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ON EXCEPTIONAL VALUES OF FUNCTIONS MEROMORPHIC OUTSIDE A SET OF POSITIVE HAUSDORFF DIMENSION

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1. Introduction

1. Let E be a closed set in the complex plane and f a non-constant meromorphic function outside E omitting a set F. We shall consider the following problem: How thick E ought to be to find f such that F is thick, too? It is known that if the Hausdorff dimension of E Dim (E) is greater than one then there exists a non-constant function f which is regular and bounded outside E. If E has linear measure zero then Dim (F) ≤ 1. If the logarithmic capacity of E Cap (E) is zero then Cap (F) $= 0$. In [3], it is given a geometrical condition under which Dim $(F) \leq$ Dim (E). Carleson [1] has proved that there exists a set E with Cap (E) > 0 such that if f omits 4 values outside E then f is rational.

In this paper, we shall prove that Dim $(E) > 0$ does not guarantee that F is thick, too. We construct a set E with Dim (E) uniformly positive such that if f is meromorphic outside E with a singularity at all points of E, then f omits at most 4 values. Here »Dim (E) uniformly positive» means that there exists $a > 0$ such that if A is open then either $A \cap E = \emptyset$ or Dim $(A \cap E) \geq a$. Then we shall prove that there exists a set E with Dim $(E) > 0$ such that if f is meromorphic and non-rational outside E omitting F then Cap $(F) = 0$.

2. Notations and lemmas

2. Given positive numbers $\xi_{n,k}$, $0 < \xi_{n,k} < 1/3$, $n = 0, 1, \ldots$, $k = 1, 2, \ldots, 2^n$, and a sequence $\{\varphi_n\}$ of real numbers, we construct the corresponding Cantor set E in the following manner.

Let $\eta_0 = 1$, $z_{0,1} = 0$ and $l_{0,1} = 1$. Inductively $(n \geq 1)$, we define $\eta_n = \eta_{n-1} e^{i\varphi_n}$ and for $k = 2p - 1$, $2p$ $(1 \leq p \leq 2^{n-1})$, we set $\mu_{n,k} = \xi_{n-1,p}$, $l_{n,k} = \xi_{n-1,p}\, l_{n-1,p}$ and

$$z_{n,k} = z_{n-1,p} + (-1)^k \eta_{n-1} (1 - \xi_{n-1,p})\, l_{n-1,p} .$$

We set

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} D_{n,k}$$

where $D_{n,k} = \{z : |z - z_{n,k}| \leq l_{n,k}\}$.

3. We need the following lemmas in our considerations. Let Σ be the Riemann sphere with radius $1/2$ touching the w-plane at the origin. The chordal distance of the images on Σ of two points w and w' in the plane is denoted by $[w, w']$ and $C(w, \delta)$ is the spherical open disc with centre at the image of w and with chordal radius δ.

We set

$$S_{n,k} = \left\{ z : l_{n,k} < |z - z_{n,k}| < \frac{l_{n,k}}{3\,\mu_{n,k}} \right\}$$

and

$$\Gamma_{n,k} = \left\{ z : |z - z_{n,k}| = \frac{l_{n,k}}{\sqrt{3\mu_{n,k}}} \right\}.$$

We have (see Carleson [1], Matsumoto [2])

Lemma 1. There exists a constant A such that if f is analytic in $S_{n,k}$ and omits 0 and 1 then $f(\Gamma_{n,k})$ is contained in a spherical disc $C_{n,k}$ with radius $\delta_{n,k}$ less than $A\sqrt{\mu_{n,k}}$.

We choose $\delta > 0$ such that $C(1, 8\delta) \cap C(0, 8\delta) = \emptyset$. Now we assume that $A\sqrt{\xi_{n,k}} < \delta$ for any n and k. Let $\triangle_{n,k}$ be the triply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. An easy modification of Matsumoto's [2] Lemma 2 gives us

Lemma 2. Let f be analytic in $\triangle_{n,k} \cup S_{n,k} \cup S_{n+1,2k-1} \cup S_{n+1,2k}$ and omit the values 0 and 1. Then only two possibilities can occur:

(1) The spherical discs $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ containing the images of the boundary components of $\triangle_{n,k}$, contain the origin, the point $w = 1$, and the point at infinity, one by one, and f takes each value outside the union of these discs once and only once in $\triangle_{n,k}$.

