} and t_i, $i = 0, 1, \ldots, n$, are, under certain conditions, algebraically independent over the field of rational functions of z.

We split the proof of lemma 1, which is a fundamental step in the proof of theorem 1, into a number of separate steps. At first, in § 5, we consider some properties of the functions t_i needed in the proof. Then, in § 6, we give two lemmas by MAHLER and begin the proof of lemma 1. The proper proof is given in § 7, § 8 and § 9. It follows the same main features as the proof of MAHLER's theorem 1 in [3].

Then, in § 10, we prove theorem 1 and, at the same time, the corollary of theorem 1.

In § 11 we present SHIDLOVSKI's theorem. By this theorem we can then, in § 12, establish the algebraic independence of the elements of every one of 2^{n+1} sets of $3n+3$ function values

$$\{A_0(\alpha)\,,\,A_0'(\alpha)\,,\,A_1(\alpha)\quad\text{or}\quad A_1'(\alpha)\,,\,\ldots\,,\,A_{2i}(\alpha)\,,\,A_{2i}'(\alpha)\,,\,A_{2i+1}(\alpha)$$
$$\text{or}\quad A_{2i+1}'(\alpha)\,,\,\ldots\,,\,A_{2n}(\alpha)\,,\,A_{2n}'(\alpha)\,,\,A_{2n+1}(\alpha)\quad\text{or}\quad A_{2n+1}'(\alpha)\}\,,$$

where $\alpha \neq 0$ is algebraic. When $n = 1$, we have MAHLER's result (p. 7).

Finally, in § 13, we obtain some transcendental expressions involving Euler's constant γ and the values $\zeta(3)\,,\ldots,\ \zeta(2n+1)$, $n = 1, 2, \ldots,$ of Riemann Zeta function. These results generalize MAHLER's theorem 3 in [3].

§ 1. Preliminaries

Let $\mathbf{C}$ be the field of complex numbers and $\mathbf{C}(z)$ the field of rational functions of z over $\mathbf{C}$. The ring of entire functions of z we denote by $\mathbf{E}$.

Let z and ν be two complex variables. Differentiation with respect to these variables will be denoted by a dash and by the symbol $\partial/\partial\nu$, respectively.

In this paper we shall mainly consider the functions ([3] p. 150)

$$(1)\qquad A_i(z) = \frac{1}{i!}\left(\frac{\partial}{\partial\nu}\right)^i K_\nu(z)\,|_{\nu=0}\,,\ B_i(z) = \frac{1}{i!}\left(\frac{\partial}{\partial\nu}\right)^i L_\nu(z)\,|_{\nu=0}\,,$$
$$i = 0, 1, \ldots,$$

where

$$K_\nu(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z^2/4)^n}{n!\,(\nu+1)\,(\nu+2)\ldots(\nu+n)}$$

and

$$L_\nu(z) = \frac{1}{2\nu}\,(K_\nu(z) - z^{-2\nu}K_{-\nu}(z)\,)\,.$$

Between the functions A_i and B_i there are the relations (see [3] p. 151)

$$(2)\qquad B_i(z) = \frac{1}{2}\,(A_{i+1}(z) + (-1)^i \sum_{h=0}^{i+1} \frac{Z^h}{h!}\,A_{i-h+1}(z)\,)\,,\ i = 0, 1, \ldots,$$

where $Z = 2\,\log\,z$.

The function $K_\nu(z) \in \mathbf{E}$ and is a meromorphic function of ν. Further, if $J_\nu(z)$ is the Bessel function of the first kind, then we have an equation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} K_\nu(z) \, .$$

The functions $K_\nu(z)$ and, if ν is not an integer, $L_\nu(z)$ satisfy the linear differential equation ([3] p. 150)

$$w'' + \frac{2\nu+1}{z} w' + w = 0 \, . \tag{3}$$

Let now

$$w = w(z\,,\nu) = \sum_{i=0}^{\infty} w_i \nu \tag{4}$$

be any integral of (3) which, for sufficiently small $|\nu|$, can be expanded into a convergent power series of ν with coefficients w_i which are functions of z. Then the coefficients $w_i = w_i(z)$ satisfy the following infinite system of differential equations ([3] p. 151),

$$w_0'' + \frac{1}{z} w_0' + w_0 = 0 \, ,$$

$$w_i'' + \frac{1}{z} w_i' + w_i + \frac{2}{z} w_{i-1}' = 0, \quad i = 1, 2, \ldots \, .$$

In this paper we shall consider the finite subsystems

$$Q(n) : \begin{cases} w_0'' + \dfrac{1}{z} w_0' + w_0 = 0 \, , \\ w_i'' + \dfrac{1}{z} w_i' + w_i + \dfrac{2}{z} w_{i-1}' = 0 \, , \\ i = 1, 2, \ldots, 2n+1 \, , \end{cases}$$

$n = 0, 1, \ldots,$ of this infinite system. We present the solutions of $Q(n)$ as row vectors

$$\mathbf{W} = (w_0\,, w_1\,, \ldots\,, w_{2n+1}) \, .$$

Now, if $|\nu|$ is small enough, we have, by (1), the convergent series

$$K_\nu(z) = \sum_{i=0}^{\infty} A_i(z)\nu^i \, , \; L_\nu(z) = \sum_{i=0}^{\infty} B_i(z)\nu^i \, .$$

Therefore, from the series in powers of ν of the functions

$$K_\nu(z)\nu^i \, , \; L_\nu(z)\nu^i \, , \; i = 0, 1, \ldots, 2n+1 \, ,$$

we obtain immediately the following special solutions for $Q(n)$,

$$\mathbf{A}_i(z) = (\overbrace{0 , \ldots , 0}^{i \text{ zeros}} , A_0(z) , A_1(z) , \ldots , A_{2n+1-i}(z)) ,$$

$$\mathbf{B}_i(z) = (\overbrace{0 , \ldots , 0}^{i \text{ zeros}} , B_0(z) , B_1(z) , \ldots , B_{2n+1-i}(z)) ,$$

$$i = 0 , 1 , \ldots , 2n + 1 .$$

The Wronski determinant $W(A_0 , B_0)$ is ([3] p. 152)

$$W(A_0 , B_0) = A_0B_0' - A_0'B_0 = 1/z.$$

Therefore the vectors $\mathbf{A}_0$ and $\mathbf{B}_0$ are linearly independent over $\mathbf{C}$. From the triangular form of the two square matrices

$$\begin{pmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_{2n+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{B}_0 \\ \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_{2n+1} \end{pmatrix}$$

it follows that the vectors $\mathbf{A}_i$ and $\mathbf{B}_i$, $i = 0, 1, \ldots , 2n + 1$, are linearly independent. From this we can deduce that every solution $\mathbf{W}$ of $Q(n)$ has a unique representation

$$\mathbf{W} = \sum_{h=0}^{2n+1} (a_h \mathbf{A}_h + b_h \mathbf{B}_h) , \tag{5}$$

where the coefficients a_h and $b_h \in \mathbf{C}$.

§ 2. The algebraic dependence of the functions w_i and w_i'

Let $\mathbf{W} = (w_0, w_1, \ldots ,w_{2n+1})$ be any solution of $Q(n)$, such that $w_0 \not\equiv 0$. We prove now that every derivate w_{2j+1}', $j = 0, 1, \ldots ,n$, can be presented as a rational function of the functions $w_0, w_1, \ldots ,w_{2j+1}$ and $w_0', w_2', \ldots ,w_{2j}'$ with coefficients in $\mathbf{C}(z)$.

Let

$$L_j (\mathbf{W}) = z\sum_{i=0}^{2j+1} (- 1)^i w_i w_{2j+1-i}' + \sum_{i=0}^{2j} (- 1)^i w_i w_{2j-i} , j = 0, 1, \ldots, n. \tag{6}$$

By means of the equations $Q(n)$ we get the following equations,

$$\frac{d}{dz} L_j (\mathbf{W}) = \sum_{i=0}^{2j+1} (- 1)^i w_i w_{2j+1-i}' + z\sum_{i=0}^{2j} (- 1)^i \Big\{ w_i' w_{2j+1-i}' +$$

$$+ w_i \Big(- \frac{1}{z} w_{2j+1-i}' - w_{2j+1-i} - \frac{2}{z} w_{2j-i}' \Big) \Big\} - z \Big(w_{2j+1}' w_0' + w_{2j+1} \Big(- \frac{1}{z} w_0' - w_0 \Big) \Big)$$

$$+ \sum_{i=0}^{2j} (-1)^i (w_i' w_{2j-i} + w_i w_{2j-i}')$$

$$= z \sum_{i=0}^{2j+1} (-1)^i (w_i' w_{2j+1-i}' - w_i w_{2j+1-i}) - 2 \sum_{i=0}^{2j} (-1)^i w_i w_{2j-i}'$$

$$+ 2 \sum_{i=0}^{2j} (-1)^i w_i w_{2j-i}' = 0 .$$

Hence

$$\frac{d}{dz} L_j (\mathbf{W}) = 0 , \quad j = 0, 1, \ldots , n .$$

So there must be complex numbers

$$C_j = C_j(\mathbf{W}) , \ j = 0, 1, \ldots , n,$$

which are independent of the variable z but depend on the vector $\mathbf{W}$, such that

$$L_j (\mathbf{W}) = C_j, \quad j = 0, 1, \ldots , n. \tag{7}$$

The equation $L_0 (\mathbf{W}) = C_0$ is due to BELOGRIVOV ([1] p. 56), and the equation $L_1 (\mathbf{W}) = C_1$ is in MAHLER's paper ([3] p. 153).

If $w_0 \not\equiv 0$, then it follows from (7) that we can express the functions w_{2j+1}', $j = 0, 1, \ldots ,n$, as rational functions of the functions w_{2i}, w_{2i+1}, w_{2i}', $i = 0, 1, \ldots ,j$, where the coefficients of these rational functions lie in $\mathbf{C}(z)$. Hence

$$w_{2j+1}' = D_{2j+1}(z , w_0 , w_1 , \ldots , w_{2j+1} , w_0' , w_2' , \ldots , w_{2j}') , \tag{8}$$

$$j = 0 , 1 , \ldots , n ,$$

where the functions D_{2j+1} are rational functions of their variables.

Further, we show that the constants $C_j = C_j (\mathbf{W})$ can be expressed by a_h and b_h, the coefficients of (5).

By (5) we have

$$w_h = \sum_{i=0}^{h} (a_i A_{h-i} + b_i B_{h-i}) , \ h = 0 , 1 , \ldots , 2n + 1 . \tag{9}$$

The functions A_h and zA_h', $h = 0, 1, \ldots ,2n + 1$, are elements of $\mathbf{E}$. The functions B_h and zB_h', $h = 0, 1, \ldots 2n + 1$, are, by (2), polynomials in $\log z$ with coefficients in $\mathbf{E}$. Therefore, it is obvious that also $L_j (\mathbf{W}), j = 0, 1, \ldots ,n$, are polynomials in $\log z$ with coefficients in $\mathbf{E}$.

If now P is any polynomial in $\log z$ with coefficients in $\mathbf{E}$, we denote by $[P]$ that term of P which has no factor $\log z$. Then it follows, by (7), that the equations

$$L_j(\mathbf{W}) = [L_j(\mathbf{W})]\,, j = 0\,,1\,,\ldots,n\,,$$

must hold. The left-hand sides of these equations are independent of z. Therefore, we can obtain the values of $L_j(\mathbf{W})$ from the equations

(10) $$L_j(\mathbf{W}) = [L_j(\mathbf{W})]_{z=0}\,,\; j = 0\,,1\,,\ldots,n\,.$$

Let us determine now

$$[w_h]_{z=0}\,,\; [zw'_h]_{z=0}\,,\; h = 0\,,1\,,\ldots,2n+1\,.$$

It is obvious from the definition (1) of $A_i(z)$, that

$$A_0(0) = 1\,, A_1(0) = A_2(0) = \ldots = A_{2n+1}(0) = 0\,.$$

Further, by (2), it follows that

$$[B_i(z)] = \frac{1}{2}\,(A_{i+1}(z) + (-1)^i A_{i+1}(z))\,,\; i = 0\,,1\,,\ldots,2n+1\,,$$

and therefore

$$[B_i(z)]_{z=0} = 0\,,\; i = 0\,,1\,,\ldots,2n+1\,.$$

From (2) and (9) it follows that

$$[w_h] = \sum_{i=0}^{h} (a_i A_{h-i} + b_i[B_{h-i}])\,,$$

$$[zw'_h] = z\sum_{i=0}^{h} (a_i A'_{h-i} + \frac{1}{2}\, b_i\,(A'_{h-i+1} + (-1)^{h-i} A'_{h-i+1})) +$$

$$+ \sum_{i=0}^{h} (-1)^{h-i} b_i A_{h-i}\,,\quad h = 0\,,1\,,\ldots,2n+1\,.$$

Hence

(11) $$[w_h]_{z=0} = a_h\,,\quad [zw'_h]_{z=0} = b_h\,,\quad h = 0\,,1\,,\ldots,2n+1\,.$$

By (6),

$$[L_j(\mathbf{W})] = \sum_{i=0}^{2j+1} (-1)^i [w_i]\,[zw'_{2j+1-i}] + \sum_{i=0}^{2j} (-1)^i [w_i]\,[w_{2j-i}]\,,$$

$$j = 0\,,1\,,\ldots,n\,.$$

Therefore, by (7), (10) and (11), we get the following equations,

(12) $$C_j(\mathbf{W}) = L_j(\mathbf{W}) = \sum_{i=0}^{2j+1} (-1)^i a_i b_{2j+1-i} + \sum_{i=0}^{2j} (-1)^i a_i a_{2j-i}\,,$$

$$j = 0\,,1\,,\ldots,n\,.$$

For example, if $\mathbf{W} = \mathbf{A}_0$, we have

$$C_0(\mathbf{A}_0) = 1\,, \; C_1(\mathbf{A}_0) = \ldots = C_n(\mathbf{A}_0) = 0\,. \tag{13}$$

§ 3. Theorem 1

In the preceding section we found that if $\mathbf{W} = (w_0, w_1, \ldots, w_{2n+1})$ is a solution of $Q(n)$, such that $w_0 \not\equiv 0$, then the functions w'_{2j+1}, $j = 0, 1, \ldots, n$, can be presented as rational functions of the functions w_0, $w_1, \ldots, w_{2j+1}$ and $w'_0, w'_2, \ldots, w'_{2j}$ with coefficients in $\mathbf{C}(z)$. In theorem 1 we prove that these later functions are, under certain conditions, algebraically independent over $\mathbf{C}(z)$. This is essential for the later applications, because we need this property in Shidlovski's theorem (see § 11).

