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LINE SETS AND ASYMPTOTIC BEHAVIOR OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISC

BY

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1. Introduction

Let f denote a complex-valued function in the open unit disc D. Let ζ be a point on the unit circle C. An *arc at* ζ is a curve $J \subset D$ such that $J \cup \{\zeta\}$ is a Jordan arc. The point ζ is an *asymptotic point* of f for the *asymptotic value* a ($a = \infty$ is admitted) if there exists an arc at ζ on which f has the limit a at ζ. Let $A(f)$ denote the set of asymptotic points of f. The class $\mathscr{A}$ consists by definition of all nonconstant holomorphic functions f for which $A(f)$ is a dense subset of C.

A set $S \subset D$ *ends at points* (*of* C) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each component of $S \cap \{1 - \delta < |z| < 1\}$ has diameter less than ε. The class $\mathscr{L}$ consists by definition of all nonconstant holomorphic functions f for which every *level set* $\{z : |f(z)| = \lambda\}$ ends at points.

The classes $\mathscr{A}$ and $\mathscr{L}$ were introduced by G. R. MacLane [6, p. 7]. One of MacLane's theorems [6, p. 10] includes the result

(M) $$\mathscr{A} = \mathscr{L} .$$

For a set S of complex numbers $f^{-1}(S)$ denotes $\{z \in D : f(z) \in S\}$. Let L_1 be a line in the complex plane. If f is a nonconstant holomorphic function for which the *line set* $f^{-1}(L_1)$ ends at points then $f^{-1}(L)$ ends at points for every line L and $e^f \in \mathscr{A}$. These results follow from Corollary 1.1. and Corollary 1.2, respectively, in Section 2.

In Section 3, classes of real-valued harmonic functions, $\mathscr{A}_r$ and $\mathscr{L}_r$, analogous to the classes $\mathscr{A}$ and $\mathscr{L}$ are introduced. By Theorem 2, $\mathscr{A}_r = \mathscr{L}_r$ and this class contains the harmonic conjugate of each of its elements. An aspect of the boundary behavior of these functions is described in Theorem 3. Theorem 4 gives a growth condition that is sufficient for a function to belong to this class.

A function f has a *linearly accessible asymptotic value* at ζ if there exists an arc J at ζ such that f maps J one-to-one into a line L. Some point of L, or possibly ∞, is an asymptotic value of f along J. Let $A_l(f)$ denote the set of asymptotic points of f for linearly accessible asymptotic values.

Let $\mathscr{E}$ consist of all f such that $e^f \in \mathscr{A}$. For a nonconstant holomorphic f, a necessary and sufficient condition for $f \in \mathscr{E}$ is that $A_l(f)$

be a dense subset of C (Theorem 5). For $f \in \mathcal{E}$ and γ a nontrivial subarc of C, Theorem 6 gives some information about the directions of accessibility of the linearly accessible asymptotic values yielded by points of $A_l(f) \cap \gamma$. These results are contained in Section 4.

A closed arc γ (possibly a point) *is the limit of a sequence of arcs* $\{\gamma_n\}$ (denoted $\gamma_n \to \gamma$) if for each $\varepsilon > 0$ each point of γ_n is within ε of γ and each point of γ is within ε of γ_n (both in the spherical metric) for all n sufficiently large. A nontrivial arc $\gamma \subset C$ is a *Koebe arc* of f for the value a ($a = \infty$ is admitted) if there exists a sequence of arcs $\{\gamma_n\}$ ($\gamma_n \subset D$, $n = 1, 2, \ldots$) such that $\gamma_n \to \gamma$ and $f(\gamma_n) \to a$. Define the class $\mathcal{K}$ as follows: $f \in \mathcal{K}$ if f is a nonconstant holomorphic function in D that has no Koebe arcs for the value ∞.

In Section 5, it is shown that the inclusion $\mathcal{K} \subset \mathcal{E}$ holds properly (Theorem 7). Thus if $f \in \mathcal{K}$ then f has no Koebe arcs [6, p. 18]. Section 5 also contains a brief summary of some well-known results concerning the classes mentioned above as well as the class of normal functions of O. Lehto and K. I. Virtanen [5].

The author's Ph.D. thesis (Rice University, 1965) contained Corollary 1.3 and part of Theorem 2.