(2) There exists a spherical disc with radius less than $2A(\sqrt{\mu_{n,k}} + 2\sqrt{\xi_{n,k}})$ which contains $f(\overline{\triangle}_{n,k})$.

Let $T_{n,k}$ be the bounded disc with $\Gamma_{n,k}$ as boundary. We denote be $L(r)$ $(r > 0)$ the union of the spherical dics $C(0, r)$, $C(1, r)$ and $C(\infty, r)$. We choose $\xi_0 > 0$ such that

$$12A\sqrt{\xi_0} < \frac{\delta}{64} \tag{1}$$

and we assume that $\xi_{n,k} \le \xi_0$ for any n and k.

Lemma 3. Let f be analytic outside E and omit the values 0 and 1. If $f(\Gamma_{n,k}) - L(\delta) \neq \emptyset$ then $f(T_{n,k} - E) \subset C_{n,k}$ where $C_{n,k}$ is the spherical disc defined in Lemma 1.

Proof. Let us suppose that $f(\Gamma_{n,k}) - L(\delta) \neq \emptyset$. Then f takes on $\Gamma_{n,k}$ a value outside $C(0, \delta)$ and we see from Lemma 1 that $C_{n,k}$ cannot contain the origin. Similarly, 1 and ∞ lie outside $C_{n,k}$, and it follows from (1) and Lemma 2 that $f(\overline{\triangle}_{n,k}) \cap L(\delta/2) = \emptyset$.

Let $\triangle_p$ be the domain bounded by $\Gamma_{n,k}$ and the circles $\Gamma_{n+p,s}$ lying in $T_{n,k}$. Then $f(\overline{\triangle}_1) \cap L(\delta/16) = \emptyset$. Let us suppose that $f(\overline{\triangle}_p) \cap L(\delta/16) = \emptyset \; (p \geq 1)$. Then it follows from (1) and Lemma 2 that $f(\overline{\triangle}_{p+1}) \cap L(\delta/32) = \emptyset$. Let $a \in \Gamma_{n+p,s} \subset T_{n,k}$. By Cauchy's integral theorem we have

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma_{n,k}} \frac{f(z)}{z-a}\, dz - \frac{1}{2\pi i} \sum \int_{\gamma_m} \frac{f(z)}{z-a}\, dz$$

where the sum is taken over all $\gamma_m = \Gamma_{n+p+1,m} \subset T_{n,k}$. On $\overline{\triangle}_{p+1}$ we have $|f(z)| \leq 32/\delta$, and on $\Gamma_{n,k}$ we have the better estimate $|f(z)| \leq 2/\delta$.

If $a \in \overline{T}_{n+q,2j-1}$, $1 \leq q \leq p$, and $\gamma_m \subset T_{n+q,2j}$, or vice-versa, then

$$\left| \frac{1}{2\pi i} \int_{\gamma_m} \frac{f(z)}{z-a}\, dz \right| \leq \frac{64\, \xi_0^{p-q+1}}{\delta}.$$

Therefore

$$|f(a)| \leq \frac{4}{\delta} + \frac{128\xi_0}{\delta} + \frac{64}{\delta} \sum_{q=1}^{p} 2^{p-q+1}\, \xi_0^{p-q+1}$$

$$< \frac{4}{\delta} + \frac{128\xi_0}{\delta}\left(1 + \frac{1}{1-2\xi_0}\right).$$

Now it follows from (1) that $|f(a)| < 5/\delta$ and we see that $f(\overline{\triangle}_p) \cap C(\infty, \delta/8) = \emptyset$. Considering the functions $1/f$ and $1/(1-f)$, we get $f(\overline{\triangle}_p) \cap L(\delta/8) = \emptyset$. Applying Lemma 2 again, we get $f(\overline{\triangle}_{p+1}) \cap L(\delta/16) = \emptyset$ and by induction, we see that $f(T_{n,k} - E) \cap L(\delta/16) = \emptyset$. Since E has linear measure zero, the lemma follows from the maximum principle.

