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ON THE DILATATION OF ISOMORPHISMS BETWEEN COVERING GROUPS

BY

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Introduction

A group G of Möbius transformations fixing a disk or half-plane D is called a *covering group* if it is discontinuous in the following sense: For each point $z \in D$ there exists a neighborhood U such that $g(z) \notin U$ whenever $g \neq id$ lies in G. Hence a covering group may contain hyperbolic and parabolic transformations only.

In [3] we introduced the *dilatation* $\delta(j)$ of an isomorphism $j : G \to G'$ defined as follows: If $\varkappa(g)$ denotes the multiplier of a Möbius transformation g, then $\delta(j)$ is the smallest number $1 \leq a \leq \infty$ for which $\varkappa(g)^{1/a} \leq \varkappa(j(g)) \leq \varkappa(g)^a$ holds for all $g \in G$. As examples of the case where $\delta(j) < \infty$ we have the isomorphisms j induced by quasiconformal mappings f, i.e. $j(g) = f \circ g \circ f^{-1}$ for all $g \in G$. On the other hand, if there is a parabolic $g \in G$ such that $j(g)$ is hyperbolic or vice versa, then $\delta(j) = \infty$.

In § 1 we consider isomorphisms j between noncyclic covering groups with $\delta(j) = \infty$. We show that the dilatation of j restricted to elements whose type is preserved is also infinite. In § 2 we consider parabolic elements under an isomorphism with a finite dilatation.

In § 3 we prove the following theorem: Let $\{g_1, g_2, \ldots\}$ be a set of generators of G. Suppose that an isomorphism j: $G \to G'$ preserves the multipliers of the elements of the type $(g_i^\alpha \circ g_k^\beta)^a \circ (g_m^\gamma \circ g_n^\varepsilon)^a$ where α, β, γ, ε are integers and $a = 1, 2$. Then j is induced by a Möbius transformation.

Let j: $G \to G'$ be an isomorphism between covering groups acting on the upper half-plane H. A homeomorphism $\varphi : R \cup \{\infty\} \to R \cup \{\infty\}$, where R is the set of the real numbers, is called a *boundary mapping* of j if $\varphi \circ g = j(g) \circ \varphi$ holds for all $g \in G$. In § 4 we characterize $\delta(j)$ in terms of the local Hölder continuity of φ and φ^{-1}. As a corollary we then obtain the following result: If φ has a K-quasiconformal extension to the extended complex plane, then $\delta(j) \leq K$.

§ 1. Isomorphisms with an infinite dilatation

For a hyperbolic transformation g, let $\varkappa(g)$ denote the multiplier and $\mathrm{P}(g)$ and $\mathrm{N}(g)$ the attracting and the repelling fixed point. The parameters $\varkappa(g)$, $\mathrm{P}(g)$, and $\mathrm{N}(g)$ determine g uniquely. We have $\varkappa(g) = (z, g(z), \mathrm{P}(g), \mathrm{N}(g)) > 1$, the cross ratio being defined as in [3, § 1]. If g is parabolic, we define $\varkappa(g) = 1$ and $\mathrm{P}(g) = \mathrm{N}(g)$ as the only fixed point of g.

Let a parabolic or hyperbolic transformation g be given in the form $z \mapsto g(z) = (az + b)/(cz + d)$ with $ad - bc = 1$. Then $a + d$ is always real, and $\chi(g) = |a + d|$ is the *trace* of g. It follows that

$$\chi(g) = \varkappa(g)^{1/2} + \varkappa(g)^{-1/2} .$$

Hence $\chi(g) \geq 2$, where the equality holds if and only if g is parabolic.

Let $j : M \to M'$ be a mapping between sets of hyperbolic and parabolic transformations. A calculation shows that the dilatation of j can also be defined in terms of $\chi(g)$.

Theorem 1. *Suppose that for any $g \in M$ the transformations g^n, $n = 2, 3, \ldots$, are in M, and suppose that $j(g^n) = j(g)^n$. If $1 \leq a \leq \infty$ is the smallest number for which $\chi(g)^{1/a} \leq \chi(j(g)) \leq \chi(g)^a$ holds for all $g \in M$, then $a = \delta(j)$.*

Proof. Let $g \in M$, $k = \varkappa(g)$ and $k' = \varkappa(j(g))$. Suppose that we have $\chi(j(g)^n) \leq \chi(g^n)^a$ for $n = 1, 2, \ldots$. Then

$$(k')^{n/2} + (k')^{-n/2} \leq (k^{n/2} + k^{-n/2})^a ,$$

and hence

$$(k')^n \leq (k')^n + (k')^{-n} + 2 \leq (k^n + k^{-n} + 2)^a \leq (2k^n)^a$$

from some $n = n_1$ on. Therefore

$$k' \leq (2k^n)^{a/n} = (2^{1/n} k)^a ,$$

and letting $n \to \infty$ we obtain $k' \leq k^a$. Similarly, if $\chi(j(g)^n) \geq \chi(g^n)^{1/a}$ for $n = 1, 2, \ldots$, then we get $k \leq (k')^a$. Thus we have $k^{1/a} \leq k' \leq k^a$.

