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ON THE EXISTENCE OF SINGULAR SOLUTIONS OF $\Delta u = Pu$ ON RIEMANN SURFACES

BY

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1. Introduction

1. Let R be a Riemann surface and P a *density*, that is, a C^1 function which depends on the local parameter so that the elliptic partial differential equation

(1) $$\Delta u = Pu$$

is invariantly defined on R. We suppose that P is *acceptable* which means that *there exists a positive P-superelliptic function ω on R*. This situation is introduced and investigated in [1] and [2]. Especially the class of densities acceptable by 1 is the class of non-negative densities.

Now we first form in a non-compact region L the ω-measures of the region, its ideal boundary and its relative boundary. Then we consider the existence of singular solutions of (1), that is, solutions with prescribed singularities on R. The local situation is known (Cf. [3]). Using the ω-measures in a similar way as the harmonic ones in the harmonic case (Cf. [6]) we are able to show that the existence of two positive linearly independent P-superelliptic functions implies the existence of a singular solution of (1). Especially the condition is necessary for the existence of a positive solution of (1) with a positive singularity. Consequently the condition is equivalent to the existence of the Green's function of (1) on R.

2. First we state some terms and results from [1] and [2] we are going to use. A density P is called *completely acceptable* if it is acceptable and has the Green's function G_P on R. A function u is said to be a *P-solution* in a region K if $u \in C^2(K)$ and it is a solution of (1) in K. By $I_V^P(f)$ we mean a P-solution in the parametric disc V with boundary values f.

A continuous function v is said to be *P-subelliptic* in a region K if to any point $z_0 \in K$ there exists a parametric disc (V_0, z_0), $\bar{V}_0 \subset K$, such that in every disc (V, z_0), $V \subset V_0$, the first boundary value problem has a unique solution and $v(z_0) \leqq I_V^P(v, z_0)$. A function v is said to be *P-superelliptic* if $-v$ is P-subelliptic. If v and v' are P-subelliptic, α a non-negative constant and V a parametric disc, then αv, $\max(v, v')$ and v_0, the P-modification of v in V,

$$v_0 = \begin{cases} v & \text{in } K - V \\ I_V^P(v) & \text{in } V, \end{cases}$$

are P-subelliptic.

The usual weak and strong forms of maximum principle are valid. Let P be acceptable by ω on a Riemann surface R and v a P-subelliptic function. If

$$0 \leqq \sup_R \frac{v}{\omega} = M < \infty ,$$

then either $v < M\omega$ or $v \equiv M\omega$ in R. If K is a compact region, $\sup_K v \geqq 0$ and

$$\overline{\lim_{z \to \zeta \in \partial K}} \frac{v(z)}{\omega(z)} \leqq M < \infty ,$$

then either $v < M\omega$ or $v \equiv M\omega$ in K.

A non-empty family $F(K)$ of P-subelliptic functions v in a region K is called a *Perron family* if the following two conditions are fulfilled:

(a) If $v_1 , v_2 \in F(K)$, then $\max (v_1 , v_2) \in F(K)$.

(b) If $v \in F(K)$, then its P-modification $v_0 \in F(K)$ for every parametric disc V, $\bar{V} \subset K$.

If $F(K)$ is a Perron family, then the function

$$u_0 = \sup \{ v \mid v \in F(K) \}$$

is either a P-solution or identically $+\infty$.

Let then K be a compact region whose boundary is the union of two disjoint sets k_1 and k_2 and let P be acceptable by ω. The P-solution $\omega_P(K , k_1)$ is said to be the ω*-measure* of k_1 with respect to K if it is identically ω on k_1 and zero on k_2. If especially k_2 is empty, we say that $\omega_P(K , \partial K) = \omega_P(K)$ is the ω-measure of K. If $\{R_n\}$ is an exhaustion of R, then the non-increasing sequence $\{\omega_P(R_n)\}$ converges to a P-solution ω_P which is called the ω-measure of R with regard to P. It is uniquely determined by being the greatest P-subelliptic function v with $v \leqq \omega$.

