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CONFORMAL CAPACITY AND QUASIREGULAR MAPPINGS

BY

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1. Introduction and terminology

1.1 By a *condenser* in R^n we mean a triple $E = (D; C_0, C_1)$ where D is a domain in R^n, and C_0, C_1 are disjoint compact sets in $\bar{D}$, the closure of D in $\bar{R}^n = R^n \cup \{\infty\}$. The *capacity* (namely the conformal capacity or the n-capacity) is defined by

$$\operatorname{cap} E = \inf_{u \varepsilon W(E)} \int_D |\nabla u|^n dm,$$

where $W(E)$ is the set of all non-negative, continuous and ACL functions $u : D \to R^1$ such that $u(x) \to j$ as $x \to x_j$ for all $x_j \varepsilon C_j$, $j = 0,1$. If $C_0 = \phi$ or $C_1 = \phi$ we set $\operatorname{cap} E = 0$. Note that $\operatorname{cap}(D; C_0, C_1) = \operatorname{cap}(D; C_1, C_0)$.

1.2. Let $f : D \to R^n$ be a non-constant *quasiregular* mapping (see [2] for terminology) and $E = (D; C_0, C_1)$ a condenser. The *cluster set* of f on a set $A \subset \bar{D}$ is denoted by $C(f, A)$, i.e. $C(f, A)$ is the set of all points $y \in \bar{R}^n$ such that $f(x) \to y$ as $x \to a$ for $a \varepsilon A$. We will show that

(a) $C(f, C_0) \cap C(f, C_1) = \phi \Rightarrow \operatorname{cap} E \leq N(f, D) K_0(f) \operatorname{cap} fE$, where $fE = (fD; C(f, C_0), C(f, C_1))$, and that

(b) $C_0, C_1 \subset D$ and $C(f, \partial D) \subset \partial fD \Rightarrow \operatorname{cap} \tilde{f}E \leq K_I(f) \operatorname{cap} E$, where $\tilde{f}E = (fD; fC_0 \setminus f(D \setminus C_0), fC_1)$.

Here $K_0(f)$ and $K_I(f)$ are the outer and inner dilatations of f in D, and $N(f, D) = \sup$ cardinality $f^{-1}(y)$ over all $y \varepsilon R^n$, see [2].

We conclude with an application of (b) to the boundary behavior of quasiregular mappings.

1.3. The notation and terminology will usually be as in [2]. Quasiconformal is abbreviated by *qc*, quasiregular by *qr*. The $L^n(D)$ norm of $|\nabla u| = \left(\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{1/2}$ will be denoted by $\|\nabla u\|_D$ or by $\|\nabla u\|$.

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2. Preliminary results on the capacity of condensers

2.1. Let $E = (D; C_0, C_1)$ be a condenser in R^n. Let $W_0(E)$ denote the set of all non-negative, continuous and ACL functions $u : D \to R^1$ such that $C_0 \cap \operatorname{spt} u = \phi$ and $C_1 \cap \operatorname{spt}(1 - u) = \phi$. Finally, let $W_0^\infty(E) = W_0(E) \cap C^\infty(D)$. Clearly $W_0^\infty(E) \subset W_0(E) \subset W(E)$.

2.2. Lemma. $\operatorname{cap} E = \inf\{\|\nabla u\|^n : u \in W_0(E)\}$.

Proof. We may assume that $\operatorname{cap} E < \infty$, since otherwise there is nothing to prove. For $u \in W(E)$ with $\|\nabla u\| < \infty$ and $0 < \delta < \frac{1}{2}$, we define $u_\delta : D \to R^1$ by setting $u_\delta(x) = 0$ iff $0 \leq u(x) \leq \delta$, $u_\delta(x) = 1$ iff $1 - \delta \leq u(x) \leq 1$ and $u_\delta(x) = \dfrac{u(x) - \delta}{1 - 2\delta}$ otherwise. It is not hard to verify that $u_\delta \in W_0(E)$ and that $\|\nabla u_\delta\| \leq \dfrac{\|\nabla u\|}{1 - 2\delta}$. Now take the infimum of $\|\nabla u\|^n$ over all $u \in W(E)$; then let $\delta \to 0$ and the result follows.