Conversely, suppose that $k^{1/a} \leq k' \leq k^a$. Then

$$\chi(j(g)) = \sqrt{k'} + 1/\sqrt{k'} \leq (\sqrt{k})^a + (1/\sqrt{k})^a \leq (\sqrt{k} + 1/\sqrt{k})^a = \chi(g)^a ,$$

and similarly $\chi(g) \leq \chi(j(g))^a$. □

Let $j : G \to G'$ be an isomorphism between covering groups G and G'. If there is a parabolic $g \in G$ such that $j(g)$ is hyperbolic or vice versa, then $\delta(j) = \infty$. By the following theorem, the dilatation of j restricted to elements whose type is preserved is also infinite.

Theorem 2. *Let $j : G \to G'$ be an isomorphism with $\delta(j) = \infty$. Define G^* as the set of all hyperbolic elements $g \in G$ for which $j(g)$ is hyperbolic. If G is not cyclic, then $\delta(j \mid G^*) = \infty$.*

Proof. It follows from Lemma 3.1 in [3] that $G^* \neq \emptyset$. If j preserves the type of all transformations of G, then there is nothing to prove. In other cases choose a hyperbolic $g_1 \in G$ such that $j(g_1)$ is parabolic (if this is not possible, then we consider the isomorphism $j^{-1} : G' \to G$) and let $g_2 \in G^*$. Then we have ([3, (4.11)]):

$$\chi(g_1^n \circ g_2) = \left| \frac{k_1^n k_2 + 1 - x(k_1^n + k_2)}{(1 - x)\,(k_1^n k_2)^{1/2}} \right|$$

where $k_i = \varkappa(g_i)$ and $x = 1 - (\mathrm{N}(g_1)\,,\ \mathrm{N}(g_2)\,,\ \mathrm{P}(g_1)\,,\ \mathrm{P}(g_2))$. Therefore

$$\lim_{n \to \infty} \frac{\chi(g_1^n \circ g_2)}{k_1^{n/2}} = k_2^{-1/2} \left| \frac{k_2 - x}{1 - x} \right| .$$

If $k_2 - x = 0$, we replace g_2 by g_2^2. Then there is a $b \geq 1$ such that we have for $n = 1, 2, \ldots$

$$(1.1) \qquad (1/b) k_1^{n/2} \leq \chi(g_1^n \circ g_2) \leq b k_1^{n/2} .$$

We now consider the group G'. Since $g_1' = j(g_1)$ is parabolic and $g_2' = j(g_2)$ hyperbolic, we may normalize such that

$$g_1'(z) = z + \omega\,, \quad g_2'(z) = (kz)/((k - 1)z + 1)\,,$$

where $k = \varkappa(g_2') > 1$. We may also assume that $\omega > 0$ since we can replace g_1' by $(g_1')^{-1}$ if necessary. Then we have

$$((g_1')^n \circ g_2')\,(z) = \frac{(k + n\omega(k - 1))z + n\omega}{(k - 1)z + 1}\,,$$

and hence

$$(1.2) \qquad \chi((g_1')^n \circ g_2') = \frac{1 + k + n\omega(k - 1)}{k^{1/2}} .$$

From (1.1) and (1.2) we conclude that $g_1^n \circ g_2 \in G^*$ from some $n = n_0$ on.

By (1.2), $\chi((g_1')^n \circ g_2') \leq 2n\omega k$ for sufficiently large n. Then we have for any $1 \leq a < \infty$

$$\chi(g_1^n \circ g_2)^{1/a} \geq (k_1^{n/2}/b)^{1/a} > 2n\omega k \geq \chi((g_1')^n \circ g_2')$$

from some $n = n_a$ on. Therefore $\delta(j \mid G^*) = \infty$ by Theorem 1. $\square$

§ 2. Distortion of parabolic transformations

Let $j : G \to G'$ be an isomorphism between covering groups which act on the upper half-plane H, and suppose that $\delta(j) < \infty$. In this section we consider the behavior of the parabolic elements of G under j.

A parabolic transformation $g \in G$ fixing ∞ is of the type

$$g(z) = z + \omega . \tag{2.1}$$

If g is parabolic with $\mathrm{P}(g) \neq \infty$, then g has a unique representation in the form

$$\frac{1}{g(z) - \mathrm{P}(g)} = \frac{1}{z - \mathrm{P}(g)} + \omega . \tag{2.2}$$

We call the number $\omega = \omega(g)$ defined by (2.1—2) the *translation vector* of g. From $g(H) = H$ it follows that $\mathrm{P}(g)$ and $\omega(g)$ are real. If the transformation g in (2.2) is given in the form $g(z) = (az + b)/(cz + d)$ with $a + d = 2$, then an elementary calculation shows that $\omega(g) = c$.