3. Let K be a region, $z_0 \in K$ and (V , z_0) a parametric disc, $\bar{V} \subset K$. By a *singularity* at z_0 we mean a function S in V which has a representation in terms of the local parameter

$$S(z , z_0) = - a_0 \log |z - z_0| + \sum_{i=1}^{n} a_i |z - z_0|^{-k_i} \tag{2}$$

where a_0 and a_i are real numbers and k_i positive integers, $i = 1 , \ldots , n$. We speak especially of a *positive singularity* if all the coefficients a_i are non-negative, $i = 0 , \ldots , n$. It is denoted by $S^{(+)}$. If we denote by $- S^{(-)}$ a singularity with non-positive coefficients, then $S^{(-)}$ is also a positive singularity and every singularity S can be represented in the form

$$S(z, z_0) = S^{(+)}(z, z_0) - S^{(-)}(z, z_0) . \tag{3}$$

As usual, we mean by a singular P-solution in K, or by a P-solution with the given singularity $S(z, z_0)$ in K, a function u which is a P-solution in $K - \{z_0\}$ so that $u(z, z_0) - S(z, z_0)$ is bounded in (V, z_0). Similarly we speak of a P-solution with a positive singularity.

2. The ω-measures of non-compact regions

4. Let R be an open Riemann surface and L a non-compact region of R. We suppose that the boundary of L in R consists at most of a countable number of analytic curves Γ_i so that $\Gamma_i \cap \Gamma_j$, $i \neq j$, is either empty or a common endpoint, and every compact set of R contains at most points from a finite number of curves. The set $\partial L = \bigcup_i \Gamma_i$ is called the relative boundary of L. The ideal boundary of L is denoted by β or, if necessary, $\beta(L)$. In the same way we denote the ideal boundary of R by β of $\beta(R)$. If $\{R_n\}$ is an exhaustion of R, we denote $L_n = R_n \cap L$, $l_n = \bar{R}_n \cap \partial L$ and $l'_n = L \cap \partial R_n$. Then $\partial L_n = l_n \cup l'_n$.

5. We define the ω-measures of L which are needed in the continuation. The ω-measures of the exhaustion of L form a non-increasing sequence $\{\omega_P(L_n)\}$ which converges to a P-solution $\omega_P(L)$ in L. It is uniquely determined by being the greatest P-subelliptic function v in L with $v \leqq \omega$ and $\overline{\lim}_{z \to z_0 \in \partial L} v(z) \leqq \omega(z_0)$. By the weak maximum principle either $\omega_P(L) < \omega$ or $\omega_P(L) \equiv \omega$.

Next we form the following family:

$$F_0(L) = \{v \mid v \ P\text{-subelliptic in } L,\ v \leqq \omega,\ \overline{\lim}_{z \to z_0 \in \partial L} v(z) \leqq 0\} . \tag{4}$$

It is easy to see that $0 \in F_0(L)$ and $F_0(L)$ is a Perron family. Because it is bounded from above by ω, the function

$$\omega_P(L, \beta) = \sup\{v \mid v \in F_0(L)\}$$

is a P-solution in L. It is called the ω-measure of the ideal boundary of L. Clearly

$$0 \leqq \omega_P(L, \beta) \leqq \omega_P(L) .$$

On the relative boundary of L, $\omega_P(L, \beta) = 0$. In fact, let $z_0 \in \partial L$ and let g be a continuous function on the boundary of $L \cap (V, z_0)$ so that

$g(z_0) = 0$, $0 \leqq g \leqq \omega$ on $\partial L \cap \bar{V}$ and $g = \omega$ on $L \cap \partial V$. The function v_0 ,

$$v_0 = \begin{cases} \omega & \text{in } L - V \\ I^P_{L \cap V}(g) & \text{in } L \cap V \end{cases}$$

is P -superelliptic in L . If $v \in F_0(L)$, then $v \leqq v_0$ on ∂L_n , $n \geqq n_0$ for a value n_0 wherefore the same must hold in L . Therefore

$$0 \leqq \underline{\lim}_{z \to z_0} \omega_P(L , \beta , z) \leqq \overline{\lim}_{z \to z_0} \omega_P(L , \beta , z) \leqq v_0(z_0) = 0 .$$

In the same way we can form $\omega_P(L , \partial L)$, the ω -measure of ∂L , by taking a family