2. Line sets

Let $\zeta \in C$. Recall that the *range of* f *at* ζ, $R(f, \zeta)$, is the set of all complex numbers α such that $\zeta \in \overline{f^{-1}(\{\alpha\})}$ (the bar denotes closure).

A sequence of arcs $\{\gamma_n\}$ is *in the set* S if $\gamma_n \subset S$ ($n = 1, 2, \ldots$). Note that it is not necessary that any γ_n be a component of S.

Lemma. *Let* f *be a nonconstant holomorphic function in* D. *Suppose that a nontrivial arc* $\gamma \subset C$ *is the limit of a sequence of arcs* $\{\gamma_n\}$ *in* $f^{-1}(L)$ *where* L *is a line. Then for each* $\zeta \in \gamma$, *the complement of* $R(f, \zeta)$ *in the plane consists of at most one complex number.*

Proof. Let $\zeta \in \gamma$. Let N be an open disc centered at ζ, and let $\Delta = N \cap D$. Let φ be a homeomorphism of $\bar{D}$ onto $\bar{\Delta}$ that is holomorphic in D. Choose θ (real) so that $\operatorname{Re}(e^{i\theta} w)$ is constant for $w \in L$, and set $g = e^{i\theta} f(\varphi)$.

The nontrivial arc $\varphi^{-1}(\gamma \cap \bar{\Delta})$ is the limit of a sequence of arcs in $\bigcup \varphi^{-1}(\gamma_n \cap \Delta)$. Since f maps each γ_n into L, it follows from the choice of θ that the union is contained in a level set of e^g. Therefore $e^g \notin \mathcal{L}$, and thus $e^g \notin \mathcal{A}$ by (M). By a result of MacLane [7, Theorem 10] either $g \notin \mathcal{A}$ or g has a Koebe arc for ∞. In either case, since g is not constant, it follows from results of F. Bagemihl and W. Seidel [2, Theorem 1 and

Theorem 3] that g is not normal. Therefore, the complement of $g(D)$ contains at most one complex number. Since $g(D) = f(\Delta)$ and N was an arbitrary disc centered at ζ, the proof is complete.

A *half-line* is a set of the form $\{w + te^{i\psi} : t \geq 0\}$ where w is a complex number and ψ is a real number. The next theorem is the main result of this section.

Theorem 1. *Let f be a nonconstant holomorphic function in D. Let L be a line and let H be a half-line. If $f^{-1}(L)$ does not end at points of C, then there exists a nontrivial subarc of C that is the limit of a sequence of arcs in $f^{-1}(L)$ and the limit of a sequence of arcs in $f^{-1}(H)$.*

Proof. Since $f^{-1}(L)$ does not end at points there exists a nontrivial arc $\gamma \subset C$ that is the limit of a sequence of arcs $\{\gamma_n\}$ in $f^{-1}(L)$. Assume (take a subarc if necessary) that $\gamma \neq C$ and let ζ be the midpoint of γ.

Suppose first that H is not a subset of L and choose a half-line $H' \subset H$ such that $H' \cap L = \emptyset$. By the lemma, there exists a sequence $\{z_n\} \subset D$ such that $z_n \to \zeta$ and $f(z_n) \in H'$ $(n = 1, 2, \ldots)$. Let Γ_n be the component of $f^{-1}(H')$ that contains z_n $(n = 1, 2, \ldots)$. Since $\gamma_m \cap \Gamma_n = \emptyset$ $(m = 1, 2, \ldots;\ n = 1, 2, \ldots)$, $\bar{\Gamma}_n \cap C \neq \emptyset$ $(n = 1, 2, \ldots)$, $\gamma_n \to \gamma$, and $z_n \to \zeta$, it follows that at least one of the two subarcs of γ determined by the removal of ζ is the limit of a sequence of arcs in $\bigcup \Gamma_n$. Since $\bigcup \Gamma_n \subset f^{-1}(H') \subset f^{-1}(H)$, this arc is the limit of a sequence of arcs in $f^{-1}(H)$ as well as the limit of a sequence of arcs in $f^{-1}(L)$.