To interpret geometrically the translation vector $\omega(g)$, consider first the transformation (2.1) with $\omega > 0$. If we define the non-euclidean metric in H by $(\mathrm{Im}\, z)^{-1}|dz|$, then the non-euclidean length of the euclidean line segment $\{x + i \mid x_0 \leq x \leq x_0 + \omega\}$ is ω. Since the non-euclidean distances are invariant under Möbius transformations, we then obtain from (2.2) the following interpretation for $\omega(g)$: Suppose that $\mathrm{P}(g) \neq \infty$ and define $K(g)$ as the circle of diameter one through $\mathrm{P}(g)$ and $\mathrm{P}(g) + i$. If $z \in K(g)$, then $|\omega(g)|$ is the non-euclidean length of the part of $K(g)$ between z and $g(z)$. From this it follows that we have $\omega(g) = \omega(h \circ g \circ h^{-1})$ for all translations $h: z \mapsto z + b$, b real.

For a hyperbolic transformation h fixing H, let $\mathrm{Ax}(h)$ be the axis of h (i.e. the circle through $\mathrm{P}(h)$ and $\mathrm{N}(h)$ orthogonal to R). If $z \in \mathrm{Ax}(h)$, then $\log \varkappa(h)$ is the non-euclidean length of the part of $\mathrm{Ax}(h)$ between z and $h(z)$. Thus $|\omega(g)|$ has some analogy with $\log \varkappa(h)$. However, if we normalize such that j fixes the translation $z \mapsto z + 1$, then $|\omega(g)|$ does not behave under j as $\log \varkappa(h)$ but like $\varkappa(h)$.

Theorem 3. *Suppose that the transformation $g_0 : z \mapsto z + 1$ lies in $G \cap G'$. Let $j : G \to G'$ be an isomorphism such that $a = \delta(j) < \infty$. If $j(g_0) = g_0$, then $|\omega(g)|^{1/a} \leq |\omega(j(g))| \leq |\omega(g)|^a$ holds for all parabolic transformations g of G.*

Proof. We first note that for any parabolic element $h \neq g_0$ of G we have $|\omega(h^{-1} \circ g_0 \circ h)| = \omega(h)^2$. To prove this, let h be the transformation $z \mapsto ((1 + \omega x)z - \omega x^2)/(\omega z + 1 - \omega x)$, where $x = \mathrm{P}(h)$ and $\omega = \omega(h)$. Then

$$(h^{-1} \circ g_0 \circ h)\,(z) = \frac{(1 + \omega - \omega^2 x)z + (1 - \omega x)^2}{-\omega^2 z + 1 - \omega + \omega^2 x} .$$

Hence $\omega(h^{-1} \circ g_0 \circ h) = -\omega^2$.

Let $g \neq g_0$ be a fixed parabolic transformation of G. Define $g_1 = g^{-1} \circ g_0 \circ g$ and inductively $g_n = g_{n-1}^{-1} \circ g_0 \circ g_{n-1}$ for $n = 2, 3, \ldots$. Then $\{g_n\}$ is a sequence of parabolic elements of G. By the above remark we have $|\omega(g_n)| = \omega(g_{n-1})^2$. Therefore

$$|\omega(g_n)| = \omega(g)^{2^n} \tag{2.3}$$

for $n = 1, 2, \ldots$.

Since $a = \delta(j) < \infty$, $\{j(g_n)\}$ is a sequence of parabolic elements of G'. Because $j(g_0) = g_0$, (2.3) holds if g_n and g are replaced by $j(g_n)$ and $j(g)$, respectively.

For any parabolic transformation $h \neq g_0$ of G we have

$$\chi(g_0 \circ h) = |2 + \omega(h)|. \tag{2.4}$$

Since $\chi(g_0 \circ h^{\pm 1}) \geq 2$, it follows that $|\omega(h)| \geq 4$. We apply (2.4) to the transformations g_n and $j(g_n)$. Then by Theorem 1

$$|2 + \omega(g_n)|^{1/a} \leq |2 + \omega(j(g_n))| \leq |2 + \omega(g_n)|^a.$$

Formula (2.3) and the triangle inequality yield

$$0 < (\omega(g)^{2^n} - 2)^{1/a} = (|\omega(g_n)| - 2)^{1/a} \leq |2 + \omega(g_n)|^{1/a} \leq$$

$$|2 + \omega(j(g_n))| \leq 2 + \omega(j(g))^{2^n}.$$

Hence

$$[(\omega(g)^{2^n} - 2)^{1/2^n}]^{1/a} \leq [2 + \omega(j(g))^{2^n}]^{1/2^n},$$

and letting $n \to \infty$ we obtain $|\omega(g)|^{1/a} \leq |\omega(j(g))|$. It follows similarly that $|\omega(j(g))| \leq |\omega(g)|^a$. $\square$

Remark. Let G be a covering group containing the transformation $g_0 : z \mapsto z + 1$. As remarked above, it follows from (2.4) that $|\omega(g)| \geq 4$ for all parabolic elements $g \neq g_0$ of G. This bound is sharp: Let $g_1(z) = z/(4z + 1)$ and let G_1 be the group generated by g_0 and g_1. Then G_1 is a covering group and we have $\omega(g_1) = 4$.