(5) $$F_{-\omega}(L) = \{v \mid v \ \ P\text{-subelliptic in } L , \ v \leqq 0 , \ \overline{\lim}_{z \to z_0 \in \partial L} v(z) \leqq -\omega(z_0)\}$$

and defining

$$\omega_P(L , \partial L) = -\sup\{v \mid v \in F_{-\omega}(L)\} .$$

This is a P -solution in L and it is equal to ω on ∂L . Moreover

$$0 \leqq \omega_P(L , \partial L) \leqq \omega_P(L) .$$

Because $\omega_P(L) - \omega_P(L , \partial L)$ belongs to $F_0(L)$ and $\omega_P(L , \beta) - \omega_P(L)$ to $F_{-\omega}(L)$ we must have

(6) $$\omega_P(L , \partial L) + \omega_P(L , \beta) = \omega_P(L) .$$

If L_1 and L_2 are non-compact regions with $L_1 \subset L_2$, then $\omega_P(L_1 , \beta(L_1)) \leqq \omega_P(L_2 , \beta(L_2))$ and $\omega_P(L_1 , \partial L_1) \geqq \omega_P(L_2 , \partial L_2)$ in L_1 .

If the relative boundary of L is compact, then

(7) $$\omega_P(L , \beta) = \lim_{n \to \infty} \omega_P(L_n , l'_n)$$

and

(8) $$\omega_P(L , \partial L) = \lim_{n \to \infty} \omega_P(L_n , l_n) .$$

6. The extremum property of the ω -measure of ∂L implies the maximum principle for L .

Lemma 1. *Let P be acceptable by ω on an open Riemann surface R , L a non-compact region and v a P -subelliptic function in L with $v \leqq M\omega_P(L , \partial L)$ for a constant M . If*

$$\overline{\lim}_{z \to z_0 \in \partial L} \frac{v(z)}{\omega(z)} \leqq m ,$$

then either $v < m\,\omega_P(L , \partial L)$ or $v \equiv m\,\omega_P(L , \partial L)$.

Proof: We suppose that $m < M$. The function

$$v_1 = (M - m)^{-1}(v - M\,\omega_P(L\,,\ \partial L))$$

is P-subelliptic and non-positive. Moreover

$$\overline{\lim_{z \to z_0 \in \partial L}}\ v_1(z) \leqq -\,\omega(z_0)\ ,$$

wherefore $v_1 \in F_{-\omega}(L)$ in (5). So $v_1 \leqq -\,\omega_P(L\,,\ \partial L)$ which implies

$$v \leqq m\,\omega_P(L\,,\ \partial L)\ .$$

If there exists a point z_0 with $v(z_0) = m\,\omega_P(L\,,\ \partial L\,,\ z_0)$, then in a parametric disc $(V\,,\ z_0)$

$$m\,\omega_P(L\,,\ \partial L\,,\ z_0) = I_V^P(m\,\omega_P(L\,,\ \partial L)\,,\ z_0) \geqq I_V^P(v\,,\ z_0) \geqq v(z_0)\ .$$

Therefore $I_V^P(m\,\omega_P(L\,,\ \partial L) - v\,,\ z_0) \equiv 0$ for every disc $(V\,,\ z_0)$ which gives the statement.

From this result we get a uniqueness property.

Lemma 2. *Let P be acceptable by ω on an open Riemann surface R and u a P-solution in a non-compact region L. If u vanishes on ∂L and*

$$|u| \leqq M\,\omega_P(L\,,\ \partial L)$$

in L for a constant M, then $u \equiv 0$.

Proof: By lemma 1 both u and $-u$ are non-positive in L.

7. By Harnack's inequalities (Cf. [1]) either $\omega_P(L\,,\ \beta)$ is positive or identically zero. In the former case

$$\overline{\lim_{\beta}}\ \frac{\omega_P(L\,,\ \beta)}{\omega} = 1\ . \tag{9}$$

In fact, if it were smaller than one we could choose a constant α, $\alpha > 1$, so that the function $\alpha\omega_P(L\,,\ \beta)$ belongs to the family $F_0(L)$ in (4). This is a contradiction with the definition of $\omega_P(L\,,\ \beta)$.

If in the latter case the non-compact region L is of the form $R - \bar{K}$, K a compact region, then its ideal boundary is for every K the same as the one of R. It would be natural to expect that the vanishing of $\omega_P(R - K\,,\ \beta)$ would not depend on the particular region K. We can, in fact, prove even little more.

Theorem 3. *Let P be acceptable by ω on an open Riemann surface R. Then the following statements are equivalent*:

(a) *For every compact region* K, $\omega_P(R - K\,,\ \beta) \equiv 0$.

(b) *For some compact region* K, $\omega_P(R - K\,,\ \beta) \equiv 0$.