Suppose next that $H \subset L$. Choose a line L_1 so that $L_1 \cap L = \emptyset$ and let H_1 be a half-line such that $H_1 \subset L_1$. By the preceding paragraph, $f^{-1}(H_1)$ (and hence $f^{-1}(L_1)$) does not end at points. Again by the preceding paragraph, a nontrivial subarc of C is the limit of a sequence of arcs in $f^{-1}(H)$ since $H \cap L_1 = \emptyset$. This is all that the Theorem claims in case $H \subset L$. The proof of Theorem 1 is complete.

The following corollary is immediate.

Corollary 1.1. *If f is a nonconstant holomorphic function in D and $f^{-1}(H)$ ends at points for some half-line H, then $f^{-1}(L)$ ends at points for every line L.*

If c is a nonzero complex number and $\lambda > 0$, then the level set $\{z : |e^{cf(z)}| = \lambda\}$ is equal to the line set $f^{-1}(L)$ where L is the line $\{w : \operatorname{Re} cw = \log \lambda\}$. Thus the following result follows from Corollary 1.1 and (M).

Corollary 1.2. *Let f be a nonconstant holomorphic function D and let c be a nonzero complex number. A necessary and sufficient condition that $e^{cf} \in \mathcal{A}$ is that $f^{-1}(H)$ end at points for some half-line H.*

Remark. In his Ph.D. dissertation (Purdue University, 1971), D. C. Haddad proved that $f \in \mathcal{A}$ if $f^{-1}(L)$ ends at points for some line L. Corollary 1.2 extends this result since $e^f \in \mathcal{A}$ implies that $f \in \mathcal{A}$.

A holomorphic function in D that omits 0 can be written in the form e^F where F is holomorphic in D, so the following result follows from Corollary 1.2 and (M).

Corollary 1.3. *Let f be a nonconstant holomorphic function in D such that f omits 0. If there exists a $\lambda > 0$ such that $L(\lambda) = \{z : |f(z)| = \lambda\}$ ends at points, then $f \in \mathcal{L}$.*

Remark. The definition of $\mathcal{L}$ requires that every level set of f end at points. K. F. Barth and W. J. Schneider [4] have given an example of a holomorphic function f in D for which $L(\lambda)$ ends at points if $0 < \lambda < 1$ but $L(\lambda)$ does not end at points for $\lambda > 1$.

3. Real harmonic functions

Let u be a nonconstant real-valued harmonic function defined in D. Let $\mathcal{A}_r$ be the set of all u such that $A(u)$ is a dense subset of C. Let $\mathcal{L}_r$ be the set of all u such that every level set $u^{-1}(\{\lambda\})$ (λ real) ends at points.

Theorem 2. $\mathcal{A}_r = \mathcal{L}_r$. *Moreover, if v is a harmonic conjugate of a function $u \in \mathcal{A}_r$, then $au + bv \in \mathcal{A}_r$ for any real numbers a and b such that $a^2 + b^2 > 0$. In particular, $\mathcal{A}_r$ contains the harmonic conjugate of each of its elements.*

Proof. If u is a real-valued harmonic function in D and v is a harmonic conjugate of u, let f be the holomorphic function such that $\mathrm{Re}\, f = u$ and $\mathrm{Im}\, f = v$. Let $H = \{w : \mathrm{Re}\, w = 0,\ \mathrm{Im}\, w \geq 0\}$ and $L = \{w : \mathrm{Re}\, w = 1\}$.

If $u \in \mathcal{L}_r$ then $f^{-1}(H)$ ends at points since $f^{-1}(H) \subset u^{-1}(\{0\})$. Therefore, by Corollary 1.2, $e^{cf} \in \mathcal{A}$ for any $c = a - ib$ with a real, b real, and $a^2 + b^2 > 0$. By the definition of $\mathcal{A}$, $\mathrm{Re}\,(cf) = au + bv \in \mathcal{A}_r$. In particular, $u \in \mathcal{A}_r$ so that $\mathcal{L}_r \subset \mathcal{A}_r$.