§ 3. Isomorphisms with dilatation one

For a set M of Möbius transformations, let $\mathrm{Fix}(M)$ denote the set of fixed points of non-identity transformations of M. If the set $\mathrm{Fix}(G)$ is dense in a circle or a straight line, then the covering group G is said to be of the first kind. If not, then G is of the second kind.

Let $j : G \to G'$ be an isomorphism with $\delta(j) = 1$. If G and G' are of the first kind, then by Theorem 4.3 in [3] there is a Möbius transformation *h inducing* j, i.e., $j(g) = h \circ g \circ h^{-1}$ for all $g \in G$. This result is valid in the following more general form also for groups of the second kind.

Theorem 4. *Let* $E = \{g_1, g_2, \ldots\}$ *be a set of generators of a covering group* G. *Let* F *consist of the transformations of the form* $(g_i^\alpha \circ g_k^\beta)^a \circ (g_m^\gamma \circ g_n^\varepsilon)^a$, *where* α, β, γ, ε *are integers and* $a = 1, 2$. *If an isomorphism* $j : G \to G'$ *preserves the multipliers of the elements of* F, *then* j *is induced by a Möbius transformation.*

Proof. It suffices to show that there is a Möbius transformation h such that $j(g_i) = h \circ g_i \circ h^{-1}$ for all $g_i \in E$.

(A) Suppose first that E contains at least one hyperbolic element.

If $E = \{g_1\}$, g_1 hyperbolic, then j is induced by any Möbius transformation sending $\mathrm{P}(g_1)$ to $\mathrm{P}(j(g_1))$ and $\mathrm{N}(g_1)$ to $\mathrm{N}(j(g_1))$. Let $E = \{g_1, g_2\}$ with g_1 parabolic and g_2 hyperbolic. We show that the Möbius transformation which sends $\mathrm{P}(g_i)$ to $\mathrm{P}(j(g_i))$, $i = 1, 2$, and $\mathrm{N}(g_2)$ to $\mathrm{N}(j(g_2))$ induces j. Since we can replace G and G' by conjugate groups $G_1 = hGh^{-1}$ and $G_1' = h'G'(h')^{-1}$, we may assume that g_2 and $j(g_2)$ both are the transformation $z \mapsto kz/((k-1)z+1)$ and that $\mathrm{P}(g_1) = \mathrm{P}(j(g_1)) = \infty$. Since we have $\chi(g_1^n \circ g_2) = \chi(j(g_1)^n \circ j(g_2))$, it follows from (1.2) that

$$|1 + k + n(k-1)\omega(g_1)| = |1 + k + n(k-1)\omega(j(g_1))|$$

for $n = 1, 2, \ldots$. Therefore $\omega(g_1) = \omega(j(g_1))$, and the assertion follows.

Let $\mathrm{Fix}(E)$ contain at least four distinct points. Choose $z_i \in \mathrm{Fix}\,(E)$ such that $(z_1, z_2, z_3, z_4) > 1$. Suppose that $z_1 = \mathrm{N}(h_1)$, $z_2 = \mathrm{N}(h_2)$, $z_3 = \mathrm{P}(h_3)$, $z_4 = \mathrm{P}(h_4)$, where for each i, $i = 1, 2, 3, 4$, either $h_i \in E$ or $h_i^{-1} \in E$. If

$$w_1 = \mathrm{N}(j(h_1)), \qquad w_2 = \mathrm{N}(j(h_2)),$$
$$w_3 = \mathrm{P}(j(h_3)), \qquad w_4 = \mathrm{P}(j(h_4)),$$

then the points w_i are well-defined and distinct. We show that

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4). \tag{3.1}$$

To prove (3.1), set $g_{1n} = h_3^n \circ h_1^n$ and $g_{2n} = h_4^n \circ h_2^n$. Then by Lemma 3.1 in [3], $\mathrm{N}(g_{in}) \to z_i$, $\mathrm{P}(g_{in}) \to z_{i+2}$ and similarly $\mathrm{N}(j(g_{in})) \to w_i$, $\mathrm{P}(j(g_{in})) \to w_{i+2}$ as $n \to \infty$, $i = 1, 2$. Thus it suffices to show that

$$(\mathrm{N}(g_{1n}), \mathrm{N}(g_{2n}), \mathrm{P}(g_{1n}), \mathrm{P}(g_{2n})) = (\mathrm{N}(j(g_{1n})), \mathrm{N}(j(g_{2n})), \mathrm{P}(j(g_{1n})), \mathrm{P}(j(g_{2n}))) \tag{3.2}$$

for sufficiently large values of n. Choose n_0 such that for $n \geq n_0$

$$(\mathrm{N}(g_{1n}) , \ \mathrm{N}(g_{2n}) , \ \mathrm{P}(g_{1n}) , \ \mathrm{P}(g_{2n})) > 1 .$$