(c) *Let* L *be a non-compact region and* c *a non-negative constant. Then every* P*-solution* u *in* L *for which* u/ω *is bounded from above in* L *and which satisfies* $u \leqq c\omega$ *on* ∂L *must also satisfy* $u \leqq c\omega$ *in* L.

Proof: (a) $\Rightarrow$ (c): Let K be a compact region, $\bar{K} \subset R - L$. Then

$$0 \leqq \omega_P(L\,,\ \beta) \leqq \omega_P(R - K\,,\ \beta) = 0\,. \tag{10}$$

If $u/\omega \leqq M$ in L, we have by the definition of $\omega_P(L)$ and by (6)

$$u \leqq M\,\omega_P(L) = M\,\omega_P(L\,,\ \partial L)\,.$$

Therefore by lemma 1

$$u \leqq c\,\omega_P(L\,,\ \partial L) \leqq c\omega\,.$$

(c) $\Rightarrow$ (b): $R - \bar{K}$ is a non-compact region and $\omega_P(R - K\,,\ \beta) = 0$ on ∂K.

(b) $\Rightarrow$ (a): Let K be a compact region with $\omega_P(R - K\,,\ \beta) = 0$ and let K_1 be another compact region. There exists a compact region K_2 containing both K and $\bar{K}_1$. For $R - K_2$ we have

$$0 \leqq \omega_P(R - K_2\,,\ \beta) \leqq \omega_P(R - K\,,\ \beta) = 0\,. \tag{11}$$

If now $\omega_P(R - K_1\,,\ \beta) > 0$, then we could by (9) find a constant α, $1 < \alpha < 2$, so that the P-subelliptic function $u_0 = \alpha\omega_P(R - K_1\,,\ \beta) - \omega$ is positive at some point $z_0 \in R - \bar{K}_2$ but is non-positive on ∂K_2. Moreover $u_0 \leqq \omega$. So u_0 belongs by (4) to the Perron family $F_0(R - K_2)$ and is positive at some point $z_0 \in R - \bar{K}_2$ which is impossible by (11). Therefore we must have $\omega_P(R - K_1\,,\ \beta) \equiv 0$.

The theorem is thus proved.

Notice that if ω is not a P-solution then the statements of theorem 3 can be shown to be equivalent to the vanishing of $\omega_P(R)$ (Cf. [2]).

8. The existence of a P-solution in a non-compact region L with given, suitable bounded, continuous boundary values on ∂L can now be shown in the usual way.

Lemma 4. *Let* P *be acceptable by* ω *on an open Riemann surface* R *and* L *a non-compact region. Let* f *be a continuous function on* ∂L *so that* $|f|/\omega \leqq M < \infty$. *Then there exists exactly one* P*-solution* u *in* L *with* $u = f$ *on* ∂L *and*

$$|u| \leqq M\,\omega_P(L\,,\ \partial L)$$

in L.

Proof: By lemma 2 there exists at most one such P-solution. Let us consider a family $F_f(L)$ of such P-subelliptic functions v in L for which

$$v \leqq M\,\omega_P(L\,,\ \partial L)\,,\quad \overline{\lim_{z \to z_0 \in \partial L}}\ v(z) \leqq f(z_0)\,.$$

$F_f(L)$ is a Perron family, $-M\,\omega_P(L\,,\ \partial L) \in F_f(L)$. So

$$u = \sup\{v \mid v \in F_f(L)\}$$

is a P-solution.

In order to see that u has the right boundary values we take a point $z_0 \in \partial L$ and a continuous function g on the boundary of $L \cap (V\,,\ z_0)$ so that $g(z_0) = f(z_0)$, $-M\,\omega_P(L\,,\ \partial L) \leqq g \leqq f$ on $\partial L \cap \bar{V}$ and $g = -M\,\omega_P(L\,,\ \partial L)$ in $L \cap \partial V$. Then the function v_0,

$$v_0 = \begin{cases} -M\,\omega_P(L\,,\ \partial L) & \text{in } L - V \\ I^P_{L\cap V}(g) & \text{in } L \cap V \end{cases}$$

belongs to $F_f(L)$. Therefore

$$\underline{\lim}_{z \to z_0}\ u(z) \geqq \underline{\lim}_{z \to z_0}\ v_0(z) = f(z_0)\,.$$

As in verifying the boundary values of $\omega_P(L\,,\ \beta)$ in 5 we see that

$$\overline{\lim_{z \to z_0}}\ u(z) \leqq f(z_0)\,.$$

So u has the right boundary values on ∂L. By construction

$$-M\,\omega_P(L\,,\ \partial L) \leqq u \leqq M\,\omega_P(L\,,\ \partial L)$$

which proves the result.