Theorem 1. *Let* $\mathbf{W} = (w_0, w_1, \ldots, w_{2n+1})$ *be any solution of* $Q(n)$ *such that* $L_0(\mathbf{W}) \neq 0$ *and, if* $n \geq 2$, $L_1(\mathbf{W}) = \ldots = L_{n-1}(\mathbf{W}) = 0$. *Then the* $3n + 3$ *functions*

$$w_{2i}\,, w_{2i+1}\,, w'_{2i}\,, \quad i = 0\,, 1\,, \ldots, n\,, \tag{14}$$

are algebraically independent over $\mathbf{C}(z)$.

From theorem 1 follows, by (13), the following corollary.

Corollary. *The functions*

$$A_{2i}\,, A_{2i+1}\,, A'_{2i}\,, \quad i = 0\,, 1\,, \ldots, n \; (n = 0\,, 1\,, \ldots)\,, \tag{15}$$

are algebraically independent over $\mathbf{C}(z)$.

Theorem 1 is a generalization of the following theorem by Mahler ([3] p. 155).

Let $\mathbf{W} = (w_0, w_1, w_2, w_3)$ *be any solution of* $Q(1)$ *such that* $L_0(\mathbf{W}) \neq 0$. *Then the six functions*

$$w_0\,, w_1\,, w_2\,, w_3\,, w'_0\,, w'_2$$

are algebraically independent over $\mathbf{C}(z)$.

In his paper [3] Mahler conjectures that this kind of generalization is possible.

In the following sections we prove lemma 1 by which the proof of theorem 1 can be established.

§ 4. A lemma on algebraic independence

Let $\mathbf{W} = (w_0, w_1, \ldots, w_{2n+1})$ be again a solution of $Q(n)$. We define the functions $t_j = t_j(z)$, $j = 0, 1, \ldots, n$, as follows,

$$(16)\quad \begin{cases} t_0 = w_0' \,, t_1 = z(w_0 w_2' - w_2 w_0') + w_0 w_1 \,, \\ t_j = z\left(\sum_{i=0}^{j-1} (-1)^i w_i w_{2j-i}' - \sum_{i=j+1}^{2j} (-1)^i w_i w_{2j-i}'\right) \\ \quad + 2\sum_{i=0}^{j-2} (-1)^i w_i w_{2j-1-i} + (-1)^{j-1} w_{j-1} w_j \,, j = 2, 3, \ldots, n \,. \end{cases}$$

The following lemma, which deals with the algebraic independence of the functions w_{2i}, w_{2i+1} and t_i, $i = 0, 1, \ldots, n$, is a fundamental step in the proof of theorem 1.

Lemma 1. *Let* $\mathbf{W} = (w_0, w_1, \ldots, w_{2n+1})$ *be any solution of* $Q(n)$ *such that* $L_0(\mathbf{W}) \neq 0$ *and, if* $n \geqq 2$, $L_1(\mathbf{W}) = \ldots = L_{n-1}(\mathbf{W}) = 0$. *Then the functions*

$$(17)\qquad w_{2i} \,, w_{2i+1} \,, t_i \,, \; i = 0 \,, 1 \,, \ldots , n \,,$$

are algebraically independent over $\mathbf{C}(z)$.

Before starting the proof of this lemma we consider some properties of the functions $t_j, j = 0, 1, \ldots, n$, which will be needed in this proof.

§ 5. Some properties of the functions t_j

In the following we shall use the notation $w(k)$ to denote the functions $w_0, w_1, \ldots, w_k$ in this order, and, respectively, $w'(k)$ to denote the functions $w_0', w_2', \ldots, w_{2k}'$, and $t(k)$ to denote the functions $t_0, t_1, \ldots, t_k$. Generally, $x(k)$ denotes $x_0, x_1, \ldots, x_k$ in this order, and according to this notation we shall write

$$f(x_0 \,, x_1 \,, \ldots , x_k) = f(x(k)) \,.$$

Further, when we write that

$$f(x(k)) = 0$$

identically in $x(k)$, this means that

$$f(x_0 \,, x_1 \,, \ldots , x_k) = 0$$

identically in the indeterminates $x_0, x_1, \ldots, x_k$.

From the definition (16) of the functions t_j we get, by (8), the equations

$$t_0 = w_0' \,, t_1 = z(w_0 w_2' - w_2 w_0') + w_0 w_1 \,,$$

$$t_j = z(w_0 w_{2j}' - w_{2j} w_0') + F_j(z, w(2j-1), w'(j-1)), \; j = 2, 3, \ldots, n,$$

where the functions F_j are rational functions of their variables. From these equations we get the recursive formulae

$$(18)\quad \begin{cases} t_0 = w_0', \; t_1 = z(w_0 w_2' - w_2 t_0) + w_0 w_1, \\ t_j = z(w_0 w'_{2j} - w_{2j} t_0) + G_j(z, w(2j-1), t(j-1)), \; j = 2, 3, \ldots, n, \end{cases}$$

where the functions G_j are rational functions of their variables.

So we get, by (8) and (18), for the derivatives w_i' the following formulae

$$(19)\quad \begin{cases} w'_{2j} = S_{2j}(z, w(2j), t(j)), \\ w'_{2j+1} = S_{2j+1}(z, w(2j+1), t(j)), \quad j = 0, 1, \ldots, n, \end{cases}$$

where the functions S_i are rational functions of their variables.

Further, by $Q(n)$ and (18),

$$t_0' = -\frac{1}{z} w_0' - w_0, \; t_1' = w_0' w_1 - w_0 w_1',$$

$$t_j' = -2 w_0 w'_{2j-1} + \frac{d}{dz} G_j(z, w(2j-1), t(j-1)), \; j = 2, 3, \ldots, n.$$

Hence we get, by (7) and (19) the recursive formulae

$$(20)\quad \begin{cases} t_0' = -\frac{1}{z} t_0 - w_0, \; t_1' = \frac{1}{z}(w_0^2 - C_0), \\ t_j' = T_j(z, w(2j-1), t(j-1)), \quad j = 2, 3, \ldots, n, \end{cases}$$

where the functions T_j are rational functions of their variables.

If we choose $\mathbf{W} = \mathbf{A}_0$, we have, by the definition (16) of t_j,

$$t_0 = A_0', \; t_1 = z(A_0 A_2' - A_2 A_0') + A_0 A_1,$$

$$t_j = z\left(\sum_{i=0}^{j-1} (-1)^i A_i A'_{2j-i} - \sum_{i=j+1}^{2j} (-1)^i A_i A'_{2j-i}\right) +$$

$$+ 2\sum_{i=0}^{j-2} (-1)^i A_i A_{2j-1-i} + (-1)^{j-1} A_{j-1} A_j, \; j = 2, 3, \ldots, n.$$

We denote these functions by $t_j(\mathbf{A}_0), j = 0, 1, \ldots, n$.

Next we present lemma 2.

Lemma 2. *If* $\mathbf{W} = a_0 \mathbf{A}_0 + a_{2l} \mathbf{A}_{2l} + b_{2l+1} \mathbf{B}_{2l+1}, l > 0,$ *is a solution of* $Q(n)$, $0 \leqq j \leqq n$ *and* $2l + 1 > j$, *then*

$$t_0 = a_0 t_0(\mathbf{A}_0),$$

$$t_j = a_0^2 t_j(\mathbf{A}_0) + \delta_{j,2l} a_0 a_{2l} \{z(A_0 A_j' - A_j A_0') + A_0 A_{j-1}\}, \; j \geqq 1.$$

If $\mathbf{W} = a_0\mathbf{A}_0 + b_{2l}\mathbf{B}_{2l}$ *is a solution of* $Q(n)$, $0 \leqq j \leqq n$ *and* $2l > j$, *then*

$$t_0 = a_0 t_0(\mathbf{A}_0)\,,$$

$$t_j = a_0^2 t_j\,(\mathbf{A}_0) + \delta_{j,l} a_0 b_{2l}\,,\quad j \geqq 1\,.$$

(Here δ denotes Kronecker's δ-symbol.)

Proof. From (9) and (16) it follows immediately that

$$t_0 = a_0 t_0\,(\mathbf{A}_0)\,,\ t_j = a_0^2 t_j\,(\mathbf{A}_0)\,,\ 1 \leqq j < l\,,$$

for both solutions $\mathbf{W} = a_0\mathbf{A}_0 + a_{2l}\mathbf{A}_{2l} + b_{2l+1}\mathbf{B}_{2l+1}$ and $\mathbf{W} = a_0\mathbf{A}_0 + b_{2l}\mathbf{B}_{2l}$. Thus lemma 2 is true if $j < l$.

If $j = l \geqq 1$ and $\mathbf{W} = a_0\mathbf{A}_0 + a_{2l}\mathbf{A}_{2l} + b_{2l+1}\mathbf{B}_{2l+1}$, then we have, by (9) and (16), $t_j = a_0^2 t_j\,(\mathbf{A}_0)$. If $j = l \geqq 1$ and $\mathbf{W} = a_0\mathbf{A}_0 + b_{2l}\mathbf{B}_{2l}$, then it follows from (2), (9), (13) and (16) that

$$t_j = a_0^2 t_j\,(\mathbf{A}_0) + a_0 b_{2l} z (A_0 B_0' - B_0 A_0') = a_0^2 t_j\,(\mathbf{A}_0) + a_0 b_{2l}\,.$$

Next, let $j \leqq n$ satisfy $l < j < 2l$. Further, let δ_p, where p is a rational integer, denote

$$\delta_p = \begin{cases} 1\,, & \text{if } p \text{ is odd,} \\ 0\,, & \text{if } p \text{ is even.} \end{cases}$$

Then we obtain both vectors $\mathbf{W} = a_0\mathbf{A}_0 + b_{2l}\mathbf{B}_{2l}$ and $\mathbf{W} = a_0\mathbf{A}_0 + a_{2l}\mathbf{A}_{2l} + b_{2l+1}\mathbf{B}_{2l+1}$ fom the vector

$$\mathbf{W} = a_0\mathbf{A}_0 + \delta_p a_{p-1}\,\mathbf{A}_{p-1} + b_p\,\mathbf{B}_p$$

by choosing $p = 2l$ or $p = 2l + 1$, respectively.

So ,for proving both cases of lemma 2 simultaneously, let

$$\mathbf{W} = a_0\mathbf{A}_0 + \delta_p a_{p-1}\mathbf{A}_{p-1} + b_p\,\mathbf{B}_p\,,$$

where $p = 2l$ or $p = 2l + 1$. Then we have, by (9),

$$\begin{cases} w_0 = a_0 A_0\,, \ldots, w_{p-2} = a_0 A_{p-2}\,, \\ w_{p-1} = a_0 A_{p-1} + \delta_p a_{p-1} A_0\,, \\ w_p = a_0 A_p + \delta_p a_{p-1} A_1 + b_p B_0\,, \\ w_{p+1} = a_0 A_{p+1} + \delta_p a_{p-1} A_2 + b_p B_1\,, \\ \quad \cdot\ \cdot\ \cdot \\ w_{2j} = a_0 A_{2j} + \delta_p a_{p-1} A_{2j-p+1} + b_p B_{2j-p}\,. \end{cases}$$

