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# CHARACTERIZATION OF THE QUASICONFORMALITY BY ARC FAMILIES OF EXTREMAL LENGTH ZERO

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## Introduction

According to J. Väisälä [7], a homeomorphism $f : D \rightleftharpoons D^*$ ($D, D^*$ domains of the Euclidean $n$-space $R^n$) is called *K-qc* (*K-quasiconformal mapping*) $(1 \leqq K < \infty)$ in $D$ if

$$\frac{M(\Gamma)}{K} \leqq M\,(\Gamma^*) \leqq KM(\Gamma) \tag{1}$$

for every arc family $\Gamma$ contained in $D$. The preceding inequality may be written also

$$\frac{\lambda(\Gamma)}{K} \leqq \lambda(\Gamma^*) \leqq K\lambda(\Gamma)\,. \tag{1'}$$

However, it is possible to characterize the quasiconformality by asking that the preceding inequalities hold only for a certain class of arc families, as for instance for the arc families joining the boundaries of a topological cylinder, or of rings, or even of spherical rings (see J. Väisälä [7, 8], F. Gehring [5]). In the present paper, we shall characterize the quasiconformality by the condition that arc families of extremal length zero are mapped into arc families of extremal length zero. As simple consequences, we derive the characterization of the quasiconformality by the invariance of the property of arc families of having extremal length $> 0$, or infinite (or finite) modulus.

This result generalizes the corresponding theorem in plane obtained by H. Renggli [6]. His proof however, (which is a little more complicated) appeals to Riemann theorem from conformal mappings (which is no more true for $n > 2$).

Finally, we show that the invariance of the property of a family of closed sets contained in a domain $D$ of having finite (or infinite) modulus implies the quasiconformality of the corresponding mapping.

## 1. Definitions and notations

Let $\Gamma$ be a family of arcs $\gamma \subset R^n$, where by an arc we mean a homeomorphic image of the segment (0, 1), and let $F(\Gamma)$ be the class of functions $\varrho(x)$ such that

1° $\varrho(x) \geqq 0$ in $R^n$;
2° $\varrho(x)$ is Borel measurable;
3° $\int_\gamma \varrho(x)\, ds \geqq 1$ for each $\gamma \in \Gamma$.

The *modulus* of $\Gamma$ is defined as

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho(x)^n d\tau \ .$$

Its inverse

$$\gamma(\Gamma) = \frac{1}{M(\Gamma)}$$

is the *extremal length* of $\Gamma$.

Clearly, if all arcs $\gamma \in \Gamma$ are contained in a Borel set $E \subset R^n$, then it is sufficient to consider the family $F_E(\Gamma)$ of functions $\varrho \in F(\Gamma)$ defined only in $E$ and satisfying conditions 1°, 2°, 3° in $E$, or in other words, to consider only the functions $\varrho$ satisfying the additional condition $\varrho_{|CE} = 0$ (where $\varrho_{|CE}$ means the restriction of $\varrho$ to the set $CE$). Then, the modulus of $\Gamma$ may be written in the form

$$M(\Gamma) = \inf_{\varrho \in F_E(\Gamma)} \int_E \varrho^n d\tau \ .$$

A *ring* is a domain $A$ homeomorphic to a spherical ring (an annulus), i.e. the domain contained between two concentric spheres. Let $C_0$, $C_1$ be the bounded, respectively unbounded, component of the complement $CA$ of $A$ and $F_k = \partial C_k$ $(k = 0, 1)$ the boundary components of the ring $A$. Let $\Gamma_A$ be the family of all arcs $\gamma$ joining the boundary components of $A$ in $A$, i.e. such that their endpoints $a_k$ belong to $F_k$ $(k = 0, 1)$, and the open arcs $\gamma \subset A$. The modulus of ring $A$ is

$$\text{mod}\, A = \left[\frac{n\omega_n}{M(\Gamma_A)}\right]^{\frac{1}{n-1}} \ . \tag{2}$$

We say that a closed set $\sigma$ separates $F_0$ from $F_1$ in $A$ if $\sigma \subset A$ and every $\gamma \in \Gamma_A$ meets $\sigma$.