Since j preserves the multipliers of g_{1n}, g_{2n}, $g_{1n} \circ g_{2n}$ and $g_{1n}^2 \circ g_{2n}^2$, we can apply the proof of Theorem 4.3 in [3] by replacing g_i by g_{in}. Then it follows that (3.2) holds for $n \geq n_0$, and (3.1) is proved. By (3.1) there is a Möbius transformation h such that $h(\mathrm{P}(g_i^{\pm 1})) = \mathrm{P}(j(g_i)^{\pm 1})$ for all $g_i \in E$. By the previous part of the proof we have $j(g_i) = h \circ g_i \circ h^{-1}$ for $g_i \in E$. Thus case (A) is proved.

(B) Suppose secondly that E contains only parabolic elements.

The case when E consists of one parabolic element is clear. Let $E = \{g_1 , g_2\}$ with g_1 and g_2 parabolic. We may suppose that g_1 and $j(g_1)$ both are the transformation $z \mapsto z + 1$ and that $\mathrm{P}(g_2) = \mathrm{P}(j(g_2))$. Since we have $\chi(g_1 \circ g_2^n) = \chi(j(g_1) \circ j(g_2)^n)$, it follows from (2.4) that

$$|2 + n\,\omega(g_2)| = |2 + n\,\omega(j(g_2))|$$

for $n = 1 , 2 , \ldots$. Therefore $\omega(g_2) = \omega(j(g_2))$, and it follows that $j = id$.

Let $E = \{g_1 , g_2 , g_3\}$ with g_1, g_2, g_3 parabolic. We show that the Möbius transformation sending $\mathrm{P}(g_i)$ to $\mathrm{P}(j(g_i))$ induces j. We normalize such that

$$\mathrm{P}(g_1) = \mathrm{P}(j(g_1)) = \infty , \quad \mathrm{P}(g_2) = \mathrm{P}(j(g_2)) = 0 , \quad \mathrm{P}(g_3) = \mathrm{P}(j(g_3)) = -1 .$$

Then it suffices to show that $j = id$.

Let $\omega_i = \omega(g_i)$, $i = 1 , 2 , 3$. Then we have (cf. 2.4))

$$\chi(g_1^n \circ g_i) = |2 + n\,\omega_1\omega_i|$$

for $i = 2 , 3$. A simple calculation yields

$$(3.3) \qquad (g_3 \circ g_2^n)(z) = \frac{(1 - n\omega_2\omega_3 - \omega_3)\,z - \omega_3}{(n\,\omega_2 + n\omega_2\omega_3 + \omega_3)\,z + \omega_3 + 1} .$$

Hence

$$\chi(g_3 \circ g_2^n) = |2 - n\,\omega_2\omega_3| ,$$

and we also obtain similar expressions for

$$\chi(j(g_1)^n \circ j(g_i)) \quad \text{and} \quad \chi(j(g_3) \circ j(g_2)^n) .$$

Let $\omega_i' = \omega(j(g_i))$. Then we have the following equations

$$|2 + n\omega_1\omega_i| = |2 + n\omega_1'\omega_i'| , \quad i = 2 , 3 ,$$

$$|2 - n\omega_2\omega_3| = |2 - n\omega_2'\omega_3'|$$

for $n = 1 , 2 , \ldots$. Hence $\omega_i\omega_k = \omega_i'\omega_k'$ holds for $i \neq k$, and we have either $\omega_i = \omega_i'$ or $\omega_i = -\,\omega_i'$ for $i = 1 , 2 , 3$. To verify that the latter case is impossible, consider the transformation $g_3 \circ g_2 \circ g_1^n$. It follows from (3.3) that

$$\chi(g_3 \circ g_2 \circ g_1^n) = |2 - \omega_2\omega_3 + n(\omega_1\omega_2 + \omega_1\omega_3 + \omega_1\omega_2\omega_3)| .$$

From $\chi(g_3 \circ g_2 \circ g_1^n) = \chi(j(g_3) \circ j(g_2) \circ j(g_1)^n)$ we infer that ω_i and ω_i' have the same sign. Hence we have $j(g_i) = g_i$ for $i = 1, 2, 3$ and it follows that $j = id$ as asserted.

Suppose finally that $\mathrm{Fix}(E)$ contains at least four points. Similarly as in (A) we can show that there is a Möbius transformation h such that $h(\mathrm{P}(g_i^{\pm 1})) = \mathrm{P}(j(g_i)^{\pm 1})$ for all $g_i \in E$. From the case of three generating transformations it then follows that h induces j. □

About results related to Theorem 4 we refer to [2] pp. 150—151.

If we only know that $\varkappa(g_i) = \varkappa(j(g_i))$ for all $g_i \in E$, then j need not be induced by any Möbius transformation. This is seen considering e.g. the case when the Riemann surfaces corresponding to G and G' are compact.