So we get, by using the definition (16) of t_j,

$$t_j = a_0^2 t_j(\mathbf{A}_0) + \delta_p a_0 a_{p-1}\left\{ z\left(\sum_{i=0}^{2j-p+1}(-1)^i A_i A'_{2j-p+1-i}\right.\right.$$

$$\left.\left. - \sum_{i=p-1}^{2j}(-1)^i A_{i-p+1}A'_{2j-i}\right) + 2\sum_{i=0}^{2j-p}(-1)^i A_i A_{2j-p-i}\right\}$$

$$+ a_0 b_p\left\{ z\left(\sum_{i=0}^{2j-p}(-1)^i A_i B'_{2j-p-i} - \sum_{i=p}^{2j}(-1)^i B_{i-p}A'_{2j-i}\right)\right.$$

$$\left. + 2\sum_{i=0}^{2j-p-1}(-1)^i A_i B_{2j-p-1-i}\right\}$$

$$= a_0^2 t_j(\mathbf{A}_0) + a_0 b_p\left\{ z\sum_{i=0}^{2j-p}(-1)^i (A_i B'_{2j-p-i} - (-1)^p B_i A'_{2j-p-i})\right.$$

$$\left. + 2\sum_{i=0}^{2j-p-1}(-1)^i A_i B_{2j-p-1-i}\right\}.$$

We now put $2j - p = m$ and use (2). So we get the following equations,

$$t_j = a_0^2 t_j(\mathbf{A}_0) + a_0 b_p\left\{\frac{z}{2}\sum_{i=0}^{m}(-1)^i\left\{A_i\left(A'_{m-i+1} + (-1)^{m-i}\sum_{h=0}^{m-i+1}\frac{Z^h}{h!}A'_{m-i-h+1}\right.\right.\right.$$

$$\left. + (-1)^{m-i}\sum_{h=1}^{m-i+1}\frac{2Z^{h-1}}{z(h-1)!}A_{m-i-h+1}\right) - (-1)^p A'_{m-i}\left(A_{i+1} + \right.$$

$$\left.\left.\left. + (-1)^i\sum_{h=0}^{i+1}\frac{Z^h}{h!}A_{i-h+1}\right)\right\} + \sum_{i=0}^{m-1}(-1)^i A_i\left(A_{m-i} + (-1)^{m-1-i}\sum_{h=0}^{m-i}\frac{Z^h}{h!}A_{m-i-h}\right)\right\}$$

$$= a_0^2 t_j(\mathbf{A}_0) + a_0 b_p\left\{\frac{z}{2}\sum_{i=0}^{m}(-1)^i(A_i A'_{m-i+1} - (-1)^p A'_{m-i}A_{i+1}) + \right.$$

$$\left. + \sum_{i=0}^{m-1}(-1)^i A_i A_{m-i}\right\} + a_0 b_p\left\{\frac{z}{2}\sum_{i=0}^{m}\left((-1)^m A_i\sum_{h=0}^{m-i+1}\frac{Z^h}{h!}A'_{m-i-h+1}\right.\right.$$

$$\left.\left. - (-1)^p A'_{m-i}\sum_{h=0}^{i+1}\frac{Z^h}{h!}A_{i-h+1}\right)\right\} + a_0 b_p\left\{\sum_{i=0}^{m}\sum_{h=0}^{m-i}(-1)^m\frac{Z^h}{h!}A_i A_{m-i-h} + \right.$$

$$\left. + \sum_{i=0}^{m-1}\sum_{h=0}^{m-i}(-1)^{m-1}\frac{Z^h}{h!}A_i A_{m-i-h}\right\} = a_0^2 t_j(\mathbf{A}_0) + a_0 b_p\left\{\frac{z}{2}\sum_{i=0}^{m}(-1)^i(A_i A'_{m-i+1}\right.$$

$$\left. - (-1)^p A'_{m-i}A_{i+1}) + \sum_{i=0}^{m}(-1)^i A_i A_{m-i}\right\}$$

$$+ \frac{1}{2} z a_0 b_p \sum_{i=0}^{m}\sum_{h=0}^{i+1}((-1)^m A_{m-i}A'_{i-h+1} - (-1)^p A'_{m-i}A_{i-h+1})\frac{Z^h}{h!}$$

$$= a_0^2 t_j(\mathbf{A}_0) + \frac{1}{2}\delta_p z a_0 b_p (A_0 A'_{m+1} - A_{m+1}A'_0)$$

$$+ \delta_{p-1} a_0 b_p \left\{ \frac{z}{2} \sum_{i=0}^{m} (-1)^i (A_i A'_{m-i+1} - A'_{m-i} A_{i+1}) + \sum_{i=0}^{m} (-1)^i A_i A_{m-i} \right\}$$

$$+ \frac{1}{2}(-1)^p z a_0 b_p \left\{ \sum_{i=-1}^{m} \sum_{h=0}^{i+1} (A_{m-i} A'_{i-h+1} - A'_{m-i} A_{i-h+1}) \frac{Z^h}{h!} - (A_{m+1} A'_0 \right.$$

$$\left. - A'_{m+1} A_0) \right\} = a_0^2 t_j(\mathbf{A}_0) + \delta_{p-1} a_0 b_p \left\{ z \sum_{i=0}^{m+1} (-1)^i A_i A'_{m+1-i} + \right.$$

$$\left. + \sum_{i=0}^{m} (-1)^i A_i A_{m-i} \right\} + \frac{1}{2}(-1)^p z a_0 b_p \sum_{h=0}^{m+1} \frac{Z^h}{h!} \sum_{i=h-1}^{m} (A_{m-i} A'_{i-h+1} - A'_{m-i} A_{i-h+1})$$

$$= a_0^2 t_j(\mathbf{A}_0) + \delta_{p-1} a_0 b_p \left\{ z \sum_{i=0}^{m+1} (-1)^i A_i A'_{m+1-i} + \sum_{i=0}^{m} (-1)^i A_i A_{m-i} \right\}$$

$$= \begin{cases} a_0^2 t_j(\mathbf{A}_0), & \text{if } p = 2l+1, \\ a_0^2 t_j(\mathbf{A}_0) + a_0 b_p L_{\frac{m}{2}}(\mathbf{A}_0), & \text{if } p = 2l. \end{cases}$$

We assumed that $j > l$. Hence $m > 0$. Therefore, by (13), $L_{\frac{m}{2}}(\mathbf{A}_0) = 0$. From this it follows that lemma 2 is true when $2l > j > l$.

If $n \geqq 2l = j > l$ and $\mathbf{W} = a_0\mathbf{A}_0 + a_j\mathbf{A}_j + b_{j+1}\mathbf{B}_{j+1}$, then we can deduce, in the same way as before, that the term with coefficient $a_0 b_{j+1}$ vanishes. Therefore,

$$t_j = a_0^2 t_j(\mathbf{A}_0) + a_0 a_j \left\{ z \left(\sum_{i=0}^{j-1} (-1)^i A_i A'_{j-i} - \sum_{i=j+1}^{2j} (-1)^i A_{i-j} A'_{2j-i} \right) \right.$$

$$\left. + 2 \sum_{i=0}^{j-2} (-1)^i A_i A_{j-1-i} - A_0 A_{j-1} \right\}$$

$$= a_0^2 t_j(\mathbf{A}_0) + a_0 a_j \{ z(A_0 A'_j - A_j A'_0) + A_0 A_{j-1} \}.$$

So lemma 2 is proved.

Lemma 3. *Let* $n \geqq 2$, *and let*

$$\mathbf{W} = a_0\mathbf{A}_0 + \delta_n a_{n-1}\mathbf{A}_{n-1} + b_n\mathbf{B}_n + \delta_n a_{2n-2}\mathbf{A}_{2n-2}$$

be a solution of $Q(n)$. *If* $a_0 \neq 0$, *then*

$$t_n = E(z) + (-1)^n \left\{ \frac{1}{a_0} (z(w_0 w'_n - w_n w'_0) + w_0 w_{n-1}) - b_n \right\} b_n \log z,$$

where $E(z)$ *is an entire function of* z. (Here, again, $\delta_n = 1$, if n is odd, $\delta_n = 0$, if n is even.)

Proof. We have, by (9),

$$(21)\quad \begin{cases} w_0 &= a_0 A_0 \,, \ldots, w_{n-2} = a_0 A_{n-2} \,, \\ w_{n-1} &= a_0 A_{n-1} + \delta_n a_{n-1} A_0 \,, \\ w_n &= a_0 A_n + \delta_n a_{n-1} A_1 + b_n B_0 \,, \\ & \quad \cdot \quad \cdot \quad \cdot \\ w_{2n-2} &= a_0 A_{2n-2} + \delta_n a_{n-1} A_{n-1} + b_n B_{n-2} + \delta_n a_{2n-2} A_0 \,, \\ & \quad \cdot \quad \cdot \quad \cdot \\ w_{2n} &= a_0 A_{2n} + \delta_n a_{n-1} A_{n+1} + b_n B_n + \delta_n a_{2n-2} A_2 \,. \end{cases}$$

From this it follows, by (16), that

$$t_n = E_1(z) + a_0 b_n \Big\{ z \Big(\sum_{i=0}^{n-1} (-1)^i A_i B'_{n-i} - \sum_{i=n+1}^{2n} (-1)^i B_{i-n} A'_{2n-i} \Big)$$

$$+ 2 \sum_{i=0}^{n-2} (-1)^i A_i B_{n-1-i} + (-1)^{n-1} A_{n-1} B_0 \Big\}$$

$$+ (-1)^{n-1} \delta_n a_{n-1} b_n \{ z (A_0 B'_1 - B_1 A'_0) + A_0 B_0 \} \,,$$

where $E_1(z)$ is an entire function of z, because the functions $A_i(z)$ and $A'_i(z) \in \mathbf{E}$.

By (2) and (13),

$$z(A_0 B'_1 - B_1 A'_0) + A_0 B_0 = - \{ z(A_0 A'_1 - A_1 A'_0) + A_0^2 \} \log z = - \log z .$$

Therefore, by using again (2), we obtain

$$t_n = E_1(z) + a_0 b_n \Big\{ z \sum_{i=0}^{n-1} (-1)^i (A_i B'_{n-i} - B_{n-i} A'_i)$$

$$+ 2 \sum_{i=0}^{n-2} (-1)^i A_i B_{n-1-i} + (-1)^{n-1} A_{n-1} B_0 \Big\} - \delta_n a_{n-1} b_n \log z$$

$$= E_2(z) + a_0 b_n \Big\{ \frac{z}{2} \sum_{i=0}^{n-1} (-1)^i \Big\{ A_i (-1)^{n-i} \Big(\sum_{h=0}^{n-i+1} \frac{Z^h}{h!} A'_{n-i-h+1}$$

$$+ \sum_{h=1}^{n-i+1} \frac{2 Z^{h-1}}{z (h-1)!} A_{n-i-h+1} \Big) - A'_i (-1)^{n-i} \sum_{h=0}^{n-i+1} \frac{Z^h}{h!} A_{n-i-h+1} \Big\}$$

$$+ \sum_{i=0}^{n-2} (-1)^i A_i (-1)^{n-1-i} \sum_{h=0}^{n-i} \frac{Z^h}{h!} A_{n-i-h} + (-1)^{n-1} A_{n-1} B_0 \Big\}$$

$$- \delta_n a_{n-1} b_n \log z$$

$$= E_2(z) + a_0 b_n \Big\{ \frac{z}{2} \sum_{i=0}^{n-1} \sum_{h=0}^{n-i+1} (-1)^n (A_i A'_{n-i-h+1} - A'_i A_{n-i-h+1}) \frac{Z^h}{h!}$$

$$+ (-1)^n \sum_{h=0}^{1} \frac{Z^h}{h!} A_{n-1} A_{1-h} + (-1)^{n-1} A_{n-1} (A_1 + A_0 \log z) \Big\}$$

$$- \delta_n a_{n-1} b_n \log z$$

$$= E(z) + (-1)^n a_0 b_n \left\{ \frac{z}{2} \sum_{i=0}^{n} \sum_{h=1}^{n-i+1} (A_i A'_{n-i-h+1} - A'_i A_{n-i-h+1}) \frac{Z^h}{h!} \right.$$

$$\left. - \frac{z}{2} (A_n A'_0 - A'_n A_0) Z + A_0 A_{n-1} \log z \right\} - \delta_n a_{n-1} b_n \log z$$

$$= E(z) + (-1)^n a_0 b_n \left\{ \frac{z}{2} \sum_{h=1}^{n+1} \frac{Z^h}{h!} \sum_{i=0}^{n+1-h} (A_i A'_{n+1-h-i} - A'_i A_{n+1-h-i}) \right.$$

$$\left. + z(A_0 A'_n - A_n A'_0) \log z + A_0 A_{n-1} \log z \right\} - \delta_n a_{n-1} b_n \log z$$

$$= E(z) + (-1)^n a_0 b_n (z(A_0 A'_n - A_n A'_0) + A_0 A_{n-1}) \log z - \delta_n a_{n-1} b_n \log z .$$

So we have

$$t_n = E(z) + \{(-1)^n a_0 (z(A_0 A'_n - A_n A'_0) + A_0 A_{n-1}) - \delta_n a_{n-1}\} b_n \log z, \tag{22}$$

where $E(z)$ is an entire function of z.

Now, because $n \geqq 2$, it follows from (21) that

$$z(w_0 w'_n - w_n w'_0) + w_0 w_{n-1}$$

$$= z(a_0 A_0 (a_0 A'_n + \delta_n a_{n-1} A'_1 + b_n B'_0) - a_0 A'_0 (a_0 A_n + \delta_n a_{n-1} A_1 + b_n B_0))$$

$$+ a_0 A_0 (a_0 A_{n-1} + \delta_n a_{n-1} A_0)$$

$$= a_0^2 \{z(A_0 A'_n - A_n A'_0) + A_0 A_{n-1}\} + a_0 b_n + \delta_n a_0 a_{n-1} ,$$

since, by (13),

$$z(A_0 B'_0 - B_0 A'_0) = z(A_0 A'_1 - A_1 A'_0) + A_0^2 = L_0 (\mathbf{A}_0) = 1 .$$

Thus we obtain

$$(-1)^n a_0 (z(A_0 A'_n - A_n A'_0) + A_0 A_{n-1}) - \delta_n a_{n-1}$$

$$= (-1)^n \{a_0 (z(A_0 A'_n - A_n A'_0) + A_0 A_{n-1}) + \delta_n a_{n-1}\}$$

$$= (-1)^n \left\{ \frac{1}{a_0} (z(w_0 w'_n - w_n w'_0) + w_0 w_{n-1}) - b_n \right\} .$$

By (22), this gives lemma 3.

§ 6. Two lemmas and the beginning of the proof of lemma 1

At first we present two lemmas by Mahler and then begin to prove lemma 1.