If $u \notin \mathcal{L}_r$, then $e^f \notin \mathcal{L}$. Thus $f^{-1}(L)$ does not end at points of C by Corollary 1.3. By Theorem 1 there exists a nontrivial arc $\gamma \subset C$ that is the limit of a sequence of arcs in $f^{-1}(L)$ and the limit of a sequence of arcs in $f^{-1}(H)$. But then no interior point of γ can be an asymptotic point of u, so $u \notin \mathcal{A}_r$. This argument shows that $\mathcal{A}_r \subset \mathcal{L}_r$ and completes the proof of Theorem 2.

Remark. A different proof of the equality $\mathcal{A}_r = \mathcal{L}_r$ can be obtained from [6, Theorem 1, p. 10].

Remark. The equivalence of the statements $e^f \in \mathscr{L}$ and $\operatorname{Re} f \in \mathscr{L}_r$ is clear from the definitions. Since $\mathscr{A}_r = \mathscr{L}_r$ and $\mathscr{A} = \mathscr{L}$, it also follows that $e^f \in \mathscr{A}$ is a necessary and sufficient condition for $\operatorname{Re} f \in \mathscr{A}_r$.

Remark. F. B. Ryan and K. F. Barth [9] have constructed functions f and g, both belonging to $\mathscr{A}$, such that $f + g$ is not constant and $f + g \notin \mathscr{A}$. An examination of their construction reveals that $\operatorname{Re} f \in \mathscr{A}_r$ and $\operatorname{Re} g \in \mathscr{A}_r$. Then $\operatorname{Re} f + \operatorname{Re} g \notin \mathscr{A}_r$ because $\operatorname{Re}(f + g) \in \mathscr{A}_r$ implies that $e^{f+g} \in \mathscr{A}$ which implies that $f + g \in \mathscr{A}$.

A *level curve* of u is a component of a level set of u. A level curve Λ is called *simple* if $f'(z) \neq 0$ for all $z \in \Lambda$ where f is a holomorphic function in D such that $\operatorname{Re} f = u$.

Theorem 3. *Let* $u \in \mathscr{L}_r$ *and let* γ *be a nontrivial open subarc of* C. *Then either*

(1) *there exists a point* $\zeta \in \gamma$ *and an arc* J *at* ζ *such that* J *is contained in a simple level curve of* u, *or*

(2) *there exists a real number* B *such that for each* $\zeta \in \gamma$, $u(z) \to B$ *as* $z \to \zeta$ $(z \in D)$.

Remark. In case (2) it follows from the reflection principle of Schwarz that u has a harmonic continuation across γ.

Proof. Suppose that (1) does not hold. It will be shown that (2) must hold.

Let γ_1 be a nontrivial closed subarc of γ. Suppose, without loss of generality, that there exist α and β, $-\pi < \alpha < \beta < \pi$, such that $\gamma_1 = \{e^{it} : a \leq t \leq \beta\}$. For each r, $0 < r < 1$, let $S(r) = \{z : r < |z| < 1,\ \alpha < \arg z < \beta\}$, $B^*(r) = \sup\{u(z) : z \in S(r)\}$, and $B_*(r) = \inf\{u(z) : z \in S(r))\}$. Let B^* (resp. B_*) denote the limit of $B^*(r)$ (resp. $B_*r())$ as $r \to 1$. It is clear that

$$B_* \leq B^* .$$

Let f be a holomorphic function in D such that $\operatorname{Re} f = u$. If $B^* = -\infty$ (resp. $B_* = +\infty$) then it follows from the reflection principle of Schwarz and the identity theorem that e^f (resp. e^{-f}) is constant. But u is not constant, so

