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ON VARIATIONAL INTEGRALS IN "THE BORDERLINE CASE"

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1. Introduction

In this paper we study variational integrals in R^n

$$I(v) = \int_G F(x, v(x), \nabla v(x))\, dm_n(x) \tag{1.1}$$

in »the borderline case» $F(x, y, z) \approx |z|^n$. The behavior of (1.1) is well-known for $n = 2$, see e.g. [3], [5]. Our main contribution is to show how conformal technics in R^2 can be replaced by the use of quasiconformal mappings in R^n, $n \geq 3$. The main result, Theorem 4.1, is an extension of [5, Theorem 4.3.5, p. 111] which states that if G is a Lipschitz domain in R^n and $u_0 \in C(\bar{G}) \cap W_n^1(G)$, then there exists $u \in C(\bar{G}) \cap W_n^1(G)$, $u \mid \partial G = u_0 \mid \partial G$, and u minimizes (1.1) among all similar v. We replace the condition »G Lipschitz» by »G quasiconformally collared». Our method is based on the fact that a quasiconformal mapping between two domains G and G' preserves the classes $C(G) \cap W_n^1(G)$ and $C(G') \cap W_n^1(G')$.

Section 2 contains notation and assumptions on F. It also includes a lower-semicontinuity theorem relevant to our case and some basic properties of quasiconformally collared domains. Equicontinuity properties of monotone functions are studied in Section 3 and Section 4 contains the main theorem.

2. Notation and preliminaries

2.1. *Notation.* The real number system is denoted by R, $R^+ = \{x \in R \mid x \geq 0\}$, and $\bar{R}^+ = R^+ \cup \{\infty\}$. We let R^n, $n \geq 1$, denote the euclidean n-space with a fixed orthonormal bases $e_1, \ldots, e_n$. For $x \in R^n$ and $r > 0$, $B^n(x, r)$ denotes the open ball centered at x with radius r, and $S^{n-1}(x, r) = \partial B^n(x, r)$. We shall use the abbreviations $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0, r)$. H_n^+ denotes the upper half-space $\{x \in R^n \mid x_n > 0\}$.

If $A \subset R^n$ we let $C(A)$ denote the class of real valued continuous functions on A. If $A \subset R^n$ is a Lebesgue measurable set and $u : A \to \bar{R}^+$ a measurable function we let

$$\int_A u \, dm_n \quad \text{or} \quad \int_A u(x) \, dm_n(x)$$

denote the integral of u over A. The integral has values in $\bar{R}^+$. $L^p(A)$, $1 \leq p < \infty$, denotes the Banach space of (equivalence classes of) measurable functions $u : A \to \{-\infty\} \cup R \cup \{\infty\}$ with the norm

$$\|u\|_{p,A} = \Big(\int_A |u|^p dm_n \Big)^{1/p} < \infty .$$

Suppose that $G \subset R^n$ is a domain and $1 \leq p < \infty$. $W_p^1(G)$ means the Sobolev space of all functions u in $L^p(G)$ with the first generalized partial derivatives $\partial_j u$, $1 \leq j \leq n$, in $L^p(G)$. We let $\nabla u = (\partial_1 u, \ldots, \partial_n u)$. The norm in $W_p^1(G)$ is denoted by

$$\|u\|_{1,p,G} = \|u\|_{p,G} + \|\nabla u\|_{p,G} = \|u\|_{p,G} + \|(\sum_{i=1}^n |\partial_i u|^2)^{1/2}\|_{p,G} .$$

We shall make use of the following reflection principle in the Sobolev space. It can be easily proved by using the ACL-properties of functions in $W_p^1(G)$, see e.g. [5, pp. 66—67].