Lemma 4. *Let* **F** *be an extension field of* $\mathbf{C}(z)$ *which is closed under differentiation. Let* f *be an element of some extension field* $\mathbf{F}^*$ *of* **F** *such that* f *is algebraic over* **F**, *while its derivative* f' *lies in* **F** *itself. Then* f *is an element of* **F**.

Proof. See [3] p. 155.

Lemma 5. *Let* **W** *be any solution of* $Q(1)$ *such that* $L_0(\mathbf{W}) \neq 0$. *Then the functions*

$$w_0\,,\, w_1\,,\, w_2\,,\, w_3\,,\, t_0\,,\, t_1$$

are algebraically independent over $\mathbf{C}(z)$.

Proof. From the proof of Mahler's lemma 5 (see [3] pp. 159—161) it follows that the functions w_0, w_1, w_2, t_0, t_1 are algebraically independent over $\mathbf{C}(z)$. By substituting the function w_2' by the function t_1 in the proof of Mahler's theorem 1 ([3] pp. 161—163) we see that lemma 5 is true.

Next we begin the proof of lemma 1, which will be performed by induction on n.

It follows from lemma 5 that lemma 1 holds when $n = 0$ and $n = 1$.

We assume that this lemma is true when $n = k - 1 \geqq 1$. This means that if

$$\mathbf{W} = (w_0\,,\, w_1\,,\, \ldots\,,\, w_{2k-1})$$

is any solution of $Q(k-1)$ such that the conditions

$$L_0(\mathbf{W}) \neq 0\,,\;\; L_1(\mathbf{W}) = \ldots = L_{k-2}(\mathbf{W}) = 0$$

are fulfilled, then the functions $w(2k-1)$ and $t(k-1)$ are algebraically independent over $\mathbf{C}(z)$.

By using this assumption we prove, in § 7, § 8 and § 9, that if

$$\mathbf{W} = (w_0\,,\, w_1\,,\, \ldots\,,\, w_{2k+1})$$

is any solution of $Q(k)$ such that it satisfies the conditions

$$L_0(\mathbf{W}) \neq 0\,,\;\; L_1(\mathbf{W}) = \ldots = L_{k-1}(\mathbf{W}) = 0, \tag{23}$$

then the functions $w(2k+1)$ and $t(k)$ are algebraically independent over $\mathbf{C}(z)$.

We split this proof into three steps. The first two of these steps we state as lemmas and then in the third step we conclude the proof of lemma 1. In all three steps $\mathbf{W} = (w_0, w_1, \ldots, w_{2k+1})$ denotes a solution of $Q(k)$ satisfying the conditions (23).

§ 7. The first step of the proof of lemma 1

Lemma 6. *The functions* $w(2k-1)$ *and* $t(k)$ *are algebraically independent over* $\mathbf{C}(z)$.

Proof. Because $\mathbf{W} = (w_0, w_1, \ldots, w_{2k+1})$ is a solution of $Q(k)$ such that the conditions (23)

$$L_0(\mathbf{W}) = C_0 \neq 0\,,\quad L_1(\mathbf{W}) = \ldots = L_{k-1}(\mathbf{W}) = 0$$

are fulfilled, the vector $(w_0, w_1, \ldots, w_{2k-1})$ is a solution of $Q(k-1)$ satisfying the conditions

$$L_0(\mathbf{W}) = C_0 \neq 0\,,\quad L_1(\mathbf{W}) = \ldots = L_{k-2}(\mathbf{W}) = 0\,.$$

Hence, by induction hypothesis, the functions $w(2k-1)$ and $t(k-1)$ are algebraically independent over $\mathbf{C}(z)$.

Assume now that the assertion of lemma 6 is false. This means that the functions $w(2k-1)$ and $t(k)$ are algebraically dependent over $\mathbf{C}(z)$.

Let $\mathbf{F}_1$ be the extension field

$$\mathbf{F}_1 = \mathbf{C}\,(z\,,\, w\,(2k-1)\,,\, t\,(k-1)\,)$$

of $\mathbf{C}\,(z)$.

Because the functions $w(2k-1)$ and $t(k-1)$, but not the functions $w(2k-1)$ and $t(k)$, are algebraically independent over $\mathbf{C}(z)$, the function t_k must be algebraic over $\mathbf{F}_1$.

By the relations (19) and (20) $\mathbf{F}_1$ is closed under differentiation. Further, by (20), t_k' lies also in $\mathbf{F}_1$. Hence, by lemma 4, t_k itself must be an element of $\mathbf{F}_1$.

Therefore, there exists a rational function

$$R = R(z\,,\, x(2k-1)\,,\, u(k-1)\,)\,,$$

where $x(2k-1)$ and $u(k-1)$ denote the indeterminates $x_0, x_1, \ldots, x_{2k-1}$ and $u_0, u_1, \ldots, u_{k-1}$, respectively, in $\mathbf{C}(z, x(2k-1), u(k-1)\,)$ such that

$$t_k = R(z, w(2k-1), t(k-1)\,) \tag{24}$$

identically in z.

We now put (as MAHLER [3] on p. 157)

$$R^*(z\,,\, x(2k-1)\,,\, u(k-1)\,) = \frac{\partial R}{\partial z} + \frac{\partial R}{\partial x_0} y_0 + \frac{\partial R}{\partial x_1} y_1 + \ldots +$$

$$+ \frac{\partial R}{\partial x_{2k-1}} y_{2k-1} + \frac{\partial R}{\partial u_0} v_0 + \frac{\partial R}{\partial u_1} v_1 + \ldots + \frac{\partial R}{\partial u_{k-1}} v_{k-1}\,,$$

where, according to (19) and (20),

$$
(25) \quad \begin{cases} y_{2i} = S_{2i}(z\,,\, x(2i)\,,\, u(i)\,)\,, \\ y_{2i+1} = S_{2i+1}(z\,,\, x(2i+1)\,,\, u(i)\,)\,, \\ i = 0\,,1\,,\ldots,k-1\,, \\ v_0 = -\dfrac{1}{z}u_0 - x_0\,,\quad v_1 = \dfrac{1}{z}(x_0^2 - C_0)\,, \\ v_j = T_j(z\,,\, x(2j-1)\,,\, u(j-1)\,)\,, \\ j = 2\,,3\,,\ldots,k-1\,. \end{cases}
$$

Let, further,

$$\overline{\mathbf{W}} = (\bar{w}_0\,,\bar{w}_1\,,\ldots,\bar{w}_{2k+1})$$

be any second solution of $Q(k)$ for which

$$L_0(\overline{\mathbf{W}}) = C_0 \neq 0\,,\quad L_1(\overline{\mathbf{W}}) = \ldots = L_{k-1}(\overline{\mathbf{W}}) = 0\,.$$

If we state, according to (16),

$$\bar{t}_0 = \bar{w}_0'\,,\quad \bar{t}_1 = z(\bar{w}_0\bar{w}_2' - \bar{w}_2\bar{w}_0') + \bar{w}_0\bar{w}_1\,,$$

$$\bar{t}_j = z\left(\sum_{i=0}^{j-1}(-1)^i\bar{w}_i\bar{w}_{2j-i}' - \sum_{i=j+1}^{2j}(-1)^i\bar{w}_i\bar{w}_{2j-i}'\right) +$$

$$+ 2\sum_{i=0}^{j-2}(-1)^i\bar{w}_i\bar{w}_{2j-1-i} + (-1)^{j-1}\bar{w}_{j-1}\bar{w}_j\,,\; j = 2,\,3,\,\ldots,\,k,$$

it follows that the equations (19) and (20) hold for the functions $\bar{w}(2k)$ and $\bar{t}(k)$, where these notations mean the functions $\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{2k}$ and $\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_k$, respectively.

So it follows that if we set

$$x(2k-1) = \bar{w}(2k-1),\quad u(k-1) = \bar{t}(k-1)\,,$$

then we have

$$(26) \quad R^*(z\,,\bar{w}(2k-1)\,,\bar{t}(k-1)\,) = \frac{d}{dz}R(z\,,\bar{w}(2k-1)\,,\bar{t}(k-1)\,)\,.$$

We have, also,

$$R^*(z\,,w(2k-1)\,,t(k-1)\,) = \frac{d}{dz}R(z\,,w(2k-1),\,t(k-1)\,)\,.$$

Hence, by (20) and (24),

$$(27) \qquad R^*(z\,,w(2k-1)\,,t(k-1)\,) = T_k(z\,,w(2k-1)\,,t(k-1)\,)$$

identically in z.

The functions $w(2k-1)$ and $t(k-1)$ are, by induction hypothesis, algebraically independent over $\mathbf{C}(z)$. Therefore, it follows from the equation (27) that

$$R^*(z, x(2k-1), u(k-1)) = T_k(z, x(2k-1), u(k-1)) \tag{28}$$

identically in the indeterminates $z, x(2k-1)$ and $u(k-1)$.

Now, by (20),

$$\frac{d\bar{t}_k}{dz} = T_k(z, \bar{w}(2k-1), \bar{t}(k-1)),$$

and so we have, by (28),

$$\frac{d\bar{t}_k}{dz} = R^*(z, \bar{w}(2k-1), \bar{t}(k-1)).$$

From this it follows, by (26), that

$$\frac{d\bar{t}_k}{dz} = \frac{d}{dz} R(z, \bar{w}(2k-1), \bar{t}(k-1))$$

identically in z. On integrating, we obtain the relation

$$\bar{t}_k = R(z, \bar{w}(2k-1), \bar{t}(k-1)) + C(\overline{\mathbf{W}}), \tag{29}$$

where $C(\overline{\mathbf{W}})$ depens on the special solution $\overline{\mathbf{W}}$ of $Q(k)$, but is independent of z.

We choose now

$$\overline{\mathbf{W}} = a_0\mathbf{A}_0 + a_{2k-2}\mathbf{A}_{2k-2} + b_{2k-1}\mathbf{B}_{2k-1},$$

where a_0, a_{2k-2} and b_{2k-1} are complex numbers satisfying

$$a_0 = \sqrt{C_0} \neq 0, \; a_0 b_{2k-1} + 2a_0 a_{2k-2} = 0, \; b_{2k-1} \neq 0.$$

Hence, by (12),

$$L_0(\overline{\mathbf{W}}) = C_0, \; L_1(\overline{\mathbf{W}}) = \ldots = L_{k-1}(\overline{\mathbf{W}}) = 0,$$

and so the equation (29) holds for this solution $\overline{\mathbf{W}}$.

It follows now from the choice of $\overline{\mathbf{W}}$ and from lemma 2 that

$$\begin{cases} \bar{w}_0 = a_0 A_0, \ldots, \bar{w}_{2k-3} = a_0 A_{2k-3}, \bar{w}_{2k-2} = a_0 A_{2k-2} + a_{2k-2} A_0, \\ \bar{t}_0 = a_0 t_0(\mathbf{A}_0), \bar{t}_1 = a_0^2 t_1(\mathbf{A}_0), \ldots, \bar{t}_{k-1} = a_0^2 t_{k-1}(\mathbf{A}_0), \\ \bar{t}_k = a_0^2 t_k(\mathbf{A}_0) + \delta_{k,2} a_0 a_2 \{z(A_0 A_2' - A_2 A_0') + A_0 A_1\}. \end{cases} \tag{30}$$

So these functions are entire functions of z, but

$$\bar{w}_{2k-1} = a_0 A_{2k-1} + (a_{2k-2} + b_{2k-1}) A_1 + b_{2k-1} A_0 \log z \, .$$

Thus the left-hand side of (29) is an entire function of z and the right-hand side of this equation has the explicit form

$$R(z \, , \bar{w}(2k-2) \, , a_0 A_{2k-1} + (a_{2k-2} + b_{2k-1}) A_1 + b_{2k-1} A_0 \log z \, , \\ \bar{t}(k-1)\,) + C\,(\overline{\mathbf{W}}) \, .$$

The solution $\mathbf{A}_0 = (A_0 \, , A_1 \, , \ldots , A_{2k-1})$ of $Q(k-1)$ satisfies, by (13), the conditions

$$L_0\,(\mathbf{A}_0) = 1 \neq 0 \, , \;\; L_1\,(\mathbf{A}_0) = \ldots = L_{k-2}\,(\mathbf{A}_0) = 0 \, .$$

So we can deduce, by induction hypothesis, that the functions

$$A_0 \, , A_1 \, , \ldots , A_{2k-1} \, , t_0\,(\mathbf{A}_0) \, , t_1\,(\mathbf{A}_0) \, , \ldots , t_{k-1}\,(\mathbf{A}_0)$$

are algebraically independent over $\mathbf{C}(z)$. From this it follows, by (30), that the entire functions

$$\bar{w}(2k-2) \, , A_{2k-1} \, , \bar{t}(k-1) \, ,$$

and so also the functions

$$\bar{w}(2k-2) \, , A_{2k-1} \, , \bar{t}(k-1) \;\text{ and }\; \log z$$

are algebraically independent over $\mathbf{C}(z)$.