The modulus of a family $\Sigma_A$ of sets $\sigma$ separating the boundary components of $A$ is defined as

$$M(\Sigma_A) = \inf_{\varrho \in F(\Sigma_A)} \int_{R^n} \varrho^n d\tau \ ,$$

where $F(\Sigma_A)$ is the class of functions satisfying conditions 1°, 2° and

3'° $\int_\sigma \sigma^{n-1}\, d\sigma \geqq 1$ for every $\sigma \in \Sigma_A$.

## 2. Characterizations of quasiconformality

A homeomorphism $f\colon D \rightleftharpoons D^*$ is said to be *$K$-qc* $(1 \leqq K < \infty)$ *in $D$ according to Gehring's metric definition* if

$$\delta_L(x) = \overline{\lim_{r \to 0}} \frac{L(x, r)}{l(x, r)},$$

where

$$L(x, r) = \max_{|x' - x| = r} |f(x') - f(x)|, \quad l(x, r) = \min_{|x' - x| = r} |f(x') - f(x)|,$$

is bounded in $D$ and $\delta_L(x) \leqq K$ a. e. (almost everywhere) in $D$.

*Proposition 1. This definition of qc is equivalent to Väisälä's geometric definition [characterized by the double inequality* (1)].

(For the proof, see for instance our monograph [2], theorem 1, p. 126 and theorem 2, p. 127, or our paper [1].)

*Theorem 1. A homeomorphism $f\colon D \rightleftharpoons D^*$ is qc in $D$ iff there exists a constant $C$ and, for every $x \in D$, there exists a positive number $r(x)$, such that $0 < r \leqq r(x)$ implies* $\operatorname{mod} A^*(x, r) \leqq C$, *where*

$$A^*(x, r) = \{x^*; l(x, r) \leqq |x^* - f(x)| \leqq L(x, r)\},$$

*and, in the case $l(x, r) = L(x, r)$, we consider* $\operatorname{mod} A^*(x, r) = 0$.

Since

$$\operatorname{mod} A^*(x, r) = \log \frac{L(x, r)}{l(x, r)},$$

the condition is clearly sufficient (it implies $\delta_L$ to be bounded). In order to prove that this is also necessary, we observe that if $f$ is $qc$, there exists a constant $1 \leqq K < \infty$, such that $\delta_L(x) \leqq K$ in $D$, and then, given $\varepsilon > 0$, to every $x \in D$, we can associate an $r(x) > 0$, such that $\{|x^* - f(x)| < L(x, r)\} \subset D^*$ and

$$\frac{L(x, r)}{l(x, r)} < K + \varepsilon$$

for every $r < r(x)$, hence

$$\operatorname{mod} A^*(x, r) = \log \frac{L(x, r)}{l(x, r)} < \log (K + \varepsilon) < \infty$$

for every $x \in D$ and $0 < r < r(x)$.

*Proposition 2.* $\mathrm{mod}\,\{f^{-1}[A^*(x, r)]\} < \varkappa$, *where* $\varkappa = \varkappa(n)$.

(For the proof, see J. Väisälä [7].)

*Proposition 3. The homeomorphism* $f^{-1}\colon D^* \rightleftharpoons D$ *is* $K$-*qc iff* $f\colon D \rightleftharpoons D^*$ *is* $K$-*qc.*

*Proposition 4. If* $A_1, \ldots, A_m$ *are disjoint rings, each of which separates the boundary components of a ring* $A$, *then*

$$\mathrm{mod}\, A \geqq \sum_{k=1}^{m} \mathrm{mod}\, A_m .$$

(For the proof, see F. Gehring [5].)

*Lemma 1. If a homeomorphism* $f\colon D \rightleftharpoons D^*$ *is not qc in* $D$, *then there is a sequence* $\{A_m\}$ *of disjoint rings* $A_m \subset\subset D$ *(i.e. with* $\bar{A}_m \subset D$*), such that*

$$\mathrm{mod}\, A_m < \frac{\varkappa}{m^2}, \quad \mathrm{mod}\, A_m^* > m^2 \quad (m = 1, 2, \ldots). \tag{3}$$