3. Functions with a singularity at all points of E

4. Let $0 < a < b < \xi_0$, and let n_k be an increasing sequence of even positive integers. We construct the Cantor set E with $\varphi_n = 0$, $n = 1, 2, \ldots$, and with the successive ratios $\xi_{n,k}$ defined in the following manner. We set $\xi_{n,k} = a$ for $0 \leq n < n_1$, $1 \leq k \leq 2^n$, $\xi_{n_i,k} = a/i$, $i = 1, 2, \ldots, 1 \leq k \leq 2^{n_i}$, and for $n_i < n < n_{i+1}$ we set

$$\xi_{n,k} = a + (b-a)\,(p-1)/2^i$$

for $(p-1)2^{n-i} < k \leq p2^{n-i}$, $p = 1, 2, \ldots, 2^i$.

Theorem 1. It is possible to choose a and b such that if f is meromorphic outside E with an essential singularity at each point of E, then f omits at most four values (The choice of a and b does not depend on the sequence $\{n_k\}$). If n_k tends to infinity with a sufficient rapidity as $k \to \infty$ then for any open set G either $E \cap G = \emptyset$ or $\mathrm{Dim}\,(E \cap G) \geq (\log (3/4))/\log a$.

The proof of the first assertion will be given in 5—9. The second assertion is quite trivial and the proof of it will be omitted.

5. Let f be meromorphic outside E with an essential singularity at all points of E omitting 5 values a_i, $i = 1, \ldots, 5$. It does not mean any essential restriction to assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. f is not bounded, and it follows from Lemma 3 that the case (1) of Lemma 2 occurs for at least one $\triangle_{n,k}$. Therefore f takes every value outside the union of the discs $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$. By means of a linear transformation, we may suppose that $a_4 \in C(0, \delta)$ and $a_5 \in C(\infty, \delta)$.

6. Let $a < b^{18}$, and let $c > 0$ be chosen such that $a < c^6$ and $c < b^3$. We choose a real number ξ, $a \leq \xi \leq b$, in the following manner. We set $a_6 = \max(|a_4|, 1/|a_5|)$ and $a_7 = \min(|a_4|, 1/|a_5|)$.

(A) If $a_7 \geq \sqrt{a}$, we set $\xi = a$.

(B) If $a_7 < \sqrt{a}$ and $a_6 \geq c$ then there exist ξ, $c^4 < \xi < c^2$, and a positive integer q such that $a_7 = \xi^{q+1/2}$.

(C) If the cases (A) and (B) do not occur then $a_7 \leq a_6 < c$. There exist $\xi_{1/2}$, $b^4 < \xi_{1/2} < b^2$, and a positive integer p such that $a_6 = \xi_{1/2}^{p+1/2}$ We set $a_6 = \xi_r^{p+r}$ where $1/4 \leq r \leq 3/4$. Then we have $b^5 \leq \xi_r \leq b$. Let now $a_7 = \xi_r^{t_r}$. We get $t_r = K(p+r)$ where $K = (\log a_7)/\log a_6 \geq 1$. Therefore there exists r, $1/4 \leq r \leq 3/4$, such that $t_r = q + s$ where q is a positive integer and $1/4 \leq s \leq 3/4$, and we choose $\xi = \xi_r$.

7. Let E' be the Cantor set with $\varphi_n = 0$ and $\xi_{n,k} = \xi$ for any n and k. In connection with E' we write $\Gamma'_{n,k}$, $T'_{n,k}$, $\ldots$, corresponding to $\Gamma_{n,k}$, $T_{n,k}$, $\ldots$ in connection with E. Let $\triangle'_p$ be the connected domain with

$$H = \left\{ z : |z| = \frac{1}{2\sqrt{3}\,\xi} \right\}$$

and $\Gamma'_{p,s}$, $s = 1, 2, \ldots, 2^p$, as boundary.