$$B^* > -\infty \text{ and } B_* < +\infty .$$

Now suppose that $B_* < B^*$. Choose λ, $B_* < \lambda < B^*$, so that $f'(z) \neq 0$ for all z such that $u(z) = \lambda$. Since $B_* < \lambda < B^*$, there exists a sequence $\{z_n\} \subset u^{-1}(\{\lambda\})$ that converges to some point $\zeta \in \gamma_1$. Let Λ be any simple level curve of u. Since $u \in \mathscr{L}_r$, $\bar{\Lambda} \cap C$ consists of either one or two points. Thus if $\zeta \in \bar{\Lambda} \cap C$, there exists an arc J at ζ such that $J \subset \Lambda$. Therefore, the assumption that (1) fails to hold implies that $\bar{\Lambda} \cap \gamma = \emptyset$. Since each level curve $\Gamma(\lambda)$ in the level set $u^{-1}(\{\lambda\})$ is simple and $z_n \to \zeta \in \gamma_1$, it follows that at most finitely many of the z_n can belong to a single level curve $\Gamma(\lambda)$. Also, for each r, $0 < r < 1$, at most finitely many of the level curves $\Gamma(\lambda)$ intersect the disc $D_r = \{|z| \leq r\}$. Since $\overline{\Gamma(\lambda)} \cap \gamma = \emptyset$ and $\overline{\Gamma(\lambda)} \cap C \neq \emptyset$ for each level curve $\Gamma(\lambda)$, it follows that at least one of the two nontrivial subarcs of γ determined by the removal of ζ is the limit of a sequence of arcs in $u^{-1}(\{\lambda\})$. But this contradicts the fact that $u \in \mathscr{L}_r$.

Therefore, $B^* = B_* = B$ and (2) holds. This completes the proof of Theorem 3.

If h is a real-valued function in D, let $h^+(z) = \max(h(z), 0)$. The following theorem is an immediate consequence of a result of MacLane [6, p. 36].

Theorem 4. *Let u be a nonconstant real-valued harmonic function in D. Suppose that there exists a set $\Theta \subset [0, 2\pi]$ such that Θ is dense in $[0, 2\pi]$ and such that*

$$\int_0^1 (1 - r)u^+(re^{i\theta})dr < \infty \quad (\theta \in \Theta) . \tag{3}$$

Then $u \in \mathscr{A}_r$.

Proof. Let f be a holomorphic function in D such that $\operatorname{Re} f = u$. Since $\log^+ |e^{f(z)}| = u^+(z)$ for each z it follows from (3) and [6, p. 36] that $e^f \in \mathscr{A}$. Thus $u \in \mathscr{A}_r$. This completes the proof of Theorem 4.

4. Linearly accessible asymptotic values

Theorem 5. *Let f be a nonconstant holomorphic function in D. A necessary and sufficient condition for $f \in \mathscr{E}$ is that $A_l(f)$ be a dense subset of C.*

Proof. Suppose first that $f \notin \mathscr{E}$. Then $e^f \notin \mathscr{L}$ by (M) and the definition of $\mathscr{E}$. Thus there exists a line L such that $f^{-1}(L)$ does not end at points of C. Let H be a half-line such that $H \cap L = \emptyset$. By Theorem 1 there exists a nontrivial arc $\gamma \subset C$ that is the limit of a sequence of arcs in $f^{-1}(L)$ and the limit of a sequence of arcs in $f^{-1}(H)$. Since $H \cap L = \emptyset$, f can not have a linearly accessible asymptotic value at an interior point of γ. Thus $A_l(f)$ is not a dense subset of C. This proves the sufficiency of the condition.

Now if $f \in \mathscr{E}$ then $e^f \in \mathscr{A}$ and thus $u = \operatorname{Re} f \in \mathscr{A}_r$. Let γ be an open arc of C. Since $u \in \mathscr{A}_r$ and $\mathscr{A}_r = \mathscr{L}_r$, Theorem 3 applies; if either

(1) or (2) holds the conclusion $A_l(f) \cap \gamma \neq \emptyset$ follows. This proves the necessity of the condition and concludes the proof of Theorem 5.

For each θ, $0 \leq \theta < \pi$, let $\mathcal{P}(\theta)$ denote the set of all lines in the w-plane that have the angle of inclination θ with respect to the positive u-axis ($w = u + iv$). Then f has an asymptotic value at ζ that is *accessible through* $\mathcal{P}(\theta)$ if there exists an arc J at ζ such that f maps J one-to-one into a line L where $L \in \mathcal{P}(\theta)$. For each θ, $0 \leq \theta < \pi$, let $A_l^{\theta}(f)$ denote the set of asymptotic points of f for asymptotic values accessible through $\mathcal{P}(\theta)$.