Notice that f/ω is bounded if ∂L is compact and f is bounded. If f is non-negative, then $0 \in F_f(L)$ and u, too, is non-negative.

3. On singular P-solutions

9. Let R be an open Riemann surface. We examine if there exists on R a P-solution with a given behaviour on the ideal boundary. This problem can then be used in the study of our main problem.

Theorem 5. *Let P be acceptable by ω on an open Riemann surface R. Let R be divided by a compact analytic curve Γ into two parts R_1 and R_2 so that*

$$\omega_P(R_1\,,\ \Gamma) < \omega \tag{12}$$

in R_1. Let moreover the functions u_i be P-solutions on R_i and continuous on Γ, $i = 1\,,\ 2$. Then there exists on R exactly one P-solution u_0 with

(13) $$|u_0 - u_i| \leqq M\,\omega_P(R_i\,,\ \Gamma)\,,\quad i = 1\,,\ 2$$

for a constant M .

Proof: In the following the index i always takes values 1 and 2. Let K be a regular region containing Γ so that $\partial K = k_1 \cup k_2$, $k_i \subset R_i$.

We first show the uniqueness. If u' and u'' are two such functions and $u = u' - u''$, then

$$|u| \leqq 2M\,\omega_P(R_i\,,\ \Gamma)\,.$$

We denote $m = \sup\limits_{\Gamma} \dfrac{u}{\omega}$. Then $m \leqq 2M$ and by lemma 1

$$u \leqq m\,\omega_P(R_i\,,\ \Gamma)\,.$$

If $m > 0$,

$$\frac{u}{\omega} \leqq \frac{m\,\omega_P(R_i\,,\,\Gamma)}{\omega} = \begin{cases} < m & \text{on } k_1 \\ \leqq m & \text{on } k_2\,. \end{cases}$$

Therefore $u < m\omega$ in K and especially on Γ, which is impossible. So $u \leqq 0$ on Γ and by lemma 1 on R , too. By replacing u with $-u$ we see that also $-u \leqq 0$ on R . This gives the uniqueness.

Now we turn to the existence of u_0 . First we change the situation so that the given functions vanish on Γ .

By lemma 4 there exists a P-solution u_i' on R_i with $u_i' = u_i$ on Γ and

(14) $$|u_i'| \leqq a\,\omega_P(R_i\,,\ \Gamma)$$

in R_i for a constant a . The function $v_i = u_i - u_i'$ is a P-solution in R_i which vanishes on Γ .

Let then v be a P-solution in R_2 with boundary values $\omega_P(R_1 \cup K, k_2)$ on Γ so that v is non-negative and bounded from above by $\omega_P(R_2\,,\,\Gamma)$. Because

(15) $$\omega_P(R_1 \cup K\,,\ k_2) < \omega_P(R_1\,,\ \Gamma) < \omega$$

in R_1 , $v < \omega$ on Γ . Therefore

(16) $$0 < v < \omega_P(R_2\,,\ \Gamma)$$

in R_2 and especially

(17) $$v < \omega_P(R_1 \cup K\,,\ k_2)$$

in $R_2 \cap \bar{K}$ by the strong maximum principle.

Using v and the ω-measures we construct two P-superelliptic functions on R . Let g be the following function

$$g = \begin{cases} \omega_P(R_1, \Gamma) + \omega_P(R_1 \cup K, k_2) & \text{in } R_1 \cup \Gamma \\ \omega_P(R_2, \Gamma) + v & \text{in } R_2 . \end{cases}$$