We choose a sequence $\{\xi_{s_i,t_i}\}$ such that $\lim \xi_{s_i,t_i} = \xi$ and $s_{i+1} > s_i$. Let i be fixed. It follows from Lemma 3 that the case (1) of Lemma 2 occurs for at least one $\triangle_{s,t} \subset T_{s_i,t_i}$. Let $n_{j-1} < s \leq n_j$. Then we can choose $q_i \leq n_j$ and $\triangle_{q_i, m_i} \subset T_{s,t}$ such that the case (1) of Lemma 2 occurs for $\triangle_{q_i,m_i}$, and for any $\triangle_{n,k} \subset T_{q_i,m_i}$, $q_i < n \leq n_j$, occurs the case (2) of Lemma 2 (possibly $q_i = n_j$).

We choose the function g being one of the functions f, $f/(1-f)$ and $1/f$ such that

$$(3) \qquad g(\Gamma'_{q_i,m_i}) \subset C(\infty\,,\ \delta)$$

for infinitely many i. Taking a subsequence, we see that we may assume that (3) is true for each i. If $g = f$ we set $a_8 = a_4$, if $g = f/(1-f)$ then $a_8 = a_4/(1-a_4)$, and if $g = 1/f$ we set $a_8 = 1/a_5$. Then g omits the values 0, 1, ∞ and a_8 outside E and $a_8 \in C(0\,,\ \delta)$.

8. We define $g_i(z) = g(l_{q_i,m_i} z + z_{q_i,m_i})$. If $g(\Gamma_{q_i+1,2m_i}) \subset C(1\,,\ \delta)$ we set $f_i(z) = g_i(z)$, otherwise we set $f_i(z) = g_i(-z)$. We set $G = \{z : 2 < |z| < 1/(12\xi)\}$. Taking any $\triangle'_p$, $p \geq 2$, we see that for sufficiently large i, f_i is defined on $G \cup \triangle'_p$. Applying Lemma 2, it follows from the definition of f_i that $f_i(H) \subset C(\infty\,,\ 8A\sqrt{\xi})$, $f_i(\Gamma'_{1,1}) \subset C(0\,,\ \delta)$, $f_i(\Gamma'_{1,2}) \subset C(1\,,\ \delta)$, and f_i takes on $\triangle'_1$ exactly once every value outside the union of the discs $C(\infty\,, 8A\sqrt{\xi})$, $C(0\,, \delta)$ and $C(1\,, \delta)$. Applying Lemma 3, it follows from the choice of the sequence $\{\triangle_{q_i,m_i}\}$ that $f_i(T'_{1,1} \cap \triangle'_p) \subset C(0\,,\ 2\delta)$ and $f_i(T'_{1,2} \cap \triangle'_p) \subset C(1\,,\ 2\delta)$ if i is large enough.

Let $D = \{z : |z| < 1/(12\xi)\}$. We can choose a subsequence $\{f_{i_k}\}$ which converges uniformly on compact subsets of $D - E'$ towards a limit function f_0, f_0 being defined in $D - E'$. It is easily seen that $f_0(H) \subset C(\infty\,,\ 9A\sqrt{\xi})$, f_0 omits the values 0, 1, ∞ and a_8 in $D - E'$ and that f_0 takes the value -1 exactly once in $\triangle' - E'$ where $\triangle'$ is the disc bounded by H. Further on, $f_0(T'_{1,1} - E') \subset C(0\,,\ 3\delta)$ and $f_0(T'_{1,2} - E') \subset C(1\,,\ 3\delta)$ and since E' has linear measure zero then f_0 has an analytic continuation in D. We denote by f_0 this continuation, too. Applying Rouché's theorem, we see that f_0 takes the values outside $C(\infty\,,\ 9A\sqrt{\xi})$ exactly once in $\triangle'$. Now we choose z_0, $z_8 \in T'_{1,1} \cap E'$ and $z_1 \in T'_{1,2} \cap E'$ such that $f_0(z_0) = 0$, $f_0(z_1) = 1$ and $f_0(z_8) = a_8$.