2.2. Lemma. *Suppose* $u \in W_p^1(B^n \cap H_n^+) \cap C(\overline{B^n \cap H_n^+})$. *Let*

$$u^*(x) = u(x) , \; x \in \overline{B^n \cap H_n^+}$$
$$= u(x - 2x_n e_n) , \; x \in \bar{B}^n \setminus \overline{B^n \cap H_n^+} .$$

Then $u^* \in C(\bar{B}^n) \cup W_p^1(B^n)$ *and*

$$2 \int_{B^n \cap H_n^+} |\nabla u|^p dm_n = \int_{B^n} |\nabla u^*|^p dm_n .$$

2.3. *Variational integrals.* Let $G \subset R^n$ be a bounded domain and $F : G \times R \times R^n \to \bar{R}^+$. We shall make use of three sets of assumptions on F:

(2.4) Either

(i) F is continuous and into R^+. $z \mapsto F(x, y, z)$ is convex for all $(x, y) \in G \times R$.

or

(ii) F is Borel-measurable. For every $\varepsilon > 0$ there exists a compact set S in G such that $m_n(G \setminus S) < \varepsilon$ and $F | S \times R^{n+1}$ is continuous. $F(x, \cdot) : R^{n+1} \to R^+$ is convex for a.e. $x \in G$.

(2.5) For a.e. $x \in G$ $F(x, y, z) \geq F(x, y, 0)$ for all $y \in R$.

There exists $C > 0$ and $w \in L^1(G)$ such that for a.e. $x \in G$

$$F(x, y, z) \geq C\, |z|^n - w(x) \tag{2.6}$$

for all $(y, z) \in R \times R^n$ (»the borderline case»).

2.7. Example. If $F(x, y, z) = |z|^n$ then F satisfies (2.4)—(2.6).

We shall use the abbreviation

$$I(u) = \int_G F(x, u(x), \nabla u(x))\, dm_n(x)$$

if $u \in W_n^1(G)$. The following lower-semicontinuity theorem will be used several times.

2.8. Lemma. *Suppose that* F *satisfies* (2.4). *If* (u_i) *is a bounded sequence in* $W_p^1(G)$ *and* $u_i \to u \in W_p^1(G)$ *in* $L^p(G)$, *then*

$$I(u) \leq \varliminf_{i \to \infty} I(u_i)\, . \tag{2.9}$$

Proof. If F satisfies (2.4) (i), then (2.9) is a special case of [9, Theorem 13] (see also [5, Theorem 4.1.2] and [8, Theorem 1.3]) and if F satisfies (2.4) (ii), then (2.9) follows from [8, Theorem 1.2].

2.10. Remark. Both sides of (2.9) may be infinite.

2.11. *Quasiconformally collared domains.* Let $G \subset R^n$ be a bounded domain. G is called *quasiconformally collared*, if every $x \in \partial G$ has arbitrary small neighborhoods U such that $U \cap G$ can be mapped quasiconformally onto $B^n \cap H_n^+$. This means that there is a homeomorphism $f: U \cap G \to B^n \cap H_n^+$ such that the coordinate functions of f belong to $W_n^1(U \cap G)$ and

$$|f'(x)|^n \leq K\, J(x, f) \quad \text{a.e. in } G$$

for some $K \geq 1$, for more details see [1] or [10].

2.12. Remarks. (a) A Lipschitz-domain is quasiconformally collared since every bi-Lipschitz homemorphism is quasiconformal but not vice versa, see Remark 4.2 (b).

(b) For $n = 2$, G is quasiconformally collared if and only if ∂G consists of a finite number of disjoint Jordan curves.

2.13. Remark. Instead of the above definition for a quasiconformally collared domain we may also use the equivalent definition: Every $x \in \partial G$

has arbitrary small neighborhoods U such that there exists a homeomorphism $f: \overline{U \cap G} \to \overline{B^n \cap H_n^+}$ with the properties (1) $f \mid U \cap G$ is quasiconformal, (2) $f(x) = 0$, and (3) $f(\partial G \cap U) = B^n \cap \partial H_n^+$. This can be seen as follows: Let U be as in the first definition. Then f can be extended to a homeomorphism of $U \cap \bar{G}$ [7, Lemma 2.3]. Denote by V the x-component of $\partial G \cap U$. Then fV is open on $\partial(B^n \cap H_n^+)$ and clearly we may assume $fV \subset \partial H_n^+$ and $f(x) = 0$. Thus $U' = f^{-1}(B^n(r) \cap H_n^+)$ for sufficiently small r can be used in the second definition with an obvious modification of f.