Clearly g is positive and continuous in R and a P-solution in R_i. Let then b be a positive constant so that

$$b(\omega - \omega_P(R_1, \Gamma)) \geqq |v_1|$$

in $\bar{K} \cap \bar{R}_1$ and

$$b(\omega_P(R_1 \cup K, k_2) - v) \geqq |v_2|$$

in $\bar{K} \cap \bar{R}_2$. This is possible by (12), (17) and because v_i is a P-solution in R_i vanishing on Γ. Now the functions $h_1 = bg + v_i$ and $h_2 = bg - v_i$ are P-superelliptic on R. This is obvious at every point $z \in R_i$. If $z \in \Gamma$ and (V, z) is a parametric disc, $V \subset K$, then

$$h_i \leqq b(\omega + \omega_P(R_1 \cup K, k_2)) \tag{18}$$

in V by the construction of g and the choice of b. The upper bound in (18) is P-superelliptic in V. Therefore

$$\begin{aligned} h_i(z) &= b(\omega(z) + \omega_P(R_1 \cup K, k_2, z)) \\ &\geqq I_V^P(b(\omega + \omega_P(R_1 \cup K, k_2)), z) \\ &\geqq I_V^P(h_i, z) \end{aligned}$$

which shows the P-superellipticy.

Let us now consider the family F of P-subelliptic functions u in R which are bounded from above by h_1. We notice that $-h_2 \in F$ and F is a Perron family. So

$$u_0 = \sup \{u \mid u \in F\}$$

is a P-solution in R. It has the required behaviour because by (14), (15) and (16) we have on one hand

$$u_0 - u_i \leqq h_1 - v_i - u_i' \leqq (2b + a)\, \omega_P(R_i, \Gamma)$$

and on the other hand

$$u_0 - u_i \geqq -h_2 - v_i - u_i' \geqq -(2b + a)\, \omega_P(R_i, \Gamma) .$$

This proves the theorem.

Notice that theorem 5 is also valid if R_1 is a compact region, $\partial R_1 = \Gamma$, provided that $\omega_P(R_1, \Gamma)$ is replaced by $\omega_P(R_1)$.

10. Let R be a Riemann surface and let a positive singularity $S^{(+)}(z, z_0)$ be given at a point $z_0 \in R$. Using theorem 5 we can now form a necessary and sufficient condition for the existence of a positive P-solution on R

with the singularity $S^{(+)}$ at z_0. This condition is related to the assumption (12). We are also able to present two formulations to this condition.

Theorem 6. *Let R be a Riemann surface and P an acceptable density. The following statements are equivalent:*

(a) *There exists on R a positive P-solution with a given positive singularity.*

(b) *There exist on R two positive linearly independent P-superelliptic functions.*

(c) *There exists on R a positive P-superelliptic function which is not a P-solution.*

Proof: (a) $\Rightarrow$ (b): Let P be acceptable by ω and $u(z, z_0)$ be the positive P-solution with the given positive singularity at z_0. We choose a constant c so that on the boundary of a parametric disc (V, z_0)

$$u(z, z_0) < c\omega(z) .$$

The function

$$v(z) = \begin{cases} \min (u(z, z_0), c\omega(z)) & \text{if } z \neq z_0 \\ c\omega(z_0) & \text{if } z = z_0 \end{cases}$$

is positive and P-superelliptic on R. In a neighbourhood of z_0, $v = c\omega < u$ and on ∂V, $v = u < c\omega$. Therefore v and ω are linearly independent.

(b) $\Rightarrow$ (c): If u_1 and u_2 are positive linearly independent P-superelliptic functions on R, then there exists a positive constant c so that cu_1 has both greater and smaller values than u_2. The function

$$v = \min (cu_1, u_2)$$

is positive and P-superelliptic on R. If v were a P-solution, the function $u = v - u_2$ would be P-subelliptic. Because u is non-positive, $u(z_1) = 0$ and $u(z_2) < 0$ at some points $z_1, z_2 \in R$, this is a contradiction with the maximum principle. So v is not a P-solution.

(c) $\Rightarrow$ (a): We first suppose that R is open. Let ω be the positive P-superelliptic function which is not a P-solution. By lemma 3.3.2 in [1] ω cannot fail to be a P-solution only at one point. So, if the positive singularity $S^{(+)}$ is given at $z_0 \in R$, there exists a parametric disc V so that $z_0 \in V$ and ω fails to be a P-solution outside $\bar{V}$. Therefore

$$\omega_P(R - V, \partial V) \leqq \omega_P(R - V) < \omega . \tag{19}$$

In V we can form a P-solution u_0' with the singularity $S^{(+)}$ at z_0 (Cf. [3]). Now we use theorem 5 in the case $\Gamma = \partial V$, $R_1 = R - \bar{V}$, $R_2 = V - \{z_0\}$, $u_1 = 0$ and $u_2 = u_0'$. By (19) we get a P-solution $u_0(z, z_0)$ in $R - \{z_0\}$ so that in V

$$|u_0 - u_0'| \leqq M\,\omega_P(V - \{z_0\}\,,\ \partial V) = M\,\omega_P(V)$$

and in $R - V$

(20) $$|u_0| \leqq M\,\omega_P(R - V\,,\ \partial V)\,.$$

The function $v_0 = u_0 + M\omega$ is a positive P-superelliptic function in $R - \{z_0\}$.