9. Let $B = \{w : |w| < r\}$ where $r = 1/(18A\sqrt{\xi})$. f_0 is schlicht in $f_0^{-1}(B) \cap \triangle'$ and it has an inverse function g which is schlicht in B. We write

$$h(w) = \frac{g(rw) - z_0}{r\, g'(0)}\,.$$

Then h is schlicht in $|w| < 1$, $h'(0) = 1$, and applying the distortion theorem for schlicht functions we see that

$$\left|\frac{z_1 - z_0}{r\, g'(0)}\right| = |h(1/r)| \leq \frac{1}{r(1 - 1/r)^2}$$

and that h takes every value of $|z| < 1/4$. Then g takes in B every value of $|z - z_0| < \frac{1}{4}|z_1 - z_0| r(1 - 1/r)^2$. We have $1/r = 18A\sqrt{\xi} < \delta/32 < 1/500$, and we see that f_0 is schlicht in $|z| < r/4$. Then it follows from the distortion theorem that $\frac{1}{4}|z_8 - z_0| < |a_8| < |z_8 - z_0|$.

Since z_8 and z_0 belong to $E' \cap T'_{1,1}$, then there exists $D'_{n,k}$, $n \geq 1$, such that $D'_{n+1,2k-1}$ contains one of the points z_0 and z_8, and $D'_{n+1,2k}$ contains the other one. Therefore we have $\xi^n \leq |z_8 - z_0| \leq 2\,\xi^n$ and we get

$$\frac{1}{4}\xi^n \leq |a_8| \leq 2\,\xi^n \tag{4}$$

where n is a positive integer.

On the other hand, it follows from the choice of ξ and the definition of a_8 that either $|a_8| \geq \frac{1}{2}\xi^{1/2}$ or $\frac{1}{2}\xi^{p+r} \leq |a_8| \leq 2\,\xi^{p+r}$ where p is a positive integer and $1/4 \leq r \leq 3/4$. In both cases we have a contradiction with (4) and the first assertion of Theorem 1 is proved.

10. **Remark.** Modifying a little the proof of Theorem 1 we see that our set E has the following local property: Let A be an open domain such that $A \cap E \neq \emptyset$, and let f be meromorphic in $A - E$ with an essential singularity at all points of $E \cap A$. Then f omits in $A - E$ at most a finite number of values.

4. Non-rational functions

11. A set A is said to have logarithmic measure zero if given $\varepsilon > 0$, then we can cover A with open spherical discs $C(b_i, \delta_i)$, $0 < \delta_i < 1$, such that

$$\sum \frac{1}{\log(1/\delta_i)} < \varepsilon .$$

Let a and b be as in the proof of Theorem 1 and let $\{n_k\}$ be a sequence of positive even integers such that $n_{k+1} > 2n_k$, $k = 1, 2, \ldots$. We construct the Cantor set E with $\xi_{n,k} = \xi_n$, $k = 1, \ldots, 2^n$, and φ_n defined in the following manner. Let $\{r_i\}$ be a sequence of all rational numbers satisfying the condition $a \leq r_i \leq b$. We set $\xi_n = r_1$ and $\varphi_n = 0$ for $0 \leq n < n_1$, $\xi_{n_i} = a/i$ and $\varphi_{n_i} = \pi/2$, $i = 1, 2, \ldots$, and $\xi_n = r_i$, $\varphi_n = 0$ for $n_{i-1} < n < n_i$, $i \geq 2$.

Theorem 2. It is possible to choose a, b and the sequence $\{n_k\}$ such that $\mathrm{Dim}(E) > 0$ and if f is meromorphic and non-rational outside E and omits F, then F has logarithmic measure zero.