Now, either $R = 0$ identically in $z \, , x(2k-1)$ and $u(k-1)$ or R can be presented in the form

$$R = \frac{P_p x_{2k-1}^p + \ldots + P_0}{Q_q x_{2k-1}^q + \ldots + Q_0} \, ,$$

where p and q are non-negative integers, $P_0, \ldots , P_p, Q_0, \ldots , Q_q$ are polynomials in z, $x(2k-2)$ and $u(k-1)$, such that the polynomials P_p and Q_q do not vanish identically in these variables. If R has the later form, then we have, by (29),

$$\begin{aligned} &(\bar{t}_k - C(\overline{\mathbf{W}})\,) \{ Q_q\,(z \, , \bar{w}(2k-2) \, , \bar{t}(k-1)\,)\, \bar{w}_{2k-1}^q + \ldots + \\ &Q_0(z \, , \bar{w}(2k-2) \, , \bar{t}\,(k-1)\,)\,\} \\ &= P_p(z \, , \bar{w}(2k-2) \, , \bar{t}(k-1)\,)\, \bar{w}_{2k-1}^p + \ldots + \\ &P_0(z \, , \bar{w}(2k-2) \, , \bar{t}(k-1)\,) \, . \end{aligned}$$

Since $\bar{t}_k - C(\overline{\mathbf{W}}) \in \mathbf{E}$ and the functions $\bar{w}(2k-2)$ and $\bar{t}(k-1)$ together with the functions A_{2k-1} and $\log z$ are algebraically independent over $\mathbf{C}(z)$, this equation implies that $p = q$ and

$$\bar{t}_k - C(\overline{\mathbf{W}}) = \frac{P_p(z, \bar{w}(2k-2), \bar{t}(k-1))}{Q_q(z, \bar{w}(2k-2), \bar{t}(k-1))}.$$

Hence,

$$R(z, \bar{w}(2k-1), \bar{t}(k-1)) = \frac{P_p(z, \bar{w}(2k-2), \bar{t}(k-1))}{Q_q(z, \bar{w}(2k-2), \bar{t}(k-1))}$$

identically in z. By the algebraic independence of the functions $\bar{w}(2k-1)$ and $\bar{t}(k-1)$ it follows that this equation must hold identically in z, $\bar{w}(2k-1)$ and $\bar{t}(k-1)$. So the function $R(z, x(2k-1), u(k-1))$ does not depend implicitly on the variable x_{2k-1}.

Therefore, $R = R(z, x(2k-2), u(k-1))$, and so the equation (29) has the form

$$\bar{t}_k = R(z, \bar{w}(2k-2), \bar{t}(k-1)) + C(\overline{\mathbf{W}}). \tag{31}$$

Now, if $2k-2 \geqq k+1$ (If $k = 2$, then we can continue from the equation (35).), we choose

$$\overline{\mathbf{W}} = a_0\mathbf{A}_0 + b_{2k-2}\mathbf{B}_{2k-2},$$

where a_0 and b_{2k-2} are complex numbers satisfying

$$a_0 = \sqrt{C_0} \neq 0, \quad b_{2k-2} \neq 0.$$

It follows now, by (12), that

$$L_0(\overline{\mathbf{W}}) = C_0, L_1(\overline{\mathbf{W}}) = \ldots = L_{k-1}(\overline{\mathbf{W}}) = 0.$$

By lemma 2

$$\bar{t}_0 = a_0 t_0(\mathbf{A}_0), \bar{t}_1 = a_0^2 t_1(\mathbf{A}_0), \ldots, \bar{t}_{k-2} = a_0^2 t_{k-2}(\mathbf{A}_0),$$

$$\bar{t}_{k-1} = a_0^2 t_{k-1}(\mathbf{A}_0) + a_0 b_{2k-2}, \bar{t}_k = a_0^2 t_k(\mathbf{A}_0).$$

These functions and the functions

$$\bar{w}_0 = a_0 A_0, \ldots, \bar{w}_{2k-3} = a_0 A_{2k-3}$$

are entire functions of z, but

$$\bar{w}_{2k-2} = a_0 A_{2k-2} + b_{2k-2}A_1 + b_{2k-2}A_0 \log z.$$

We can now see, as before, that the equation (31) can hold only if the indeterminate x_{2k-2} does not occur in the function R. Hence $R = R(z, x(2k-3), u(k-1))$, and so the equation (31) must be of the form

$$\bar{t}_k = R(z, \bar{w}(2k-3), \bar{t}(k-1)) + C(\overline{\mathbf{W}}). \tag{32}$$

Now, if $2k - 3 \geqq k + 1$, we take alternatively

$$\overline{\mathbf{W}} = a_0 \mathbf{A}_0 + a_{2s-2} \mathbf{A}_{2s-2} + b_{2s-1} \mathbf{B}_{2s-1} \,,$$

where

$$a_0 = \sqrt{C_0} \neq 0 \,,\; a_0 b_{2s-1} + 2a_0 a_{2s-2} = 0 \,,\; b_{2s-1} \neq 0 \,, \tag{33}$$

and

$$\overline{\mathbf{W}} = a_0 \mathbf{A}_0 + b_{2s-2} \mathbf{B}_{2s-2} \,,$$

where

$$a_0 = \sqrt{C_0} \neq 0 \,,\, b_{2s-2} \neq 0 \,, \tag{34}$$

$s = k - 1,\; k - 2, \ldots,$ until $2s - 1 = k + 1$ or $2s - 2 = k + 1$.

By (33) and (34) these solutions satisfy the conditions

$$L_0 (\overline{\mathbf{W}}) = C_0 \,,\, L_1 (\overline{\mathbf{W}}) = \ldots = L_{k-1} (\overline{\mathbf{W}}) = 0 \,.$$

Further, by lemma 2, the functions $\bar{t}(k)$ are entire functions of z for all these solutions. In the same way as before (see p. 27) we can now deduce that none of the indeterminates x_{2k-3}, $x_{2k-4} \ldots, x_{k+1}$ can occur implicitly in R.

Hence

$$R = R(z \,,\, x(k) \,,\, u(k - 1)\,) \,.$$

So the equation (32) has the form

$$\bar{t}_k = R(z \,,\, \bar{w}(k) \,,\, \bar{t}(k - 1)\,) + C(\overline{\mathbf{W}}) \,. \tag{35}$$

If k is odd, we choose

$$\overline{\mathbf{W}} = a_0 \mathbf{A}_0 + a_{k-1} \mathbf{A}_{k-1} + a_{2k-2} \mathbf{A}_{2k-2} + b_k \mathbf{B}_k \,,$$

where a_0, a_{k-1}, a_{2k-2} and b_k are complex numbers satisfying

$$\begin{cases} a_0 = \sqrt{C_0} \neq 0 \,, \quad a_0 b_k + 2a_0 a_{k-1} = 0 \,, \\ a_{k-1} b_k + 2a_0 a_{2k-2} + a_{k-1}^2 = 0 \,,\, b_k \neq 0 \,. \end{cases} \tag{36}$$

From these equations it follows, by (12), that

$$L_0 (\overline{\mathbf{W}}) = C_0 \,,\, L_1 (\overline{\mathbf{W}}) = \ldots = L_{k-1} (\overline{\mathbf{W}}) = 0 \,. \tag{37}$$

Because $L_{\frac{k-1}{2}} (\mathbf{W}) = 0$, we have, by (6) and (19),

$$z(w_0' w_k - w_0 w_k') = z \sum_{i=1}^{k-1} (-1)^i w_i w_{k-i}' + \sum_{i=0}^{k-1} (-1)^i w_i w_{k-1-}$$

$$= r\left(z \,,\, w(k - 1) \,,\, t\left(\frac{k-1}{2}\right)\right),$$

where r is a rational function of its variables. It follows, by (37), that also

$$z(\bar{w}_0'\bar{w}_k - \bar{w}_0\bar{w}_k') = r(z\,,\bar{w}(k-1)\,,\bar{t}\left(\frac{k-1}{2}\right))\,.$$

Therefore, by lemma 3, the left-hand side of (35) is

$$E(z) + \left\{\frac{1}{a_0}(r(z\,,\bar{w}(k-1)\,,\bar{t}\left(\frac{k-1}{2}\right)) - \bar{w}_0\bar{w}_{k-1}) + b_k\right\} b_k \log z\,.$$

By lemma 2

$$\bar{t}_0 \;= a_0 t_0\,(\mathbf{A}_0)\,, \bar{t}_1 = a_0^2 t_1\,(\mathbf{A}_0)\,, \ldots, \bar{t}_{k-2} = a_0^2 t_{k-2}\,(\mathbf{A}_0)\,,$$

$$\bar{t}_{k-1} = a_0^2 t_{k-1}\,(\mathbf{A}_0) + a_0 a_{k-1}\{z(A_0 A_{k-1}' - A_{k-1}A_0') + A_0 A_{k-2}\}\,.$$

Further, it follows from (18) that

$$\bar{t}_{k-1} = a_0^2 t_{k-1}\,(\mathbf{A}_0) + \frac{a_{k-1}}{a_0}(\bar{t}_{\frac{k-1}{2}} - G_{\frac{k-1}{2}}(z\,,\bar{w}(k-2)\,,\bar{t}\left(\frac{k-3}{2}\right)) + \bar{w}_0\bar{w}_{k-2})\,.$$

The functions $A_0, A_1, \ldots, A_k$ and $t_0\,(\mathbf{A}_0), t_1\,(\mathbf{A}_0), \ldots, t_{k-1}\,(\mathbf{A}_0)$ are, by induction hypothesis (see p. 26), algebraically independent over $\mathbf{C}(z)$. Therefore, the entire functions

$$\bar{w}(k-1)\,, A_k\,, \bar{t}(k-1)$$

and the logarithmic function, $\log z$, are algebraically independent over $\mathbf{C}(z)$.

Thus, since the right-hand side of (35) is

$$R(z\,,\bar{w}(k-1)\,, a_0 A_k + (a_{k-1} + b_k)\frac{\bar{w}_1}{a_0} + \frac{b_k}{a_0}\bar{w}_0 \log z, \bar{t}(k-1)\,) + C(\overline{\mathbf{W}})\,,$$

where $R = R(z\,, x(k)\,, u(k-1)\,)$ is a rational function of its arguments, the equation (35) cannot hold unless R is an entire linear function of the argument x_k. This means that

$$R(z\,, x(k)\,, u(k-1)\,) = R_1(z\,, x(k-1)\,, u(k-1)\,) +$$

$$R_2(z\,, x(k-1)\,, u(k-1)\,)x_k\,,$$

where rational functions R_1 and R_2 do not depend on x_k.

The equation (35) becomes now

$$E\,(z) + \left\{\frac{1}{a_0}(r(z\,,\bar{w}(k-1)\,,\bar{t}\left(\frac{k-1}{2}\right)) - \bar{w}_0\bar{w}_{k-1}) + b_k\right\} b_k \log z$$

$$= R_1\,(z\,,\bar{w}(k-1)\,,\bar{t}(k-1)\,) + R_2\,(z\,,\bar{w}(k-1)\,,\bar{t}(k-1)\,)\,(a_0 A_k +$$

$$+ (a_{k-1} + b_k)A_1 + b_k A_0 \log z) + C(\overline{\mathbf{W}})$$

and requires that

$$R_2\,(z\,,\bar{w}(k-1)\,,\bar{t}(k-1)\,)\,b_kA_0\log z$$
$$=\left\{\frac{1}{a_0}\,(r(z\,,\bar{w}(k-1)\,,\bar{t}\left(\frac{k-1}{2}\right))-\bar{w}_0\bar{w}_{k-1})+b_k\right\}b_k\log z.$$

From this it follows, because $b_k\neq 0$, that

$$R_2(z\,,\bar{w}(k-1)\,,\bar{t}(k-1)\,)=\frac{1}{\bar{w}_0}\,r(z\,,\bar{w}(k-1),\bar{t}\left(\frac{k-1}{2}\right))-\bar{w}_{k-1}$$
$$+\frac{\sqrt{C_0}b_k}{\bar{w}_0}\,.$$

In this equation the functions $\bar{w}(k-1)$ and $\bar{t}(k-1)$ are, by induction hypothesis, algebraically independent over $C(z)$. So we have

$$\begin{aligned}R_2(z\,,x(k-1)\,,u(k-1)\,)&=\frac{1}{x_0}\,r\,(z\,,x(k-1)\,,u\left(\frac{k-1}{2}\right))-x_{k-1}\\&\quad+\frac{\sqrt{C_0}b_k}{x_0}\,.\end{aligned}\tag{38}$$

From the definitions of R, R_2 and r it follows that these functions do not depend on the special choice of the constant b_k. It is possible, on the other hand, to satisfy the conditions (36) for infinitely many choices of $b_k\neq 0$. Hence we have a contradiction in (38).

Next we prove that the same kind of contradiction arises in the even case. We choose

$$\overline{\mathbf{W}}=a_0\mathbf{A}_0+b_k\mathbf{B}_k\,,$$

where

$$a_0=\sqrt{C_0}\neq 0\,,\,b_k\neq 0\,.\tag{39}$$

It follows, by (12), that $\overline{\mathbf{W}}$ satisfies (37)

$$L_0\,(\overline{\mathbf{W}})=C_0\,,L_1\,(\overline{\mathbf{W}})\;=\;\ldots\;=\;L_{k-1}\,(\overline{\mathbf{W}})=0\,.$$

The left-hand side of (35) is thus, by lemma 3 and (18),

$$E(z)+\left\{\frac{1}{a_0}\,(\bar{t}_{\frac{k}{2}}-G_{\frac{k}{2}}(z\,,\bar{w}(k-1)\,,\bar{t}\left(\frac{k}{2}-1\right))+\bar{w}_0\bar{w}_{k-1})-b_k\right\}b_k\;\log\,z.$$

The right-hand side of (35) has the explicit form $R(z\,,a_0A_0,\ldots,$ $a_0A_{k-1},\,a_0A_k+b_kA_1+b_kA_0\log z,\,\bar{t}(k-1)\,)+C(\overline{\mathbf{W}})$, where, by lemma 2,

$$\bar{t}_0 = a_0 t_0 (\mathbf{A}_0) \, , \bar{t}_j = a_0^2 t_j (\mathbf{A}_0) + \delta_{j,k/2} a_0 b_k \, , j = 1 \, , 2 \, , \ldots , k - 1 \, .$$

Here, by induction hypothesis (see p. 26), the functions $A_0, A_1, \ldots, A_k$ and $t_0(\mathbf{A}_0), t_1(\mathbf{A}_0), \ldots, t_{k-1}(\mathbf{A}_0)$ are algebraically independent over $\mathbf{C}(z)$. Hence also the entire functions

$$\bar{w}(k-1) \, , A_k \, , \bar{t}(k-1)$$

and the logarithmic function, $\log z$, are algebraically independent over $\mathbf{C}(z)$. From this it follows that the equation (35) cannot hold unless R is an entire linear function of the argument x_k. This means that

$$R(z \, , x(k) \, , u(k-1)\,) = R_1(z \, , x(k-1) \, , u(k-1)\,) +$$
$$+ R_2(z \, , x(k-1) \, , u(k-1)\,) x_k \, ,$$

where rational functions R_1 and R_2 do not depend on x_k.