Indeed, by theorem 1, there exist two sequences $\{x_m\}$ and $\{r_m\}$, such that

$$\mathrm{mod}\, A^*(x_m, r_m) > m^4. \tag{4}$$

Now, let us prove that the rings $A_m = A(x_m, r_m) = f^{-1}\{A^*(x_m, r_m)\}$ can be chosen disjoint. Suppose first that there is an index $m_0$ such that $x_m = x_{m0}$ for an infinity of indices $m$. Then, considering this sequence and relabeling, we shall have a sequence $\{x_m\}$ with $x_1 = x_2 = \ldots = x_0$. Since $L(x, r) \to 0$ as $r \to 0$, it follows that for a fixed $r_m$, it is possible to choose $r_{m+1}$ so that $r_{m+1} < r_m$ and $L(x_0, r_{m+1}) < l(x_0, r_m)$, and then, again $A_m \,.\, A_{m+1} = \emptyset$, hence $A_m \,.\, A_p = \emptyset$ for all indices $m \neq p$.

Next, we can suppose (without loss of generality) that the sequence $\{x_m\}$ is such that $x_m \neq x_p$ for $m \neq p$. We may assume even that $\{x_m\}$ is discrete. Indeed, if it is not discrete, it has a limit point $x_0$, and we may pass to a subsequence which converges to $x_0$ without containing $x_0$.

Then, according to proposition 2, $\mathrm{mod} f^{-1}[A^*(x_m, r_m)] < \varkappa$. Finally, for a fixed $m$, decompose the spherical ring $A^*(x_m, r_m)$ in $m^2$ spherical rings $A_q^*(x_m, r_m)$ such that

$$\mathrm{mod}\, A_q^*(x_m, r_m) = \frac{1}{m^2} \mathrm{mod}\, A^*(x_m, r_m) < m^2 \quad (q = 1, \ldots, m^2),$$

which is clearly possible. If we denote by $A_m$ the ring $f^{-1}[A_q^*(x_m, r_m)]$ satisfying the condition

$$\mathrm{mod}\, A_m = \min\{\mathrm{mod} f^{-1}[A_1^*(x_m, r_m)], \ldots, \mathrm{mod} f^{-1}[A_{m^2}^*(x_m, r_m)]\},$$

then, by the preceding proposition the following is evident:

$$\operatorname{mod} A_m \leqq \frac{1}{m^2} \operatorname{mod} f^{-1}[A^*(x_m, r_m)] < \frac{\varkappa}{m^2}.$$

*Proposition 5. if* $\Gamma = \bigcup_m \Gamma_m$ *and* $\Gamma_m$ *are separate, then*

$$M(\Gamma) = \sum_m M(\Gamma_m).$$

We recall that the arc families $\Gamma_m$ ($m = 1, 2, \ldots$) are said to be *separate* if there exists a sequence of disjoint Borel sets $\{E_m\}$ such that $H^1(\gamma_m - E_m) = 0$ for every $\gamma_m \in \Gamma_m$, where $H^1$ is the linear Hausdorff measure.

(For the proof of proposition 5, see for instance B. Fuglede [3].)

*Theorem 2. A necessary and sufficient condition for a homeomorphism* $f: D \rightleftharpoons D^*$ *to be qc in* $D$ *is that* $\lambda(\Gamma) = 0$ ($> 0$) *iff* $\lambda(\Gamma^*) = 0$ ($> 0$) *for every* $\Gamma$ *contained in* $D$.

(1′) allows us to conclude that the condition is necessary.

In order to prove that the condition is also sufficient, let us suppose $f$ is not *qc*. But then, according to the preceding lemma, there exists a sequence $\{A_m\}$ of disjoint rings so that (3) hold, hence, on account of (2),

$$M(\Gamma_{A_m}) = \frac{n\omega_n}{(\operatorname{mod} A_m)^{n-1}} > \frac{n\omega_n\, m^{2(n-1)}}{\varkappa^{n-1}}, \quad M(\Gamma_{A_m^*}) < \frac{n\omega_n}{m^{2(n-1)}},$$

and then, by the preceding proposition, since $\Gamma_m$ ($m = 1, 2, \ldots$) are separate,

$$M(\Gamma) = \sum_m M(\Gamma_{A_m}) > \frac{n\omega_n}{\varkappa^{n-1}} \sum_m m^{2(n-1)} = \infty,$$

$$M(\Gamma^*) = \sum M(\Gamma_{A_m^*}) < n\omega_n \sum_m \frac{n\omega_n}{m^{2(n-1)}} < \infty,$$

where $\Gamma = \bigcup_m \Gamma_{A_m}$, $\Gamma^* = \bigcup_m \Gamma_{A_m^*}$. Hence,

$$\lambda(\Gamma) = 0, \quad \lambda(\Gamma^*) > 0.$$

In a similar way, we can obtain also arc families $\Gamma$, such that $\lambda(\Gamma) > 0$ and $\lambda(\Gamma^*) = 0$, taking into account that, by proposition 3, if $f$ is not *qc*, neither $f^{-1}$ is, and the proof of the theorem is complete.