12. *Proof.* Let f be meromorphic and non-rational outside E omitting a set F with positive logarithmic measure. We may assume that $\{0,$

$1, \infty\} \subset F$. It follows from Lemma 3 and Lemma 2 that f takes every value outside $C(0, \delta) \cup C(1, \delta) \cup C(\infty, \delta)$. Making a linear transformation, if necessary, we may assume that the sets $F \cap C(0, \delta)$ and $F \cap C(\infty, \delta)$ have positive logarithmic measure.

We choose a sequence $\{\triangle_{s_i,t_i}\}$ such that the case (1) of Lemma 2 occurs for any $\triangle_{s_i,t_i}$ and $s_{i+1} > s_i$. Let $n_{p_i-1} < s_i \leq n_{p_i}$. If $\liminf (n_{p_i} - s_i) < \infty$ then it follows from Lemma 3 and Rouché's theorem that two of the discs $C(0, \delta)$, $C(1, \delta)$ and $C(\infty, \delta)$ contain only a finite number of points of F, because $f(\Gamma_{n_{p_i}+1,k})$ is contained in a small spherical disc if i is large. Therefore $\lim (n_{p_i} - s_i) = \infty$, and we assume that $\{s_i\}$ is chosen such that for any n and k, $s_i < n \leq n_{p_i}$, $1 \leq k \leq 2^n$, the case (2) of Lemma 2 occurs for $\triangle_{n,k}$.

Let now $\{\triangle_{s_i,t_i}\}$ be chosen such that $\lim \xi_{s_i} = \xi$. Since f omits at least 5 values it follows from the proof of Theorem 1 that all values of ξ, $a \leq \xi \leq b$, are not allowed. In fact, we can choose c and d, $a < c < d < b$, such that if $c \leq \xi_n \leq d$ then the case (2) of Lemma 2 occurs for $\triangle_{n,k}$. Therefore we may assume that the sequence $\{\triangle_{s_i,t_i}\}$ is chosen such that the case (2) of Lemma 2 occurs for $\triangle_{n,k}$ if $n_{p_i} \leq n \leq n_{p_i+1}$. Further on, since a linear transformation does not essentially change the logarithmic measure, we may assume that $f(\Gamma_{s_i,t_i}) \subset C(\infty, \delta)$, $f(\Gamma_{s_i+1,2t_i-1}) \subset C(0, \delta)$ and $f(\Gamma_{s_i+1,2t_i}) \subset C(1, \delta)$.

13. As in the proof of Theorem 1, we now construct the Cantor set E' with $\varphi_n = 0$ and $\xi_{n,k} = \xi$ for any n and k, and setting

$$f_i(z) = f(\eta_{s_i} l_{s_i} z + z_{s_i,t_i}),$$

we find a limit function f_0 which is schlicht in $D = \{z : |z| < r\}$ where $r = 1/(72A\sqrt{\xi})$. We have $f_0(D - E') \cap F = \emptyset$, $0 \in f_0(T'_{1,1} \cap E')$ and $1 \in f_0(T'_{1,2} \cap E')$, and therefore F has the following property: If $w_j \in F$, $|w_j| \leq 2$, $j = 1, 2$, $w_1 \neq w_2$ and $\operatorname{Re} w_1 \leq \operatorname{Re} w_2$ then

$$|\arg (w_2 - w_1)| \leq \pi/12. \tag{5}$$

14. For the sake of simplicity, we write s, t and p instead of s_i, t_i and p_i. Let $\triangle$ be the domain bounded by $\Gamma_{s,t}$ and $\gamma_v = \Gamma_{n_{p+1},v}$, $v = \alpha, \ldots, \omega$, where $\alpha = 2^{n_{p+1}-s}(t-1) + 1$ and $\omega = 2^{n_{p+1}-s}t$. We write $f = g + h$ where

$$g(z) = \frac{1}{2\pi i} \int_{\Gamma_{s,t}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

and

$$h(z) = \frac{-1}{2\pi i} \sum_{v=\alpha}^{\omega} \int_{\gamma_v} \frac{f(\zeta)}{\zeta - z} d\zeta .$$

Then g and h are regular in $\triangle$.