§ 4. The boundary mapping of an isomorphism with a finite dilatation

Let G and G' be covering groups acting on the upper half-plane H. A homeomorphism $\varphi : R \cup \{\infty\} \to R \cup \{\infty\}$ is called a *boundary mapping* of an isomorphism $j : G \to G'$ if $\varphi \circ g = j(g) \circ \varphi$ for all $g \in G$. Thus we have $\varphi(\mathrm{P}(g)) = \mathrm{P}(j(g))$ for $g \in G$. (Therefore, if G and G' are of the first kind, an isomorphism $j : G \to G'$ has at most one boundary mapping.) In this section we consider the interrelation between φ and $\delta(j)$.

Let K_1 and K_2 be circles or straight lines and $\psi : K_1 \to K_2$ a homeomorphism. Let $z_0 \in K_1$ be a finite point such that $\psi(z_0) \neq \infty$. We say that ψ is *Hölder continuous with the exponent* α, $0 < \alpha \leq 1$, at z_0 if there is a constant $A \geq 1$ and a neighborhood $U \subset K_1$ of z_0 such that

$$(1/A)|z - z_0|^{1/\alpha} \leq |\psi(z) - \psi(z_0)| \leq A|z - z_0|^{\alpha}$$

for all $z \in U$. The mapping ψ is Hölder continuous with the exponent α at the point ∞ or at a point z_0 where $\psi(z_0) = \infty$ if $\psi(1/z)$ has this property at the origin or $1/\psi(z)$ at z_0, respectively. If ψ is Hölder continuous with the exponent $\alpha = 1$ at z_0, then we say that ψ is a *Lipschitz mapping* at z_0.

The Hölder continuity of ψ is invariant under Möbius transformations, i.e., if h_1 and h_2 are Möbius transformations and ψ is Hölder continuous with the exponent α at z_0, then the same is true of $h_2 \circ \psi \circ h_1^{-1}$ at the point $h_1(z_0)$.

Theorem 5. *Suppose that* φ *is a boundary mapping of an isomorphism* $j : G \to G'$. *Let* $B(j)$ *be the set of the real numbers* α , $0 < \alpha \leq 1$, *such that* φ *is Hölder continuous with the exponent* α *at the fixed points of all hyperbolic elements of* G . *Then* $B(j) \neq \emptyset$ *if and only if* $\delta(j) < \infty$. *If* $B(j) \neq \emptyset$, *then we have* $1/\delta(j) = \max \alpha$, $\alpha \in B(j)$.

Proof. Let $g \in G$ be hyperbolic. From the existence of φ we conclude that $j(g)$ is also hyperbolic. Since the α-Hölder continuity of φ at a point is invariant under Möbius transformations, we may assume that

$$\mathrm{N}(g) = \mathrm{N}(j(g)) = 0 , \quad \mathrm{P}(g) = \mathrm{P}(j(g)) = \infty$$

and $\varphi(1) = 1$.

Suppose that $\alpha \in B(j)$. Then there is an $A \geq 1$ such that

$$\begin{aligned} |\varphi(g^{-n}(1)) - \varphi(0)| &= \varphi(g^{-n}(1)) = j(g)^{-n}(1) \\ &= \varkappa(j(g))^{-n} \leq A\,|g^{-n}(1) - 0|^{\alpha} = A\varkappa(g)^{-n\alpha} \end{aligned}$$

from some $n = n_0$ on. Thus $\varkappa(j(g)) \geq A^{-1/n}\varkappa(g)^{\alpha}$, and letting $n \to \infty$ we obtain $\varkappa(j(g)) \geq \varkappa(g)^{\alpha}$. Similarly it follows that $\varkappa(g) \geq \varkappa(j(g))^{\alpha}$. Hence $\delta(j) \leq 1/\alpha$.

Conversely, suppose that $a = \delta(j) < \infty$. Choose t such that $0 < t < 1$ and let n be the natural number for which $1/\varkappa(g)^{n+1} \leq t < 1/\varkappa(g)^{n}$. Since $\varphi(1) = 1$, we have $1/\varkappa(j(g))^{n+1} \leq \varphi(t) < 1/\varkappa(j(g))^{n}$. Hence

$$\frac{\varphi(t)}{t^{1/a}} \leq \frac{\varkappa(g)^{(n+1)/a}}{\varkappa(j(g))^{n}} \leq \frac{\varkappa(g)^{(n+1)/a}}{\varkappa(g)^{n/a}} = \varkappa(g)^{1/a} ,$$

and similarly $\varphi(t)/t^{a} \geq 1/\varkappa(g)^{a}$. If $-1 < t < 0$, then we obtain $|\varphi(t)|/|t|^{1/a} \leq |\varphi(-1)|\varkappa(g)^{1/a}$ and $|\varphi(t)|/|t|^{a} \geq |\varphi(-1)|/\varkappa(g)^{a}$. Hence

$$1/\delta(j) \in B(j)$$

and the first assertion is proved. Moreover, by the first part of the proof we have $\delta(j) \leq 1/\alpha$ for all $\alpha \in B(j)$. Thus $1/\delta(j) = \max \alpha$, $\alpha \in B(j)$. □