2.3. Lemma. $\operatorname{cap} E = \inf\{\|\nabla u\|^n : u \in W_0^\infty(E)\}$.

Proof. We may assume that $\operatorname{cap} E < \infty$. Given $\varepsilon > 0$ and $u \in W_0(E)$ with $\|\nabla u\| < \infty$, let

$$\delta = \min\{\operatorname{dist}(C_0, \operatorname{spt} u)\,,\ \operatorname{dist}(C_1\,, \operatorname{spt}(1 - u)\}\,.$$

We now continue with the techniques of [3]. Define $D_0 = D_{-1} = \phi$ and

$$D_i = \left\{x \in D : |x| < i \quad \text{and} \quad \operatorname{dist}(x\,, \partial D) > \frac{1}{i}\right\},\ i = 1, 2, \ldots\,.$$

Choose a partition of unity $\sum_{i=1}^{\infty} \psi_i \equiv 1$ on D such that

$$\operatorname{spt} \psi_i \subset D_{i+1} \setminus D_{i-1}\,,\quad i = 1\,, 2\,, \ldots\,,$$

and let $\varphi_i : R^n \to R^1$, $i = 1, 2, \ldots$, be non-negative C^∞ functions such that

$$\operatorname{spt} \varphi_i \subset B^n\left(\frac{1}{(i+1)\,(i+2)}\right) \cap B^n\left(\frac{\delta}{2}\right),\ \int_{R^n} \varphi_i(x)\,dm(x) = 1 \quad \text{and}$$

$$\|\nabla(\varphi_i * \psi_i u) - \nabla(\psi_i u)\| < \frac{\varepsilon}{2^i}\,.$$

$\operatorname{spt} \varphi_i * \psi_i u \subset D_{i+2} \setminus D_{i-2}$; hence the series

$$v = \sum_{i=1}^{\infty} \varphi_i * \psi_i u$$

converges and defines a function in $W_0^\infty(E)$. Finally for $k = 1, 2, \ldots$ we have:

$$\|\nabla u - \nabla v\|_{D_k} = \|\sum_{i=1}^{k+1} \nabla(\varphi_i * \psi_i u) - \nabla(\psi_i u)\|_{D_k} \leq \sum_{i=1}^{k+1} \|\nabla(\varphi_i * \psi_i u) - \nabla(\psi_i u)\| < \varepsilon$$

Letting $k \to \infty$, we conclude by Lebesgue monotone convergence theorem and Minkowski's inequality that $\|\nabla v\| \leq \|\nabla u\| + \varepsilon$. Now take the infimum of $\|\nabla u\|$ over $W_0(E)$, let $\varepsilon \to 0$ and the result follows by 2.2 and the inclusion $W_0^\infty(E) \subset W_0(E)$.

3. Capacity inequalities

3.1. Theorem. *Let $f : D \to R^n$ be non-constant and qr, and let $E = (D; C_0, C_1)$ be a condenser. If $C(f, C_0) \cap C(f, C_1) = \phi$, then*

$$\operatorname{cap} E \leq N(f, D) K_0(f) \operatorname{cap} fE,$$

where $fE = (fD; C(f, C_0), C(f, C_1))$.

Proof. We may assume that $N(f, D) < \infty$ and that $\operatorname{cap} fE < \infty$. (Actually, the capacity of a condenser is always finite.) Given $v \in W_0^\infty(fE)$ with $\|\nabla v\|_{fD} < \infty$, we define $u = v \circ f$. Clearly u is non-negative and continuous in D. Let $U' = fD \setminus \operatorname{spt} v$ and $U = f^{-1}(U')$. Then $\operatorname{spt} u = \overline{D \setminus U}$; and since $C(f, C_0) \cap \operatorname{spt} v = \phi$ it follows, by the definition of a cluster set and the nature of U and U', that $C_0 \cap \operatorname{spt} u = \phi$. In the same way $C_1 \cap \operatorname{spt}(1 - u) = \phi$. Finally f is ACL and differentiable a.e. in D, cf. [2, 2.26], and $v \in C^\infty(fD)$, hence u is ACL and

$$|\nabla u|^n \leq |(\nabla u) \circ f|^n |f'|^n \leq K_0(f) |(\nabla v) \circ f|^n J(f) \quad \text{a.e. in } D,$$

where $J(f)$ denotes the Jacobian of f in D. Consequently $u \in W_0(E)$ and

$$\operatorname{cap} E \leq \|\nabla u\|^n \leq K_0(f) \int_D |(\nabla v) \circ f|^n J(f) dm \leq K_0(f) N(f, D) \|\nabla v\|_{fD}^n .$$

Now take the infimum of $\|\nabla v\|_{fD}^n$ over all $v \in W_0^\infty(fD)$ and the result follows by virtue of 2.3.