Therefore, the equation (35) has the form

$$E(z) + \left\{ \frac{1}{a_0} (\bar{t}_{\frac{k}{2}} - G_{\frac{k}{2}}(z \, , \bar{w}(k-1) \, , \bar{t}\left(\frac{k}{2} - 1\right)) + \bar{w}_0 \bar{w}_{k-1}) - b_k \right\} b_k \log z$$

$$= R_1(z \, , \bar{w}(k-1) \, , \bar{t}(k-1)\,) + R_2(z \, , \bar{w}(k-1) \, , \bar{t}(k-1)\,) \times$$
$$(a_0 A_k + b_k A_1 + b_k A_0 \log z) + C(\overline{\mathbf{W}}) \, .$$

This implies that

$$R_2(z \, , \bar{w}(k-1) \, , \bar{t}(k-1)\,) \, b_k A_0 \log z$$

$$= \left\{ \frac{1}{a_0} (\bar{t}_{\frac{k}{2}} - G_{\frac{k}{2}}(z \, , \bar{w}(k-1) \, , \bar{t}\left(\frac{k}{2} - 1\right)) + \bar{w}_0 \bar{w}_{k-1}) - b_k \right\} b_k \log z \, .$$

Because $b_k \neq 0$, it follows that

$$R_2(z \, , \bar{w}(k-1) \, , \bar{t}(k-1)\,) = \frac{1}{\bar{w}_0} (\bar{t}_{\frac{k}{2}} - G_{\frac{k}{2}}(z \, , \bar{w}(k-1) \, , \bar{t}\left(\frac{k}{2} - 1\right))\,) + \bar{w}_{k-1}$$
$$- \frac{\sqrt{C_0} b_k}{\bar{w}_0} \, .$$

Since, by induction hypothesis, the functions $\bar{w}(k-1)$ and $\bar{t}(k-1)$ are algebraically independent over $\mathbf{C}(z)$, R_2 has the explicit form

$$R_2(z \, , x(k-1) \, , u(k-1)\,) = \frac{1}{x_0} (u_{\frac{k}{2}} - G_{\frac{k}{2}}(z \, , x(k-1) \, , u\left(\frac{k}{2} - 1\right))\,) + x_{k-1}$$
$$- \frac{\sqrt{C_0} b_k}{x_0} \, .$$

From the definitions of the functions R, R_2 and $G_{k/2}$ it follows that these functions do not depend on the special choice of the constant b_k. By (39), it is possible to choose $b_k \neq 0$ arbitrarily. So we have a contradiction which proves lemma 6.

§ 8. The second step of the proof of lemma 1

Lemma 7. *The functions* $w(2k)$ *and* $t(k)$ *are algebraically independent over* $\mathbf{C}(z)$.

Proof. Assume, against lemma 7, that the functions $w(2k)$ and $t(k)$ are algebraically dependent over $\mathbf{C}(z)$. Further, let F_2 be the extension field

$$\mathbf{F}_2 = \mathbf{C}(z\, , w(2k-1)\, , t(k)\,)$$

of $\mathbf{C}(z)$. The functions $w(2k-1)$ and $t(k)$ are, by lemma 6, algebraically independent over $\mathbf{C}(z)$. Therefore, w_{2k} and hence also

$$q = w_{2k} \,/\, w_0$$

are algebraic over $\mathbf{F}_2$.

By the relations (19) and (20) $\mathbf{F}_2$ is closed under differentiation. Further, by (18),

$$q' = \frac{1}{w_0^2}\,(w_0 w_{2k}' - w_{2k} w_0') = \frac{1}{z w_0^2}\,(t_k - G_k(z\, , w(2k-1)\, , t(k-1)\,)\,)\, , \tag{40}$$

where G_k is a rational function of its arguments. Hence q' lies in $\mathbf{F}_2$. This implies, by lemma 4, that q itself is an element of $\mathbf{F}_2$.

Hence there exists a rational function

$$R = R(z\, , x(2k-1)\, , u(k)\,)$$

in $\mathbf{C}(z, x(2k-1), u(k)\,)$ such that

$$q = R(z\, , w(2k-1)\, , t(k)\,) \tag{41}$$

identically in z.

We put again

$$R^*(z\, , x(2k-1)\, , u(k)\,) = \frac{\partial R}{\partial z} + \frac{\partial R}{\partial x_0} y_0 + \frac{\partial R}{\partial x_1} y_1 + \ldots + \frac{\partial R}{\partial x_{2k-1}} y_{2k-1} +$$

$$\frac{\partial R}{\partial u_0} v_0 + \frac{\partial R}{\partial u_1} v_1 + \ldots + \frac{\partial R}{\partial u_k} v_k\, ,$$

where y_{2i}, y_{2i+1} and v_i, $i = 0, 1, \ldots, k-1$, denote the expressions (25) and

$$v_k = T_k(z\,,\, x(2k-1)\,,\, u(k-1)\,)$$

according to (20).

Let

$$\overline{\mathbf{W}} = (\bar{w}_0\,,\, \bar{w}_1\,,\, \ldots\,,\, \bar{w}_{2k+1})$$

be any second solution of $Q(k)$ satisfying

$$L_0\,(\overline{\mathbf{W}}) = C_0 \neq 0\,,\; L_1\,(\overline{\mathbf{W}}) = \ldots = L_{k-1}\,(\overline{\mathbf{W}}) = 0\,.$$

Then the equations (19) and (20) hold for the functions $\bar{w}(2k)$ and $\bar{t}(k)$.

If we now choose

$$x(2k-1) = \bar{w}(2k-1)\,,\; u(k) = \bar{t}(k)\,,$$

we have, by the definition of R^*,

$$R^*(z\,,\, \bar{w}(2k-1)\,,\, \bar{t}(k)\,) = \frac{d}{dz} R(z\,,\, \bar{w}(2k-1),\, \bar{t}(k)\,)\,. \tag{42}$$

This equation holds, of course, for the solution $\mathbf{W}$, too. Therefore, by using the equations (40) and (41) we get the equation

$$R^*(z\,,\, w(2k-1)\,,\, t(k)\,) = \frac{1}{zw_0^2}\,(t_k - G_k(z\,,\, w(2k-1)\,,\, t(k-1)\,)\,)\,.$$

Here the functions $w(2k-1)$ and $t(k)$ are, by lemma 6, algebraically independent over $\mathbf{C}(z)$. So this equation implies that

$$R^*(z\,,\, x(2k-1),\, u(k)\,) = \frac{1}{zx_0^2}\,(u_k - G_k(z\,,\, x(2k-1)\,,\, u(k-1)\,)\,) \tag{43}$$

identically in z, $x(2k-1)$ and $u(k)$.

We put, in analogy to q,

$$\bar{q} = \bar{w}_{2k} \,/\, \bar{w}_0\,.$$

Then we evidently have, in analogy to (40),

$$\bar{q}' = \frac{1}{z\bar{w}_0^2}\,(\bar{t}_k - G_k\,(z\,,\, \bar{w}(2k-1)\,,\, \bar{t}(k-1)\,)\,)\,,$$

whence, by (42) and (43),

$$\frac{d\bar{q}}{dz} = \frac{d}{dz} R(z\,,\, \bar{w}(2k-1)\,,\, \bar{t}(k)\,)\,.$$

When we integrate this equation with respect to z, we obtain the relation

$$\bar{q} = R(z\,,\, \bar{w}(2k-1)\,,\, \bar{t}(k)\,) + C(\overline{\mathbf{W}})\,, \tag{44}$$

where $C(\overline{\mathbf{W}})$ denotes a quantity that depends on $\overline{\mathbf{W}}$, but is independent of the variable z.

The proof of lemma 7 will now be concluded by deducing a contradiction from this equation (44).

We choose, therefore, the special solution $\overline{\mathbf{W}}$ as follows

$$\overline{\mathbf{W}} = a_0 \mathbf{A}_0 + b_{2k} \mathbf{B}_{2k} ,$$

where a_0 and b_{2k} are complex numbers satisfying

$$a_0 = \sqrt{C_0} \neq 0 , \ b_{2k} \neq 0 .$$

This choice, by (12), satisfies the conditions

$$L_0 (\overline{\mathbf{W}}) = C_0 , \ L_1 (\overline{\mathbf{W}}) = \ldots = L_{k-1} (\overline{\mathbf{W}}) = 0$$

and so also the equation (44) holds for this solution.

The functions $\bar{w}_0 = a_0 A_0, \bar{w}_1 = a_0 A_1, \ldots, \bar{w}_{2k-1} = a_0 A_{2k-1}$ and, by lemma 2, the functions $\bar{t}(k)$, too, are entire functions of z. Hence the right-hand side of (44) is a meromorphic function of z, while the left-hand side of this equation,

$$\bar{q} = \frac{A_{2k}}{A_0} + \frac{b_{2k}}{a_0}\left(\frac{A_1}{A_0} + \log z\right),$$

where $b_{2k} \neq 0$, has a logarithmic singularity at $z = 0$. This contradiction proves that lemma 7 is true.

§ 9. The end of the proof of lemma 1

We now conclude the proof of lemma 1 by proving that the functions $w(2k+1)$ and $t(k)$ are algebraically independent over $\mathbf{C}(z)$.

Let us assume that these functions are algebraically dependent over $\mathbf{C}(z)$ and denote by $\mathbf{F}_3$ the extension field

$$\mathbf{F}_3 = \mathbf{C}(z , w(2k) , t(k))$$

of $\mathbf{C}(z)$. Then it follows, by our hypothesis and lemma 7, that w_{2k+1} and so also the quotient

$$q = w_{2k+1} / w_0$$

are algebraic over $\mathbf{F}_3$.

By the relations (19) and (20) $\mathbf{F}_3$ is closed under differentiation. Further, because $L_k(\mathbf{W}) = C_k$, we have, by (6) and (19),

$$z(w_0 w'_{2k+1} - w'_0 w_{2k+1}) = r(z\,,\, w(2k)\,,\, t(k)\,)\,,$$

where r is a rational function of its variables. From this it follows that

$$(45) \qquad q' = \frac{1}{w_0^2}(w'_{2k+1} w_0 - w_{2k+1} w'_0) = \frac{1}{z w_0^2} r(z\,,\, w(2k)\,,\, t(k)\,)\,,$$

and this implies that q' lies in $\mathbf{F}_3$. On applying once more lemma 4, it follows then that q is an element of $\mathbf{F}_3$.

So there exists a rational function

$$R = R(z\,,\, x(2k)\,,\, u(k)\,)$$

in $\mathbf{C}(z, x(2k), u(k)\,)$ such that

$$(46) \qquad q = R(z\,,\, w(2k)\,,\, t(k)\,)$$

identically in z.

Put again

$$R^*(z\,,\, x(2k)\,,\, u(k)\,) = \frac{\partial R}{\partial z} + \frac{\partial R}{\partial x_0} y_0 + \frac{\partial R}{\partial x_1} y_1 + \ldots + \frac{\partial R}{\partial x_{2k}} y_{2k} + $$
$$+ \frac{\partial R}{\partial u_0} v_0 + \frac{\partial R}{\partial u_1} v_1 + \ldots + \frac{\partial R}{\partial u_k} v_k\,,$$

where y_{2i}, y_{2i+1} and v_i, $i = 0, 1, \ldots, k-1$, denote the expressions (25) and

$$y_{2k} = S_{2k}(z\,,\, x(2k)\,,\, u(k)\,)\,,\; v_k = T_k(z\,,\, x(2k-1)\,,\, u(k-1)\,)$$

according to (19) and (20).

We now proceed as in the proofs of lemmas 6 and 7. Let

$$\overline{\mathbf{W}} = (\bar{w}_0\,,\, \bar{w}_1\,,\, \ldots\,,\, \bar{w}_{2k+1})$$

be an arbitrary second solution of $Q(k)$ satisfying

$$(47) \qquad L_0(\overline{\mathbf{W}}) = C_0 \neq 0\,,\, L_1(\overline{\mathbf{W}}) = \ldots = L_{k-1}(\overline{\mathbf{W}}) = 0\,,\, L_k(\overline{\mathbf{W}}) = C_k.$$

From this it follows that the equations (19) and (20) hold for this solution.