As in Theorem 4, let $E = \{g_1 , g_2 , \ldots\}$ be a set of generators of G and let F be the set of the transformations $(g_i^{\alpha} \circ g_k^{\beta})^a \circ (g_m^{\gamma} \circ g_n^{\varepsilon})^a$. Then we have the following generalization for Theorem 5.1 in [3]:

Theorem 6. *If an isomorphism* $j : G \to G'$ *has a boundary mapping which is a Lipschitz mapping at the points of* Fix (F) , *then* j *is induced by a Möbius transformation.*

Theorem 6 follows from Theorem 4 and the proof of Theorem 5.

The following theorem shows that the Hölder continuity of a boundary mapping φ of $j : G \to G'$ at the fixed points of the parabolic elements of G does not depend on $\delta(j)$.

Theorem 7. *If* $g \in G$ *is parabolic, then all boundary mappings of an isomorphism* $j: G \to G'$ *are Lipschitz mappings at* $\mathrm{P}(g)$.

Proof. We may assume that g and $j(g)$ both are the transformation $z \mapsto z/(z+1)$ and that $\varphi(\infty) = \infty$. Choose t such that $0 < t < 1$ and let n be the natural number for which $1/(n+1) < t \leq 1/n$. Since $g^n(\infty) = j(g)^n(\infty) = 1/n$, we have $1/(n+1) < \varphi(t) \leq 1/n$. Therefore $n/(n+1) \leq \varphi(t)/t \leq (n+1)/n$, and it follows that $t/2 \leq \varphi(t) \leq 2t$. Replacing g by g^{-1} we obtain $|t|/2 \leq |\varphi(t)| \leq 2|t|$ for $-1 < t < 0$. □

By Theorem 4.1 in [3] we have $\delta(j) \leq K$ if j is induced by a K-quasiconformal mapping $f\colon H \to H$. This theorem is a special case of the following more general result:

Theorem 8. *Let* $\varphi: R \cup \{\infty\} \to R \cup \{\infty\}$ *be a boundary mapping of* $j: G \to G'$. *If there is a K-quasiconformal mapping* $f: H \to H$ *such that* $f|(R \cup \{\infty\}) = \varphi$, *then* $\delta(j) \leq K$.

Proof. Let h and h' be Möbius transformations mapping H onto the unit disk such that $f_1 = h' \circ f \circ h^{-1}$ fixes the origin. By Theorem II. 3.2 in [1], the restriction of f_1 to the unit circle is Hölder continuous with the exponent $1/K$. Then the same holds true of φ at every point of $R \cup \{\infty\}$ and we have $\delta(j) \leq K$ by Theorem 5. □

Let $\psi: R \cup \{\infty\} \to R \cup \{\infty\}$ be an increasing homeomorphism fixing ∞. If for an interval $I \subset R$ there is a constant λ, $1 \leq \lambda < \infty$, such that

$$1/\lambda \leq \frac{\psi(x+t) - \psi(x)}{\psi(x) - \psi(x-t)} \leq \lambda \tag{4.1}$$

holds whenever $x \pm t \in I$, we say that ψ is *λ-quasisymmetric* on I. The mapping ψ is called λ-quasisymmetric if (4.1) holds for all x and t. Note that ψ is 1-quasisymmetric if and only if ψ is the restriction of a Möbius transformation $z \mapsto az + b$ with $a > 0$ and b real.

If an isomorphism $j: G \to G'$ has a λ-quasisymmetric boundary mapping φ, then

$$\delta(j) \leq \log 2/\log(1 + 1/\lambda) \tag{4.2}$$

by Theorem 4.2 in [3]. On the other hand, there is a K-quasiconformal extension $f: H \to H$ of φ with $K = \min(8\lambda, \lambda^2)$ (see [1, II.6.5]). Hence we have $\delta(j) \leq \min(8\lambda, \lambda^2)$ by Theorem 8. However, one can verify by calculation that $\log 2/\log(1 + 1/\lambda) \leq \min(8\lambda, \lambda^2)$ for all values $\lambda \geq 1$. Hence Theorem 8 implies (4.2) only if a λ-quasisymmetric φ always has a $(\log 2/\log(1 + 1/\lambda))$-quasiconformal extension $f: H \to H$.

By the following theorem, (4.2) can be deduced also from the local λ-quasisymmetry of φ.

Theorem 9. *Let* $\varphi : R \cup \{\infty\} \to R \cup \{\infty\}$ *be a boundary mapping of an isomorphism* $j : G \to G'$. *If for every hyperbolic* $g \in G$ *satisfying* $\mathrm{P}(g) \neq \infty$ *there is an interval* $I \ni \mathrm{P}(g)$ *on which* φ *is* λ*-quasisymmetric, then* $\delta(j) \leq \log 2/\log(1 + 1/\lambda)$.