3.2. Theorem. *Let $f: D \to R^n$ be qr and $E = (D; C_0, C_1)$ a condenser with $C_0, C_1 \subset D$ and $C(f, \partial D) \subset \partial fD$. Then*

$$\operatorname{cap} \tilde{f}E \leq K_I(f) \operatorname{cap} E,$$

where $\tilde{f}E = (\tilde{f}D; fC_0 \setminus f(D \setminus C_0), fC_1)$.

Proof. We may assume that $\operatorname{cap} E < \infty$. Given $u \varepsilon W_0^\infty(E)$ with $\|\nabla u\| < \infty$ we define $v : fD \to R^1$ by $v(y) = \sup \{u(x) \colon x \in f^{-1}(y)\}$. Clearly $v(y) \geq 0$ for all $y \varepsilon D' = fD$, $v(y) = 0$ for all $y \varepsilon fC_0 \setminus f(D \setminus C_0)$ and $v(y) = 1$ for all $y \varepsilon fC_1$. f is qr, hence f is open and discrete and since $C(f, \partial D) \subset \partial fD$, it follows that f is closed. Thus, by [1, 3.3], $N(f, D) < \infty$. Consequently v is continuous in D'. Indeed, given $\varepsilon > 0$ and $y \varepsilon D'$ with $f^{-1}(y) = \{x_1, \ldots, x_k\}$, choose neighborhoods $U_i \subset D$ of x_i, $i = 1, \ldots, k$, such that $|u(x) - u(x_i)| < \varepsilon$ for all $x \varepsilon U_i$, $i = 1, \ldots, k$. Then $A = \bigcap_{i=1}^{k} fU_i$ is open in D', $B = f(D \setminus \bigcup_{i=1}^{k} U_i)$ is closed rel. D', and so $V = A \setminus B$ is a neighborhood of y and $|v(y) - v(y')| < \varepsilon$ for all $y' \varepsilon V$. Applying the arguments of [2, 7.8—7.13] we conclude that v is ACL in D'. Thus $v \varepsilon W(\tilde{f}E)$. Then the arguments and computations of [2, 7.15—7.17] yield $\|\nabla v\|_{D'}^n \leq K_I(f) \|\nabla u\|^n$. Take the infimum of $\|\nabla u\|^n$ over all $u \varepsilon W_0^\infty(D)$ and the result follows by 2.3.

4. An application

4.1. A domain $D \subset R^n$ is said to be *quasiconformally accessible* at a point $b \varepsilon \partial D$ iff for every neighborhood U of b and for every continuum $C_0 \subset D \setminus U$, there is a positive number δ, and a neighborhood V of b with $\bar{V} \subset U$ such that $\operatorname{cap}(D; C_0, C_1) > \delta$ for every continuum C_1 in D which meets both ∂V and ∂U. cf. [4, 1.7].

4.2. Theorem. *Let* $f \colon D \to R^n$ *be* qr *with* $C(f, \partial D) \subset \partial fD$. *If* D *is locally connected at a point* $b \varepsilon \partial D$ *and* $D' = fD$ *is* qc *accessible at some point* $y \varepsilon C(f, b)$, *then* $C(f, b) = \{y\}$.

Proof. For qc mappings see [4, 2.4]. Suppose that $C(f, b)$ contains more than one point. Then there are sequences $\{x_i\}$ and $\{x_i'\}$ in D with $x_i \to b$, $x_i' \to b$, $f(x_i) \to y$ and $f(x_i') \to y'$ with $y \neq y'$. Let C_0' be any continuum in D'. $C(f, \partial D) \subset \partial fD$ implies that $C_0 = f^{-1}C'_0$ is compact in D. Choose a neighborhood U of y such that $y' \notin \bar{U}$ and $C_0' \cap \bar{U} = \phi$. $y \varepsilon \partial D'$ and D' is qc accessible at y; hence there is $\delta > 0$ and a neighborhood V of y with the properties stated in 4.1. D is locally connected at b, therefore x_i and x_i' may be joined by an arc γ_i in D whose diameter tends to 0 as $i \to \infty$. $f(x_i) \varepsilon V$ and $f(x_i') \varepsilon D' \setminus \bar{U}$ for all i sufficiently large, hence $\operatorname{cap}(D'; C_0', f\gamma_i) > \delta$

while cap $(D; C_0, \gamma_i) \to 0$ as $i \to \infty$, violating Theorem 3.1. Thus $C(f, b) = \{y\}$.

4.3. Corollary *Let D be a Jordan domain in R^n and $f\colon D \to R^n$ qr with $fD \subset B^n$ and $C(f, \partial D) \subset \partial B^n$. Then f has a continuous extension on $\bar{D}$.*

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