On choosing now

$$x(2k) = \bar{w}(2k)\,,\; u(k) = \bar{t}(k)\,,$$

we get, by the definition of R^*,

$$(48) \qquad R^*(z\,,\, \bar{w}(2k)\,,\, \bar{t}(k)\,) = \frac{d}{dz} R(z\,,\, \bar{w}(2k)\,,\, \bar{t}(k)\,)\,.$$

Also,

$$R^*(z\,,\,w(2k)\,,\,t(k)\,) = \frac{d}{dz} R(z\,,\,w(2k)\,,\,t(k)\,)\,.$$

This implies, by (45) and (46), that

$$R^*\,(z\,,\,w(2k)\,,\,t(k)\,) = \frac{1}{zw_0^2}\,r(z\,,\,w(2k)\,,\,t(k)\,)\,.$$

Because the functions $w(2k)$ and $t(k)$ are, by lemma 7, algebraically independent over $\mathbf{C}(z)$, this equation requires that

$$R^*(z\,,\,x(2k)\,,\,u(k)\,) = \frac{1}{zx_0^2}\,r(z\,,\,x(2k)\,,\,u(k)\,) \tag{49}$$

identically in the indeterminates $z, x(2k)$ and $u(k)$.

If we put, in analogy to q,

$$\bar{q} = \bar{w}_{2k+1}\,/\,\bar{w}_0\,,$$

then, by (47), the equation (45) holds for the solution $\overline{\mathbf{W}}$. This means that

$$\bar{q}' = \frac{1}{z\bar{w}_0^2}\,r(z\,,\,\bar{w}(2k)\,,\,\bar{t}(k)\,)\,,$$

whence, by (48) and (49),

$$\frac{d\bar{q}}{dz} = \frac{d}{dz}\,R(z\,,\,\bar{w}(2k)\,,\,\bar{t}(k)\,)\,.$$

When we now integrate with respect to z, we find that

$$\bar{q} = R(z\,,\,\bar{w}(2k)\,,\,\bar{t}\,(k)\,) + C(\overline{\mathbf{W}})\,, \tag{50}$$

where $C(\overline{\mathbf{W}})$ again denotes a quantity that depends on $\overline{\mathbf{W}}$, but is independent of z.

Next we choose

$$\overline{\mathbf{W}} = a_0\mathbf{A}_0 + a_{2k}\mathbf{A}_{2k} + b_{2k+1}\mathbf{B}_{2k+1}\,,$$

where a_0, a_{2k} and b_{2k+1} are complex numbers satisfying

$$a_0 = \sqrt{C_0} \neq 0\,,\; a_0b_{2k+1} + 2a_0a_{2k} = C_k\,,\; b_{2k+1} \neq 0\,.$$

This choice implies, by (12), that

$$L_0\,(\overline{\mathbf{W}}) = C_0\,,\, L_1\,(\overline{\mathbf{W}}) = \ldots = L_{k-1}\,(\overline{\mathbf{W}}) = 0\,,\, L_k\,(\overline{\mathbf{W}}) = C_k\,.$$

Hence the equation (50) holds for this solution.

Because the functions

$$\bar{w}_0 = a_0 A_0\,,\ \bar{w}_1 = a_0 A_1\,,\ \ldots,\ \bar{w}_{2k-1} = a_0 A_{2k-1}\,,\ \bar{w}_{2k} = a_0 A_{2k} + a_{2k} A_0\,,$$

$$\bar{t}_0 = a_0 t_0(\mathbf{A}_0)\,,\ \bar{t}_1 = a_0^2 t_1(\mathbf{A}_0)\,,\ \ldots,\ \bar{t}_k = a_0^2 t_k(\mathbf{A}_0)$$

are entire functions of z, the right-hand side of (50) is a meromorphic function of z. On the other hand, the left-hand side of this equation,

$$\bar{q} = \frac{A_{2k+1}}{A_0} + \frac{a_{2k} + b_{2k+1}}{a_0}\frac{A_1}{A_0} + \frac{b_{2k+1}}{a_0}\log z\,,$$

has a logarithmic singularity at the point $z = 0$. This contradiction proves that the functions

$$w(2k+1)\,,\ t(k)$$

are algebraically independent over $\mathbf{C}(z)$.

So lemma 1 is true when $n = k$. From this we can deduce, by induction, that lemma 1 is generally true.

§ 10. The proof of theorem 1

We can now at last prove theorem 1 itself.

Let $\mathbf{W} = (w_0, w_1, \ldots, w_{2n+1})$ be a solution of $Q(n)$ such that

$$L_0(\mathbf{W}) = C_0 \neq 0\,,\ L_1(\mathbf{W}) = \ldots = L_{n-1}(\mathbf{W}) = 0\,.$$

From lemma 1 it follows that the functions $w(2n+1)$ and $t(n)$ are algebraically independent over $\mathbf{C}(z)$. To prove theorem 1 we show now that this implies the algebraic independence of the functions $w(2n+1)$ and $w'(n)$ over $\mathbf{C}(z)$.

We prove this by induction on k, the number of the functions $w'(k-1)$.

From the algebraic independence of the functions $w(2n+1)$ and $t(n)$ it follows that the functions $w(2n+1)$ and w_0' are algebraically independent.

Assume now that the functions $w(2n+1)$ and $w'(k-1)$, $1 \leqq k \leqq n$, are algebraically independent.

Let the functions $w(2n+1)$ and $w'(k)$ be algebraically dependent. Then there is a polynomial $P = P(z, x(2n+1), y(k))$ which is not identically zero in z, $x(2n+1)$ and $y(k)$, such that

$$P(z\,, w(2n+1)\,, w'(k)) = 0 \tag{51}$$

identically in z.

By induction hypothesis,

$$P = P_m y_k^m + \ldots + P_0\,,\ m \geqq 1\,,$$

where $P_0, \ldots, P_m$ are polynomials in the indeterminates $z, x(2n+1)$ and $y(k-1)$, and P_m does not vanish identically in these indeterminates.

From the equation (51) it follows, by (18), that

$$(52)\qquad R_m(z\,,\, w(2n+1)\,,\, t(k-1)\,)\left(\frac{t_k}{zw_0}\right)^m + \ldots + R_0(z\,,\, w(2n+1)\,,\, t(k-1)\,) = 0\,,$$

where $R_0, \ldots, R_m$ are rational functions of their variables. Particularly,

$$(53)\qquad P_m(z\,,\, w(2n+1)\,,\, w'(k-1)\,) = R_m\,(z\,,\, w(2n+1)\,,\, t(k-1)\,)$$

identically in z.

The algebraic independence of the functions $w(2n+1)$ and $t(n)$ implies, by (52), that

$$(54)\qquad R_m\,(z\,,\, x(2n+1)\,,\, u(k-1)\,) = 0$$

identically in the indeterminates z, $x(2n+1)$ and $u(k-1)$.

From the equations (53) and (54) it follows that

$$P_m(z\,,\, w(2n+1)\,,\, w'(k-1)\,) = 0$$

identically in z. This is impossible, since the functions $w(2n+1)$ and $w'(k-1)$ are, by induction hypothesis, algebraically independent.

Therefore, the functions $w(2n+1)$ and $w'(k)$ are algebraically independent. This proves theorem 1.

§ 11. Shidlovski's theorem

In the following section we apply SHIDLOVSKI's theorem, which he, generalizing SIEGEL's work [7], [8], presented in [6] (pp. 898—899). This theorem deals with the transcendency and algebraic independence of the values of E-functions, which SIEGEL defines in the following way ([8] p. 33, [5] p. 339).

An entire function

$$f(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$$

is called an E-function, if it satisfies the following three conditions:

1) *all coefficients* c_n *belong to the same algebraic number field of finite degree over the rational number field* **Q**;

2) *for every* $\varepsilon > 0$ *we have* $\overline{|c_n|} = O(n^{\varepsilon n})$;

3) *there exists a sequence* $\{q_n\}$ *of natural numbers such that the numbers* $q_n c_k$, $k = 0, 1, \ldots, n$, *are algebraic integers, and* $q_n = O(n^{\varepsilon n})$ *for every* $\varepsilon > 0$.

Shidlovski's theorem. *Let* $w_1 = f_1(z), \ldots, w_m = f_m(z)$ *be finitely many E-functions satisfying a system of differential equations*

$$w_i' = q_{i0} + \sum_{j=1}^{m} q_{ij} w_j \ , \ i = 1 \ , \ldots , \ m \ ,$$

where the coefficients q_{ij}, $i = 1, \ldots, m$, $j = 0, 1, \ldots, m$, *are rational functions in* $\mathbf{C}(z)$, *say with the least common denominator* $d(z)$. *Let* α *be any algebraic number such that*

$$\alpha d(\alpha) \neq 0 \ .$$

Assume that N_1, *but no more, of the functions* $f_1(z), \ldots, f_m(z)$ *are algebraically independent over* $\mathbf{C}(z)$, *the field of rational functions of* z, *and that* N_2, *but no more, of the numbers* $f_1(\alpha), \ldots, f_m(\alpha)$ *are algebraically independent over* $\mathbf{Q}$, *the field of rational numbers. Then*

$$N_1 = N_2 \ .$$

§ 12. On the transcendence and algebraic independence of the numbers $A_i(\alpha)$ and $A_i'(\alpha)$

MAHLER proves ([3] pp. 163—165) that the functions $A_i(z)$, $i = 0, 1, \ldots$, are E-functions. Since the ring of E-functions is closed under differentiation ([8] p. 33), the derivatives $A_i'(z)$, $i = 0, 1, \ldots$, too, are E-functions. Further, the system of $4n + 4$ functions

$$A_i(z) \ , \ A_i'(z) \ , \ i = 0 \ , 1 \ , \ldots \ , 2n + 1 \ ,$$

is, by $Q(n)$, a solution of the following system Q_n of differential equations,

$$Q_n : \begin{cases} w_0' = W_0 \ , \ W_0' = -\dfrac{1}{z} W_0 - w_0 \ , \\[2ex] w_i' = W_i \ , \ W_i' = -\dfrac{1}{z} W_i - w_i - \dfrac{2}{z} W_{i-1} \ , \\[2ex] i = 1 \ , 2 \ , \ldots \ , 2n + 1 \ . \end{cases}$$

The coefficients of these equations are rational functions in $\mathbf{C}(z)$ and in each system (for every value of $n = 0, 1, \ldots$) the least common denominator of the coefficients is the same function

$$d(z) = z \ .$$

It follows that we can use SHIDLOVSKI's theorem, when we now begin to investigate the algebraic independence of the function values

$$A_i(\alpha)\,,\,A_i'(\alpha)\,,\ i = 0\,,1\,,\ldots,2n+1\,.$$

We prove the following theorem which is a generalization of MAHLER's theorem 2 ([3] p. 167).

Theorem 2. *Let* α *be any algebraic number distinct from zero. Then the elements of each of the following* 2^{n+1} *sets of* $3n+3$ *function values*

$$\begin{aligned}&\{A_0(\alpha)\,,A_0'(\alpha)\,,A_1(\alpha)\quad or\quad A_1'(\alpha)\,,\ldots,A_{2i}(\alpha)\,,A_{2i}'(\alpha)\,,A_{2i+1}(\alpha)\\ (55)\qquad &or\quad A_{2i+1}'(\alpha)\,,\ldots,A_{2n}(\alpha)\,,A_{2n}'(\alpha)\,,A_{2n+1}(\alpha)\quad or\quad A_{2n+1}'(\alpha)\}\\ &(n = 0\,,1\,,\ldots)\,,\end{aligned}$$

are algebraically independent over the rational number field $\mathbf{Q}$. *In particular,* $4n+4$ *function values*

$$A_i(\alpha)\,,\,A_i'(\alpha)\,,\ i = 0\,,1\,,\ldots,2n+1\,,$$

are transcendental numbers.

Proof. From MAHLER's theorem 2 it follows that our theorem is true when $n = 0$ or $n = 1$. Therefore, in particular, the numbers $A_0(\alpha)$ and $A_0'(\alpha)$ are transcendental and so distinct from zero.

From the equations (8) it follows that all functions $A_{2j+1}'(z)$, $j = 0, 1, \ldots, n$, can be expressed rationally by the functions $A_{2j}(z)$, $A_{2j+1}(z)$, $A_{2j}'(z)$, $j = 0, 1, \ldots, n$, where the coefficients of these expressions lie in $\mathbf{C}(z)$. The later functions are, by the corollary of theorem 1, algebraically independent over $\mathbf{C}(z)$. From this it follows that the integer N_1 in SHIDLOVSKI's theorem has for the system Q_n the value $N_1 = 3n+3$. Therefore, by SHIDLOVSKI's theorem, also $N_2 = 3n+3$.

Let us assume now, in contradiction to theorem 2, that the numbers of at least one of the sets (55) of $3n+3$ function values are algebraically dependent over $\mathbf{Q}$.