3. Monotone functions

Let $A \subset R^n$, $u \in C(A)$, and $S \subset A$. The oscillation of u on S is denoted by

$$\omega(f, S) = \sup_{x \in S} f(x) - \inf_{x \in S} f(x) .$$

Suppose that $G \subset R^n$ is a domain. A function $u \in C(\bar{G})$ is called *monotone* (in the sense of Lebesgue) if

$$\sup_{x \in D} f(x) = \sup_{x \in \partial D} f(x) \quad \text{and} \quad \inf_{x \in D} f(x) = \inf_{x \in \partial D} f(x)$$

for every domain $D \subset G$.

3.1. Lemma. ([6, Lemma 4.1], [1, Lemma 1]) *Let $u \in C^1(G)$ and let $\bar{B}^x(x, r) \subset G$. Then*

$$\omega(u, S^{n-1}(x, r))^n \leq Ar \int_{S^{n-1}(x, r)} |\nabla u|^n \, dS \tag{3.2}$$

where $A = A(n)$ and dS denotes the $(n-1)$-measure on $S^{n-1}(x, r)$.

By approximation Fubini's theorem implies

3.3. Corollary. *Let $u \in W_n^1(G) \cap C(G)$. If $B^n(x_0, r_0) \subset G$, then* (3.2) *holds for u for a.e. $r \in (0, r_0)$.*

3.4. Theorem. *Suppose that G is a bounded quasiconformally collared domain and that $\mathfrak{M}$ is a family of functions u on $\bar{G}$ such that*
(1) $u \in C(\bar{G}) \cap W_n^1(G)$.
(2) *u is monotone.*
(3) $\|\nabla u\|_{n, G} \leq M$ *for all* $u \in \mathfrak{M}$.
Then $\mathfrak{M}$ is equicontinuous if and only if $\mathfrak{M} \mid \partial G$ is equicontinuous.

Proof. The only if part is trivial. For the other direction we split the proof into two parts.

A. *Equicontinuity in* G. Let $x \in G$ and pick $\delta > 0$ such that $\bar{B}^n(x, \delta) \subset G$. Choose $\alpha > \delta$ so that $B^n(x, \alpha) \subset G$. Since $u \in \mathcal{M}$ is monotone, Corollary 3.3 implies for a.e. $r \in (\delta, \alpha)$

$$\omega(u, B^n(x, \delta))^n \leq \omega(u, B^n(x, r))^n = \omega(u, S^{n-1}(x, r))^n \leq A\, r \int_{S^{n-1}(x, r)} |\nabla u|^n \, dS .$$

Multipling by r^{-1} and integrating from δ to α yields

$$\omega(u, B^n(x, \delta))^n \log \alpha/\delta \leq A \int_G |\nabla u|^n \, dS \leq A M^n .$$

This shows that $\mathcal{M}$ is equicontinuous at x.

B. *Equicontinuity at points on* ∂G. Let $x \in \partial G$. By Remark 2.12 there exists a neighborhood U of x and a homeomorphism $f : \overline{U \cap G} \to \overline{B^n \cap H_n^+}$ with properties (1)—(3) in 2.11. For $r \in (0, 1)$ let $D(r)$ denote the disk $B^n(r) \cap \partial H_n^+$.

It is enough to show that $\mathcal{M}f^{-1} = \{v \mid v = u \circ f^{-1},\ u \in \mathcal{M}\}$ is equicontinuous at 0. Suppose that $\mathcal{M}f^{-1}$ is not equicontinuous at 0. Then for all $\delta \in (0, 1)$ there exists $v \in \mathcal{M}f^{-1}$ such that

$$\omega(v, B^n(\delta) \cap H_n^+) \geq \varepsilon > 0 . \tag{3.5}$$

Pick $r_0 > 0$ so small that

$$\omega(v, D(r_0)) < \varepsilon/2 \tag{3.6}$$

for all $v \in \mathcal{M}f^{-1}$. This is possible since $\mathcal{M} \mid \partial G$ is equicontinuous and hence $\mathcal{M}f^{-1} \mid B^n \cap \partial H_n^+$ is equicontinuous. Since every $u \in \mathcal{M}$ is monotone and f is a homeomorphism, $v = u \circ f^{-1}$ is also monotone, and (3.5) and (3.6) imply