And now, we shall give some other characterizations of the quasiconformality as simple consequences of the preceding theorem.

First, we remind that

A homeomorphism $f: D \rightleftharpoons D^*$ is $K$-*qc according to Gehring's geometric definition* iff

$$\frac{\operatorname{mod} A}{K} \leqq \operatorname{mod} A^* \leqq K \operatorname{mod} A$$

for every $A \subset\subset D$.

*Theorem 3. A homeomorphism* $f\colon D \rightleftharpoons D^*$ *is qc in* $D$ *iff for every sequence* $\{A_m\}$ *of disjoint rings* $A_m \subset\subset D$, $\sum_m \operatorname{mod} A_m$ *is divergent (convergent) iff* $\sum_m \operatorname{mod} A_m^*$ *is divergent (convergent).*

It is a direct consequence of the preceding lemma and Gehring's geometric definition.

Let us consider a sequence $\{\Sigma_m\}$ of families of closed sets. $\Sigma_m$ are said to be separate if there exists a sequence of Borel sets $\{E_m\}$ such that $H^{n-1}(\sigma_m - E_m) = 0$ for every $\sigma_m \in \Sigma_m$, where $H^{n-1}$ means the $(n-1)$-dimensional Hausdorff measure.

*Proposition 6. If* $\Sigma = \bigcup_m \Sigma_m$ *and* $\Sigma_m$ *are separate families of closed sets, then*

$$M(\Sigma) = \sum_m M(\Sigma_m) .$$

(For the proof, see B. Fuglede [3]).

*Lemma 2.* $M(\Sigma_A) = \dfrac{1}{M(\Gamma_A)^{\frac{1}{n-1}}}$.

This follows, for example, from [2, Corollary, p. 124] and [1, (49)].

*Theorem 4. If a homeomorphism* $f\colon D \rightleftharpoons D^*$ *has the property that:* $M(\Sigma) = \infty\ (< \infty)$ *iff* $M(\Sigma^*) = \infty\ (< \infty)$ *for every family* $\Sigma$ *of closed sets contained in* $D$, *then* $f$ *is qc in* $D$.

Indeed, suppose $f$ is not $qc$, then, by theorem 3, there exists a sequence of disjoint rings $\{A_m\}$, such that $\sum_m \operatorname{mod} A_m < \infty$ and $\sum_m \operatorname{mod} A_m^* = \infty$, but then, on account of the definition of the modulus of a ring and by the preceding lemma and proposition 6,

$$(n\omega_n)^{\frac{1}{n-1}} M(\Sigma) = \sum_m (n\omega_n)^{\frac{1}{n-1}} M(\Sigma_{A_m}) = \sum_m \left[\frac{n\omega_n}{M(\Gamma_{A_m})}\right]^{\frac{1}{n-1}} = \sum_m \operatorname{mod} A_m < \infty$$

and

$$(n\omega_n)^{\frac{1}{n-1}} M(\Sigma^*) = m\Sigma (n\omega_n)^{\frac{1}{n-1}} M(\Sigma_{A_m^*}) = \sum_m \left[\frac{n\omega_n}{M(\Gamma_{A_m^*})}\right]^{\frac{1}{n-1}} = \sum_m \operatorname{mod} A_m^* = \infty ,$$

as desired, where $\Sigma = \bigcup_m \Sigma_{A_m}$.

*Corollary. If a homeomorphism* $f: D \rightleftharpoons D^*$ *has the property that:* $\lambda(\Sigma) = 0\ (> 0)$ *iff* $\lambda(\Sigma^*) = 0\ (> 0)$ *for every family* $\Sigma$ *of closed sets contained in* $D$, *then* $f$ *is* $qc$.

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