By (13) we have

$$\alpha(A_0(\alpha)A_1'(\alpha) - A_0'(\alpha)A_1(\alpha)\,) + A_0^2(\alpha) = 1\,,$$

$$\alpha\sum_{i=0}^{2j+1}(-1)^iA_i(\alpha)A_{2j+1-i}'(\alpha) + \sum_{i=0}^{2j}(-\,1)^iA_i(\alpha)A_{2j-i}(\alpha) = 0\,,$$

$$j = 1\,,2\,,\ldots,n\,.$$

Hence, because $\alpha \neq 0$, $A_0(\alpha) \neq 0$ and $A_0'(\alpha) \neq 0$, we have the following recursive relations,

$$A_1'(\alpha) = \frac{A_0'(\alpha)}{A_0(\alpha)} A_1(\alpha) + \frac{1}{\alpha A_0(\alpha)} (1 - A_0^2(\alpha)) ,$$

$$A_1(\alpha) = \frac{A_0(\alpha)}{A_0'(\alpha)} A_1'(\alpha) - \frac{1}{\alpha A_0'(\alpha)} (1 - A_0^2(\alpha)) ,$$

$$A_{2j+1}'(\alpha) = \frac{A_0'(\alpha)}{A_0(\alpha)} A_{2j+1}(\alpha) + \frac{1}{\alpha A_0(\alpha)} R^*_{2j+1}(*) ,$$

$$A_{2j+1}(\alpha) = \frac{A_0(\alpha)}{A_0'(\alpha)} A_{2j+1}'(\alpha) - \frac{1}{\alpha A_0'(\alpha)} R^*_{2j+1}(*) , j = 1, 2, \ldots, n ,$$

where R^*_{2j+1} is a rational function with rational numerical coefficients in variables (*), which are the numbers of one of the sets

(56)
$$\{\alpha, A_0(\alpha), A_0'(\alpha), A_1(\alpha) \text{ or } A_1'(\alpha), \ldots, A_{2j-2}(\alpha), A_{2j-2}'(\alpha), A_{2j-1}(\alpha) \text{ or } A_{2j-1}'(\alpha), A_{2j}(\alpha), A_{2j}'(\alpha)\} .$$

There are 2^j sets of these numbers and, respectively, 2^j functions R^*_{2j+1}. Therefore each of the function values $A_{2j+1}(\alpha)$ and $A_{2j+1}'(\alpha)$, $j = 0, 1, \ldots, n$, can be expressed rationally in terms of the other one and the numbers of any of 2^j sets (56).

Hence, our assumption implies, against SHIDLOVSKI's theorem, that $N_2 < 3n + 3$. This contradiction proves the truth of theorem 2.

§ 13. The transcendence and algebraic independence of some expressions involving Euler's constant γ and the constants $\zeta(i)$

In analogy to $A_i(z)$ and $B_i(z)$, we define further functions $C_i(z)$ by the formulae ([3] p. 167)

(57)
$$C_i(z) = \frac{1}{i!} \left(\frac{\partial}{\partial \nu}\right)^i J_\nu(z)|_{\nu=0} , \; i = 0, 1, \ldots ,$$

where $J_\nu(z)$ denotes the Bessel function of the first kind. Particularly, we have

$$C_0(z) = J_0(z) , C_1(z) = \frac{\pi}{2} Y_0(z) ,$$

where $Y_\nu(z)$ denotes the Bessel function of the second kind.

Let now $\zeta = \log(z/2)$. Let, further,

$$s_1 = \gamma$$

denote Euler's constant. The values of Riemann Zeta function will be denoted by

$$s_i = \zeta(i) = \sum_{n=1}^{\infty} n^{-i}\ , \ i = 2\ , 3\ , \ldots\ .$$

Moreover, we put

$$\zeta_i = \sum_{j=0}^{i} \gamma_{i-j} \frac{\zeta^j}{j!}\ , \ i = 0\ , 1\ , \ldots\ , \tag{58}$$

where the coefficients γ_i are connected with s_i by the recursive formulae

$$\gamma_0 = 1\ , \ (i+1)\,\gamma_{i+1} = \sum_{j=0}^{i} (-1)^j s_{j+1}\gamma_{i-j}\ , \ i = 0\ , 1\ , \ldots\ . \tag{59}$$

From the equations (58) and (59) it follows for $i \geqq 1$ that

$$\begin{aligned}
i\,\zeta_i &= i \sum_{j=0}^{i} \gamma_{i-j} \frac{\zeta^j}{j!} = i \sum_{j=0}^{i} \gamma_j \frac{\zeta^{i-j}}{(i-j)!} \\
&= \sum_{j=0}^{i} (i-j)\,\gamma_j \frac{\zeta^{i-j}}{(i-j)!} + \sum_{j=0}^{i} j\,\gamma_j \frac{\zeta^{i-j}}{(i-j)!} \\
&= \zeta \sum_{j=0}^{i-1} \gamma_j \frac{\zeta^{i-1-j}}{(i-1-j)!} + \sum_{j=0}^{i-1} (j+1)\,\gamma_{j+1} \frac{\zeta^{i-1-j}}{(i-1-j)!} \\
&= \zeta\zeta_{i-1} + \sum_{j=0}^{i-1} \frac{\zeta^{i-1-j}}{(i-1-j)!} \sum_{k=0}^{j} (-1)^k s_{k+1}\gamma_{j-k} \\
&= \zeta\zeta_{i-1} + \sum_{k=0}^{i-1} (-1)^k s_{k+1} \sum_{j=k}^{i-1} \gamma_{j-k} \frac{\zeta^{i-1-j}}{(i-1-j)!} \\
&= \zeta\zeta_{i-1} + \sum_{k=0}^{i-1} (-1)^k s_{k+1} \sum_{j=0}^{i-k-1} \gamma_j \frac{\zeta^{i-k-1-j}}{(i-k-1-j)!} \\
&= \zeta\zeta_{i-1} + \sum_{k=0}^{i-1} (-1)^k s_{k+1} \zeta_{i-k-1} \\
&= \zeta\zeta_{i-1} + \sum_{j=1}^{i} (-1)^{j-1} s_j \zeta_{i-j}\ .
\end{aligned}$$

So we have the recursive formulae

$$\zeta_0 = 1\ , \ i\zeta_i = \zeta\zeta_{i-1} + \sum_{j=1}^{i} (-1)^{j-1} s_j \zeta_{i-j}\ , \ i = 1\ , 2\ , \ldots\ . \tag{60}$$

If we set

$$\chi = \zeta + \gamma = \log\,(z/2) + \gamma\ ,$$

we have, by (60),

(61) $\zeta_0 = 1$, $\zeta_1 = \chi$, $i\zeta_i = \chi\zeta_{i-1} + \sum_{j=2}^{i} (-1)^{j-1} s_j \zeta_{i-j}$, $i = 2, 3, \ldots$.

From this it follows that ζ_i is a polynomial of degree i in χ with coefficients that are polynomials in the values $\zeta(2)$, $\zeta(3), \ldots, \zeta(i)$ of Riemann zeta function with rational numerical coefficients.

Further, we have $\chi' = \frac{1}{z}$. Therefore, by (61),

$$\zeta_0' = 0 \, , \, \zeta_1' = \frac{1}{z} = \frac{1}{z} \zeta_0 \, .$$

Now, let us assume that

$$\zeta_i' = \frac{1}{z} \zeta_{i-1} \, , \; 1 \leqq i \leqq k \, .$$

Then we obtain, by (61), the following equations,

$$(k+1)\, \zeta_{k+1}' = \frac{1}{z} \zeta_k + \chi \zeta_k' + \sum_{j=2}^{k+1} (-1)^{j-1} s_j \zeta_{k+1-j}'$$
$$= \frac{1}{z} \zeta_k + \frac{1}{z} (\chi \zeta_{k-1} + \sum_{j=2}^{k} (-1)^{j-1} s_j \zeta_{k-j}) = \frac{k+1}{z} \zeta_k \, .$$

Hence, we have

$$\zeta_0' = 0 \, , \; \zeta_i' = \frac{1}{z} \zeta_{i-1} \, , \; i = 1, 2, \ldots \, .$$

It can be shown (see [3] p. 168) that between the functions $A_i(z)$ and $C_i(z)$ there is the relation

(62) $$C_i(z) = \sum_{j=0}^{i} \zeta_{i-j} A_j(z) \, , \; i = 0, 1, \ldots ,$$

where the coefficients ζ_i can be determined by (60) or (61).

Further it follows that

(63) $$C_0'(z) = A_0'(z) \, , \, C_i'(z) = \sum_{j=0}^{\cdot} \zeta_{i-j} A_j'(z) + \frac{1}{z} C_{i-1}(z) \, , \, i = 1, 2, \ldots ,$$

(for the explicit formulaes, when $i = 0, 1, 2$ or 3, see [3] p. 169).

By using the equations (62) and (63) we get the expressions

$$A_i(z) = \sum_{j=0}^{i} (-1)^j \chi_j(z) C_{i-j}(z) \, , \, i = 0, 1, \ldots ,$$

$$A_0'(z) = C_0'(z) , A_i'(z) = \sum_{j=0}^{i} (-1)^j \chi_j(z) C_{i-j}'(z) - \frac{1}{z} A_{i-1}(z) ,$$

$$i = 1 , 2 , \ldots ,$$

where $\chi_0(z) = 1$, $\chi_1(z) = \chi = \log (z/2) + \gamma$ and $\chi_j(z)$, $j = 2 , 3, \ldots i$, are polynomials of degree j in $\chi_1(z)$, with coefficients that are polynomials in the values $\zeta(2), \zeta(3), \ldots ,\zeta(j)$ of Riemann Zeta function with rational numerical coefficients.

Now, by theorem 2, we get some results concerning the transcendence and algebraic independence of the values of the expressions $A_i(z)$ and $A_i'(z)$, $i = 0 , 1 , \ldots , 2n + 1$. Here we can omit the terms $A_{i-1}(\alpha)/\alpha$ from the expressions of $A_i'(\alpha)$, because α is assumed distinct from zero and the numbers $A_i(\alpha)$ can be expressed rationally by the elements of any set (55). Let $A_i^*(\alpha)$ be the expressions obtained from $A_i'(\alpha)$ in this way. So we have

$$(64) \quad \begin{cases} A_i(\alpha) = C_i(\alpha) - \chi_1 C_{i-1}(\alpha) + \chi_2 C_{i-2}(\alpha) - + \ldots + (-1)^i \chi_i C_0(\alpha) , \\ A_i^*(\alpha) = C_i'(\alpha) - \chi_1 C_{i-1}'(\alpha) + \chi_2 C_{i-2}'(\alpha) - + \ldots + (-1)^i \chi_i C_0'(\alpha) , \end{cases}$$

where $\chi_j = \chi_j(\alpha)$, $j = 1, 2, \ldots ,i$.

From theorem 2 it follows now the following generalization of MAHLER's [3] theorem 3 conjectured by MAHLER.

Theorem 3. *Let* α *be any algebraic number distinct from zero. Let* $\chi_1 = \log (\alpha/2) + \gamma$, *where* γ *is Euler's constant. Let, further, the numbers*

$$A_i(\alpha) , A_i^*(\alpha) , i = 0 , 1 , \ldots , 2n + 1 \ (n = 0, 1, \ldots) ,$$

be defined by (64). *Then the elements of each of the following* 2^{n+1} *sets of* $3n + 3$ *numbers*

$$\{A_0(\alpha) , A_0^*(\alpha) , A_1(\alpha) \text{ or } A_1^*(\alpha) , \ldots , A_{2i}(\alpha) , A_{2i}^*(\alpha) ,$$

$$A_{2i+1}(\alpha) \text{ or } A_{2i+1}^*(\alpha) , \ldots , A_{2n}(\alpha) , A_{2n}^*(\alpha) , A_{2n+1}(\alpha) \text{ or } A_{2n+1}^*(\alpha)\}$$

are algebraically independent over the rational number field **Q**. *In particular, all* $4n + 4$ *numbers*

$$A_i(\alpha) , A_i^*(\alpha) , i = 0 , 1 , \ldots , 2n + 1 ,$$

are transcendental.

The coefficients χ_j in (64) are polynomials of degree j in $\chi_1 = \log (\alpha/2) + \gamma$, with coefficients that are polynomials in the values of $\zeta(2)$, $\zeta(3), \ldots ,\zeta(j)$ of Riemann Zeta function with rational numerical coefficients. For example,

$$\chi_2 = \frac{1}{2}(\chi_1^2 + \zeta(2)) \, , \; \chi_3 = \frac{1}{3!}(\chi_1^3 + 3\zeta(2)\chi_1 + 2\zeta(3)),$$

$$\chi_4 = \frac{1}{4!}(\chi_1^4 + 6\,\zeta(2)\chi_1^2 + 8\,\zeta(3)\chi_1 + 6\,\zeta(4) + 3\,\zeta^2(2)).$$

Because of the fact

$$\zeta(2j) = (-1)^{j-1}\frac{(2\pi)^{2j}}{2(2j)!}B_{2j}\,, j = 1\,,2\,,\ldots,$$

where B_{2j} are Bernoulli numbers, the transcendence of $\zeta(2), \zeta(4), \ldots,$ $\zeta(2n)$, $n = 1, 2, \ldots,$ is due to LINDEMANN. We have now got transcendental expressions $A_i(\alpha)$ and $A_i^*(\alpha)$ involving Euler's constant γ and the values $\zeta(3), \zeta(5), \ldots, \zeta(2n+1)$, $n = 1, 2, \ldots,$ of Riemann Zeta function. For example, the expressions

$$A_1(2) = \frac{\pi}{2}Y_0(2) - \gamma J_0(2)\,,$$

$$A_2(2) = C_2(2) - \frac{\pi}{2}\gamma Y_0(2) + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right)J_0(2)\,,$$

$$A_3(2) = C_3(2) - \gamma C_2(2) + \frac{\pi}{4}\left(\gamma^2 + \frac{\pi^2}{6}\right)Y_0(2) - \frac{1}{3!}\left(\gamma^3 + \frac{1}{2}\pi^2\gamma + 2\zeta(3)\right)J_0(2)\,,$$

$$A_4(2) = C_4(2) - \gamma C_3(2) + \frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right)C_2(2) - \frac{\pi}{2\cdot 3!}\left(\gamma^3 + \frac{1}{2}\pi^2\gamma + 2\zeta(3)\right)Y_0(2) + \frac{1}{4!}\left(\gamma^4 + \pi^2\gamma^2 + 8\zeta(3)\gamma + \frac{3\pi^4}{20}\right)J_0(2)$$

are transcendental.

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