$$\omega(v, S^{n-1}(r) \cap H_n^+) \geq \varepsilon/2$$

for all $r \in [\delta, 1]$. On the other hand $v \in C(\overline{B^n \cap H_n^+}) \cap W_n^1(B^n \cap H_n^+)$ [11]. Hence v has an extension v^* to $\bar{B}^n$ described in Lemma 2.2. Now Corollary 3.3 yields for a.e. $r \in [\delta, 1]$

$$(\varepsilon/2)^n \leq \omega(v, S^{n-1}(r) \cap H_n^+)^n = \omega(v^*, S^{n-1}(r))^n \leq A\, r \int_{S^{n-1}(r)} |\nabla v^*|^n \, dS .$$

Multipling by r^{-1} and integrating from δ to 1 gives

$$(3.7) \qquad (\varepsilon/2)^n \log 1/\delta \le A \int_{B^n} |\nabla v^*|^n \, dm_n \le 2A \int_{B^n \cap H_n^+} |\nabla v|^n \, dm_n .$$

Since $f \mid U \cap G$ is quasiconformal, $f^{-1} \mid B^n \cap H_n^+$ is also quasiconformal for some $K \ge 1$ [1, Theorem 4]. This gives

$$\int_{B^n \cap H_n^+} |\nabla v|^n \, dm_n \le \int_{B^n \cap H_n^+} |\nabla u(f^{-1}(y))|^n \, |(f^{-1})'(y)|^n \, dm_n(y)$$

$$\le K \int_{B^n \cap H_n^+} |\nabla u(f^{-1}(y))|^n \, J(y, f^{-1}) \, dm_n(y)$$

$$= K \int_{U \cap G} |\nabla u|^n \, dm_n \le K \int_G |\nabla u|^n \, dm_n \le K M^n .$$

This estimate together with (3.7) implies

$$(\varepsilon/2)^n \le 2 A K M^n (\log 1/\delta)^{-1}$$

which is a contradiction for δ small enough.

3.8. Remark. Theorem 3.4 is an extension of [5, Theorem 4.3.4].

The idea in the next theorem is due to Lebesgue [3].

3.9. Theorem. *Suppose that F satisfies* (2.4) *and* (2.5). *Let $u_0 \in C(\bar{G}) \cap W_n^1(G)$. Then there exists $u \in C(\bar{G}) \cap W_n^1(G)$ such that u is monotone, $u \mid \partial G = u_0 \mid \partial G$, and $I(u) \le I(u_0)$.*

Proof. Using Lebesgue's method [3] (see also [1], [5], and [6]) it is possible to construct a sequence of functions u_i, $i = 0, 1, \ldots$, such that (i) $u_i \in C(\bar{G})$, (2) $u_i \mid \partial G = u_0 \mid \partial G$, (3) $u_{i-1} = u_i$ except on an open set $V_i \subset G$ and u_i is a constant on the components of V_i, and (4) u_i converges uniformly on $\bar{G}$ to a monotone function u. From (3) it follows that $\omega(u_i, \triangle) \le \omega(u_0, \triangle)$ on each line segment $\triangle \subset G$. This implies that each u_i is ACL since u_0 is. Moreover, if U is a component of V_i, then u_i is a constant on $\bar{U}$, and hence by [4, p. 254] $\nabla u_i = 0$ a.e. in $\bar{V}_i$. This implies

$$(3.10) \qquad \int_G |\nabla u_i|^n \, dm_n \le \int_G |\nabla u_0|^n \, dm_n < \infty ,$$

and since F satisfies (2.5), it also follows that

$$I(u_i) \leq I(u_0) . \tag{3.11}$$

Since u_i is ACL and (3.10) holds, $u_i \in W_n^1(G)$. Furthermore, by (4) and (3.10) (u_i) is a bounded sequence in $W_n^1(G)$ converging in $L^n(G)$ to u . Consequently $u \in W_n^1(G)$. Finally Lemma 2.8 and (3.11) yield

$$I(u) \leq \varliminf_{i\to\infty} I(u_i) \leq I(u_0) .$$

4. Main theorem

4.1. Theorem. *Suppose that* G *is a bounded quasiconformally collared domain and* F *satisfies* (2.4)—(2.6). *Let* $u_0 \in C(\bar{G}) \cap W_n^1(G)$. *Then there exists* $u \in C(\bar{G}) \cap W_n^1(G)$ *such that* $u \mid \partial G = u_0 \mid \partial G$ *and* u *minimizes the integral* $I(v)$ *among all similar* v .