Proof. Let $g \in G$ be hyperbolic, $\mathrm{P}(g) \neq \infty$ and h, h' Möbius transformations fixing H such that $h(\mathrm{P}(g)) = h'(\mathrm{P}(j(g))) = 0$, $h(\mathrm{N}(g)) = h'(\mathrm{N}(j(g))) = \infty$. For every $\varepsilon > 0$ there is an interval I containing the origin such that the mapping $\varphi_1 = h' \circ \varphi \circ h^{-1}$ is $(\lambda + \varepsilon)$-quasisymmetric on I. Then there are 1-quasisymmetric mappings h_1 and h_1' fixing the origin such that $\varphi_1' = h_1' \circ \varphi_1 \circ h_1^{-1}$ is $(\lambda + \varepsilon)$-quasisymmetric on the closed unit interval. Replacing φ by φ_1' and λ by $\lambda + \varepsilon$ in the proof of Theorem 4.2 in [3] we can show that $\varkappa(g)^{1/a} \leq \varkappa(j(g)) \leq \varkappa(g)^a$ holds for

$$a = \log 2/\log(1 + 1/(\lambda + \varepsilon))\,. \ \square$$

Suppose that all boundary mappings of an isomorphism $j : G \to G'$ are increasing and fix the point ∞. To our knowledge, it is an open question whether $\delta(j) < \infty$ then implies that j has a boundary mapping which is λ-quasisymmetric for some fixed $\lambda \geq 1$ in a neighborhood of the attracting fixed point of every hyperbolic element of G. However, the following theorem tells that all boundary mappings of j have a quasisymmetry property at the fixed points of the parabolic elements of G.

Theorem 10. *Suppose that the transformation* $g_0 : z \mapsto z + 1$ *lies in* $G \cap G'$. *Let* $\varphi : R \cup \{\infty\} \to R \cup \{\infty\}$ *be a boundary mapping of an isomorphism* $j : G \to G'$ *for which* $j(g_0) = g_0$. *If* $g \neq g_0$ *is a parabolic element of* G, $x_0 = \mathrm{P}(g)$ *and* $a = \delta(j) < \infty$, *then we have for all* $t > 0$

$$\omega(g)^{-a} \leq \frac{\varphi(x_0 + t) - \varphi(x_0)}{\varphi(x_0) - \varphi(x_0 - t)} \leq \omega(g)^a\,.$$

Proof: It means no restriction to consider only the case when φ is increasing. Using 1-quasisymmetric mappings of the type $z \mapsto z + b$ we normalize such that $\mathrm{P}(g) = \mathrm{P}(j(g)) = 0$. Then $\omega(g)$, $\omega(j(g))$ and $\omega(g_0)$ are not changed. We may assume that $\omega(g)$ and $\omega(j(g))$ are positive. Then by Theorem 3, $\omega(g)^{1/a} \leq \omega(j(g)) \leq \omega(g)^a$.

Let $t > 1$ and n be the natural number for which $n \leq t < n + 1$. From $\pm n = g_0^{\pm n}(0) = j(g_0)^{\pm n}(0)$ we infer that $n \leq \pm\varphi(\pm t) < n + 1$. It follows that $n/(n + 1) \leq \varphi(t)/(-\varphi(-t)) \leq (n + 1)/n$, and we have $1/2 \leq \varphi(t)/(-\varphi(-t)) \leq 2$.

Let $1/\omega(g) < t \leq 1$. Since $g(\infty) = 1/\omega(g)$, we obtain

$$1/\omega(j(g)) < \varphi(t) \leq 1\,,$$

and similarly $-1/\omega(j(g)) > \varphi(-t) \geq -1$. Hence

$$\omega(g)^{-a} \leq \varphi(t)/(-\varphi(-t)) \leq \omega(g)^{a} .$$

Finally, let $0 < t \leq 1/\omega(g)$ and n be the natural number for which $1/((n+1)\omega(g)) < t \leq 1/(n\omega(g))$. From $g^{\pm n}(\infty) = 1/(\pm n\omega(g))$ it follows that $1/((n+1)\omega(j(g))) \leq \pm \varphi(\pm t) \leq 1/(n\omega(j(g)))$. Hence

$$\frac{n\omega(j(g))}{(n+1)\omega(j(g))} \leq \frac{\varphi(t)}{-\varphi(-t)} \leq \frac{(n+1)\omega(j(g))}{n\omega(j(g))} ,$$

and we conclude that $1/2 \leq \varphi(t)/(-\varphi(-t)) \leq 2$.

Since $\omega(g) \geq 4$ (cf. *Remark* in § 2), it follows that

$$\omega(g)^{-a} \leq \varphi(t)/(-\varphi(-t)) \leq \omega(g)^{a}$$

for all $t > 0$. □

Observe that Theorem 10 does not follow from Theorem 7.

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