Theorem 6. *Let $f \in \mathcal{E}$. Let γ be a nontrivial open subarc of C. If there exists a ψ, $0 \leq \psi < \pi$, such that $A_l^{\psi}(f) \cap \gamma = \emptyset$, then f has an analytic continuation across γ and the continuation maps γ one-to-one into a line L where $L \in \mathcal{P}(\psi)$.*

Proof. Let $c = ie^{-i\psi}$. By Theorem 5 (or Corollary 1.2) $cf \in \mathcal{E}$ and it follows that $\mathrm{Re}\,(cf) \in \mathcal{A}_r$. The transformation $T(w) = cw$ maps the family $\mathcal{P}(\psi)$ one-to-one onto the family $\mathcal{P}(\pi/2)$. By the hypothesis on ψ, $\mathrm{Re}\,(cf)$ must satisfy condition (2) of Theorem 3. Thus f has an analytic continuation F across γ and $\mathrm{Re}\,(cF)$ is constant on γ. Therefore F maps γ into a line L where $L \in \mathcal{P}(\psi)$.

If the derivative of F vanished at some point ζ of γ then it would follow from local properties of analytic functions that $\zeta \in A_l^{\psi}(f)$ contradicting the hypothesis. Since F' does not vanish on γ and F maps γ into a line it follows that F is one-to-one on γ. This completes the proof of Theorem 6.

5. Koebe arcs

Theorem 7. *$\mathcal{K} \subset \mathcal{E}$ and the inclusion is proper.*

Proof. Let f be a nonconstant holomorphic function in D such that $f \notin \mathcal{E}$. Then by Corollary 1.2 there exists a nontrivial arc $\gamma \subset C$ that is the limit of a sequence of arcs in $f^{-1}(L)$ where $L = \{w : \mathrm{Re}\, w = 0\}$.

Assume $\gamma \neq C$ (take a subarc if necessary) and let ζ be the midpoint of γ. By the Lemma, there exists a sequence $\{z_n\} \subset D$ such that $z_n \to \zeta$ and $\mathrm{Re} f(z_n) \to +\infty$. For each $n = 1, 2, \ldots$, let L_n be the line $\{w : \mathrm{Re}\, w = \mathrm{Re} f(z_n)\}$ and let Λ_n be the component of $f^{-1}(L_n)$ that contains z_n. Since $z_n \to \zeta$, $\Lambda_n \cap \gamma_m = \emptyset$ ($m = 1, 2, \ldots; n = 1, 2, \ldots$), $\bar{\Lambda}_n \cap C \neq \emptyset$ ($n = 1, 2, \ldots$), and $\gamma_n \to \gamma$, it follows that at least one of the two subarcs of γ determined by the removal of ζ is the limit of a sequence of arcs $\{\gamma_n'\}$ in $\cup \Lambda_n$ such that $f(\gamma_n') \to \infty$. Thus $f \notin \mathcal{K}$. This proves that $\mathcal{K} \subset \mathcal{E}$.

Let $M(f, r)$ denote the maximum modulus of f on the circle $\{|z| = r\}$. Let $\mu(r)$ be a positive, increasing function on $[0, 1)$ such that $\mu(r) \to +\infty$ as $r \to 1$. It follows from F. Bagemihl, P. Erdös and W. Seidel [1, Theorem 3 and Theorem 5] that there exists a function f holomorphic in D such that $M(f, r) < \mu(r)$, $0 \leq r < 1$, and f has a Koebe arc for ∞. If $\mu(r)$ is chosen so that $(1 - r)\mu(r)$ is integrable on the interval $[0, 1)$ then it follows from Theorem 4 that $\mathrm{Re} f \in \mathcal{A}_r$ or equivalently $f \in \mathcal{E}$. Thus the inclusion is proper and the proof of Theorem 7 is complete.

Let $\mathcal{N}$ denote the set of nonconstant normal holomorphic functions in D. Then

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{E} \subset \mathcal{A}$$

and each inclusion is proper.

The first inclusion was obtained by Bagemihl and Seidel [2]. Let $w(z) = c(1 + z)/(1 - z)$ where $c = \pi i/4$ and let $g(z) = e^{w(z)}$. The restriction of g to D has both the asymptotic values 0 and ∞ at 1. Therefore it follows from Lehto and Virtanen [5] (or see [8, p. 86]) that g is not normal. However, it is clear that $g \in \mathcal{K}$; so $\mathcal{N} \neq \mathcal{K}$.

The final inclusion follows immediately from the definitions. The propriety of the third inclusion was proved by Barth and Schneider [3].

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