Proof. Let

$$\mathcal{F} = \{v \in C(\bar{G}) \cap W_n^1(G) \mid v \mid \partial G = u_0 \mid \partial G\}$$

and denote

$$I_0 = \inf_{v\in\mathcal{F}} I(v) .$$

If $I_0 = + \infty$, we may take $u = u_0$. Suppose $I_0 < \infty$. Then there exists a sequence (u_i) , $u_i \in \mathcal{F}$, such that

$$I(u_i) \to I_0 . \tag{4.2}$$

We may assume

$$I_0 \leq I(u_i) < I_0 + 1 . \tag{4.3}$$

By Theorem 3.9 we can replace (u_i) by (u_i^*) such that $u_i^* \in \mathcal{F}$, u_i^* is monotone, and u_i^* satisfies both (4.2) and (4.3). Then (2.6) and (4.3) imply

$$\| \nabla u_i^* \|_{n,G}^n \leq (I_0 + 1 + \|w\|_{1,G})/C , \quad i = 1 , 2 , \ldots . \tag{4.4}$$

Hence by Theorem 3.4 $\{u_i^*\}$ is equicontinuous and since the functions u_i^* are monotone, $\{u_i^*\}$ is also bounded. By Ascoli's theorem there exists a subsequence $(u_{i_j}^*)$ converging uniformly on $\bar{G}$ to $u \in C(\bar{G})$. Since, by (4.4), the sequence $(u_{i_j}^*)$ is bounded in $W_n^1(G)$ and $u_{i_j}^* \to u$ in $L^n(G)$, $u \in W_n^1(G)$. Thus $u \in \mathcal{F}$. Finally Lemma 2.8 implies

$$I_0 = \lim_{j \to \infty} I(u_{i_j}^*) \geq I(u) \geq I_0 \, .$$

This completes the proof.

4.2. Remarks. (a) Theorem 4.1 (and Theorems 3.4 and 3.9) can be extended to vector valued functions $u : G \to R^m$.

(b) For instance a smooth domain in R^n, $n \geq 3$, with an outward directed spire is quasiconformally collared (for details see [2]). However, such a domain is not a Lipschitz-domain.

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References

[1] GEHRING, F. W.: Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103 (1962), 353–393.

[2] GEHRING, F. W. and J. VÄISÄLÄ: The coefficients of quasiconformality of domains in space. - Acta Math. 114 (1965), 1–70.

[3] LEBESGUE, H.: Sur le problème de Dirichlet. - Rend. Circ. Mat. Palermo 24 (1907), 371–402.

[4] MORREY, C. B.: Existence and differentiability theorems for variational problems for multiple integrals. - Partial Differential Equations and Continuum Mechanics, Univ. of Wisconsin Press, Madison, 1961.

[5] —»— Multiple integrals in the calculus of variations. - Springer-Verlag, 1966.

[6] MOSTOW, G. D.: Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. - Inst. Hautes Études Sci. Publ. Math. 34 (1968), 53–104.

[7] NÄKKI R.: Continuous boundary extension of quasiconformal mappings. - Ann. Acad. Sci. Fenn. AI 511 (1972), 1–10.

[8] REŠETNJAK, J. G.: General theorems on semicontinuity and on convergence with a functional. - Sib. Math. J. 8.5 (1967), 801–806 (English translation).

[9] SERRIN, J.: On the definition and properties of certain variational integrals. - Trans. Amer. Math. Soc. 101 (1961), 139–167.

[10] VÄISÄLÄ, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture notes in mathematics 229, Springer-Verlag, 1971.

[11] ZIEMER, W.: Change of variables for absolutely continuous functions. - Duke Math. J. 36 (1969), 171–178.

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