Let $\triangle_1$ be the domain bounded by $\Gamma_{s,t}$ and the circles $\Gamma_{n_p+m,q}$ lying in $T_{s,t}$ where $m = (n_{p+1} - n_p)/2$. It follows from Lemma 3 that $|f(z)| < 3$ on each $\gamma_v \subset T_{s,t}$, and on $\overline{\triangle}_1$ we get

$$\text{(i)} \qquad |h(z)| \leq 6(\omega - \alpha + 1) l_{n_{p+1}} / l_{n_p+m} < (2b^{1/4})^{n_{p+1}} .$$

Now we suppose that n_{p+1} is so large that $(2b^{1/4})^{n_{p+1}} < \sqrt{a}/1000$. Then it follows from Rouché's theorem that g takes every value outside $C(\infty, 8A\sqrt{\xi_s}) \cup C(0, 2\delta) \cup C(1, 2\delta)$ exactly once in $\triangle_{s,t}$. g is analytic in $T_{s,t}$ and applying Rouché's theorem again, we see that g takes every value outside $C(\infty, 8A\sqrt{\xi_s})$ exactly once in $T_{s,t}$. Now $0 \in g(T_{s+1,2t-1})$ and $1 \in g(T_{s+1,2t})$ because if for instance $0 \in g(\overline{\triangle}_{s,t})$ then we see that f takes the value 0 in $\triangle_1$. Furthermore, we see that g is schlicht in

$$\triangle_4 = \left\{ z : |z - z_{s,t}| < \frac{l_s}{72A\sqrt{\xi_s}} \right\} .$$

Let $\Gamma_{n_p,j} \subset T_{s,t}$. We set

$$L_j = \{ z : z = z_{n_p,j} + \lambda \eta_{n_p} l_{n_p} , \; -2 \leq \lambda \leq 2 \} .$$

Then $g(L_j)$ has the following property: If $w_k \in g(L_j)$, $k = 1, 2$, $w_1 \neq w_2$ and $\operatorname{Im} w_1 \leq \operatorname{Im} w_2$, then

$$\text{(6)} \qquad |\pi/2 - \arg (w_2 - w_1)| \leq \pi/12 .$$

Furthermore, if $j \neq k$ then the distance between $g(L_j)$ and $g(L_k)$ is at least $l_{n_p-1}/(4l_s) > \xi_s^{n_p}$. We denote

$$U_j(r) = \{ w : \text{distance between } w \text{ and } g(L_j) \leq r \} .$$

Let $\beta_q = \Gamma_{n_p+m,q} \subset T_{n_p,j}$. Then $\beta_q \cap L_j \neq \emptyset$ and we see that $g(\beta_q) \subset U_j(b^{n_p+1/4})$. Then it follows from (i) that $f(\beta_q) \subset U_j(r)$ where $r = 2(2b^{1/4})^{n_{p+1}}$.

Now we assume that the sequence $\{n_k\}$ is chosen such that

$$\text{(7)} \qquad \lim \frac{2^{n_k}}{n_{k+1}} = 0 .$$

Then for large i, $U_j(r) \cap U_k(r) = \emptyset$ if $j \neq k$ and we see that f takes in $\triangle_1$ every value outside $C(\infty, \delta) \cup (\cup U_j(r))$ where the number of the sets $U_j(r)$, $\Gamma_{n_p,j} \subset T_{s,t}$, is at most 2^{n_p}. We see that $F \cap C(0, \delta) \subset \cup U_j(r)$. It follows from (5) and (6) that $F \cap U_j(r)$ is contained in a disc B_j with radius $4r$, and we get from (7) that

$$\frac{2^{n_p}}{\log \frac{1}{4r}} = \frac{2^{n_p}}{-n_{p+1} \log (2b^{1/4}) - \log 8} \to 0$$

as $i \to \infty$. Therefore $F \cap C(0, \delta)$ has logarithmic measure zero. We are led to a contradiction and the second assertion of Theorem 2 is proved. The proof of the assertion concerning $\operatorname{Dim}(E)$ will be omitted.

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