Let F be the family of those P-subelliptic functions in $R - \{z_0\}$ which are bounded from above by v_0. The function $\max(u_0\,,\ 0) \in F$ and F is seen to be a Perron family. Therefore

$$u = \sup\{v \mid v \in F\}$$

is a P-solution in $R - \{z_0\}$ with $\max(u_0\,,\ 0) \leqq u \leqq v_0$. So u is positive by Harnack's inequalities and has the singularity $S^{(+)}$ at z_0.

If R is closed the proof is the same except that $\omega_P(R - V\,,\ \partial V)$ is replaced by $\omega_P(R - V)$.

The proof is thus complete.

Corollary 7. *Let P be acceptable by ω on an open Riemann surface R. If $\omega_P(R - K\,,\ \beta) > 0$ for some compact region K, then the statements* (a), (b) *and* (c) *of theorem* 6 *are valid on R.*

Proof: By (6) and theorem 3

(21) $$\omega_P(R - V\,,\ \partial V) < \omega_P(R - V) \leqq \omega$$

for every parametric disc V. Now we get the statement (a) quite as in the last part of the previous proof by replacing (19) by (21). This implies then (b) and (c) by theorem 6.

It is to be noticed that the condition of corollary 7 is not a necessary one.

11. In the preceding paragraph we considered only positive singularities. The situation is easily generalised.

Theorem 8. *Let R be a Riemann surface and P an acceptable density. Let there be given at m points z_i singularities $S_i(z\,,\ z_i)$ of the form* (2). *If one of the statements* (a)—(c) *of theorem* 6 *or the conditions of corollary* 7 *are valid, then there exists a P-solution with these singularities on R.*

Proof: We express each singularity in the form $S_i(z\,,\ z_i) = S_i^{(+)}(z\,,\ z_i) - S_i^{(-)}(z\,,\ z_i)$ where $S_i^{(+)}$ and $S_i^{(-)}$ are positive singularities. The assumptions guarantee, by theorem 6 and corollary 7, the existence of P-solutions $u_i^{(+)}$ and $u_i^{(-)}$ with singularities $S_i^{(+)}$ and $S_i^{(-)}$ respectively at the point z_i, $i = 1\,, \ldots, m$. The P-solution

$$u = \sum_{i=1}^{m} (u_i^{(+)} - u_i^{(-)})$$

has the required properties.

12. We now treat the existence of the Green's function on R for an acceptable density P. L. Myrberg has proved in [4] that all non-negative densities P, $P \not\equiv 0$, are completely acceptable. All acceptable densities, however, are not completely acceptable (Cf. [5]). It was shown in [1] and [2] that a density P is completely acceptable if either there exists an acceptable density Q, $Q \leqq P$, $Q \not\equiv P$, or if R is open and P is acceptable by ω so that the ω-measure of the ideal boundary of R is positive. Now we get directly from theorem 6 a necessary and sufficient condition, even in two formulations, for a density P to be completely acceptable.

Theorem 9. *Let P be a density on a Riemann surface R. Then the following statements are equivalent:*

(a) *P is completely acceptable.*

(b) *There exist on R two positive linearly independent P-superelliptic functions.*

(c) *There exists on R a positive P-superelliptic function which is not a P-solution.*

Proof: We have only to show that (c) implies (a). By theorem 6, (c) implies the existence of a positive P-solution with a logarithmic singularity. On the other hand the Green's function can be defined as a minimum of such functions (Cf. [1] and [2]). This proves the theorem.

This result implies those mentioned earlier as one easily notices.

Finally we remark that the situation now is not the same as for harmonic functions even if theorems 3 and 9 together with corollary 7 seem to indicate it. In fact, Royden has shown in [7] that a non-negative density P, $P \not\equiv 0$, which is always completely acceptable, may have a vanishing 1-measure of the ideal boundary even on a hyperbolic surface.

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