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SOME REMARKS ON PSEUDOCOMPACT SPACES

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Some remarks on pseudocompact spaces

A topological space E is *pseudocompact* if it satisfies one of the following equivalent conditions:

(P1) Every continuous real function of E is bounded,
(P2) If $f : E \to [0,1]$ is continuous, then $f(E)$ is closed ([10]),
(P3) Every countable cozero-set cover of E has a finite subcover ([6]),
(P4) Every locally finite collection of cozero-sets of E is finite ([6]),
(P5) Every continuous[1]) pseudometric of E is precompact ([5]).

Moreover, a uniformisable topological space E is pseudocompact if and only if

(P6)′ Every compatible uniformity of E is precompact ([3]).

We generalize (P6)′ to arbitrary topological spaces and present two conditions similar to (P5).

Proposition 1 *For a topological space E, the following conditions are equivalent:*

(P) *E is pseudocompact,*
(P6) *Every continuous uniformity of E is precompact,*
(P7) *Every continuous pseudometric of E is compact,*
(P8) *Every continuous pseudometric of E is complete.*

Proof: To prove (P) $\to$ (P6) we assume (P). Let $\mathcal{U}$ be a continuous uniformity of E. Since pseudocompactness is preserved under continuous maps, $\mathcal{U}$ is pseudocompact and, consequently, precompact by (P6)′. Obviously, (P6) $\to$ (P5). Hence (P6) $\to$ (P). To prove (P) $\to$ (P7), we again note that pseudocompactness is preserved under continuous maps and then use the fact that a pseudocompact pseudometric space is compact ([7]). The implication (P7) $\to$ (P8) is immediate. Thus it remains to prove, e.g., that (P8) $\to$ (P2). Suppose $f : E \to [0, 1]$ is continuous. The equation $d(x,y) = |f(x) - f(y)|$ defines a continuous pseudometric d, which,

[1]) A pseudometric (or uniformity) of a topological space (E, τ) is *continuous* if it induces a topology that is weaker than τ.

by (P8), is conplete. From this it easily follows that $f(E)$ is complete and hence closed.

A subspace A of a topological space E is *C-embedded* (*C*-embedded*) if each continuous (and bounded) real function of A can be extended to a continuous real function of E. Analogously, A is *P-embedded* (*T-embedded*) if each continuous (and precompact) pseudometric of A can be extended to a continuous pseudometric of E. Suppose A is a pseudocompact C*-embedded subspace of a topological space E. By (P1) A is C-embedded and hence T-embedded, by Corollary 3.5 of [1]. Finally A is P-embedded by (P5). Thus we have obtained a short proof of Theorem 3.7 in [9]. On the other hand, a P-embedded subspace is known to be C*-embedded ([9]). Thus, for a pseudocompact subspace, the notions of P-embedding, T-embedding, C-embedding and C*-embedding are all equivalent. This remark can be used to avoid the cardinality assumption in Theorem 3.6 of [9]. The reformulation of this theorem reads: a uniformisable space E is pseudocompact if and only if it is P-embedded in βE. This is proved as follows. A uniformisable pseudocompact space E is C*-embedded and hence P-embedded in βE, by the above remark. On the other hand, a P-embedded subspace is always C-embedded by Theorem 3.2 of [9] and hence pseudocompact (pseudocompactness is clearly inherited by C-embedded subspaces).

The following proposition contains some modifications of

(P9) *If* $f : E \to \mathbf{R}$ *is continuous, then* $f(E)$ *is compact* ([4]),

which is a combination of (P1) and (P2) and equivalent to them.

Proposition 2 *For a topological space* E, *the following conditions are equivalent:*

(P) E *is pseudocompact,*

(P10) *If* E' *is a Lindelöf (or paracompact)* T_3*-space and* $f : E \to E'$ *is continuous, then* $f(E)$ *is relatively compact,*

(P11) *If* E' *is a hereditarily Lindelöf* T_3*-space and* $f : E \to E'$ *is continuous, then* $f(E)$ *is compact,*

(P12) *If* E' *is a second countable* T_3*-space and* $f : E \to E'$ *is continuous, then* $f(E)$ *is compact,*

(P13) *If* E' *is a uniform space and* $f : E \to E'$ *is continuous, then* $f(E)$ *is precompact,*

(P14) *If* E' *is a complete uniform space and* $f : E \to E'$ *is continuous, then* $f(E)$ *is relatively compact.*

Proof: To prove (P) $\to$ (P10), we assume (P). Suppose E' is a Lindelöf (or paracompact) T_3-space and $f : E \to E'$ is continuous. Since a

Lindelöf T_3-space is paracompact, we may assume that E' is paracompact. As a continuous image of a pseudocompact space, $f(E)$ is pseudocompact. It is easily seen that the closure of a pseudocompact subspace of a topological space is pseudocompact. Thus $\overline{f(E)}$ is pseudocompact. As a closed subset of a paracompact T_3-space, $\overline{f(E)}$ is a paracompact $T_{3\frac{1}{2}}$-space. Consequently, $\overline{f(E)}$ is compact (Corollary 2 of [7]). Finally, we note that the relative compactness of $f(E)$ is (in the T_3-space E') equivalent to the compactness of $\overline{f(E)}$. We next prove (P10) $\to$ (P11). Suppose E' is a hereditarily Lindelöf T_3-space and $f: E \to E'$ is continuous. Then $\overline{f(E)}$ is a Lindelöf T_3-space and hence (by (P10)) relatively compact in $\overline{f(E)}$, i.e. compact. Obviously (P11) $\to$ (P12) $\to$ (P9). To prove (P) $\to$ (P13), assume (P). Suppose E' is a uniform space and $f: E \to E'$ is continuous. Again, $f(E)$ is pseudocompact and hence precompact, by (P6)$'$. Thus we have (P13). (P13) implies (P14), since a precompact subspace of a complete uniform space is relatively compact. The final implication (P14) $\to$ (P) is trivial.

A family $\mathcal{U}$ is *star-countable* if each member of $\mathcal{U}$ intersects at most countably many members of $\mathcal{U}$. The notion of star-countability can be used to generalize the condition (P3).

Proposition 3 *A topological space is pseudocompact if and only if every star-countable cozero-set cover of the space has a finite subcover.*

Proof: Suppose E is a pseudocompact topological space and $\mathcal{V}$ is a star-countable cozero-set cover of E. For each $V \in \mathcal{V}$ we define inductively

$$S^0(V) = V\,,$$
$$S^{n+1}(V) = \bigcup \{U \in \mathcal{V} \mid U \cap S^n(V) \neq \emptyset\}\,,$$
$$S^\omega(V) = \bigcup_{n<\omega} S^n(V)\,.$$

Let
$$\mathcal{S} = \{S^\omega(V) \mid V \in \mathcal{V}\}\,.$$

It is easily verified that $\mathcal{S}$ is a discrete open cover of E. $\mathcal{S}$ is finite by (P4) (note that each member of $\mathcal{S}$ is closed and open, hence a cozero-set). On the other hand, the star-countability of $\mathcal{V}$ implies that each member of $\mathcal{S}$ is a countable union of members of $\mathcal{V}$. Hence $\mathcal{V}$ itself is countable and, therefore, has a finite subcover. The converse is obvious (by (P3)), since a countable cover is always star-countable.

A topological space E is *lightly compact* if every locally finite collection of open subsets of E is finite or, equivalently, if every countable open cover of E has a finite subfamily the union of which is dense. The proof of Proposition 3 applies also, mutatis mutandis, to the case of a lightly compact space. Thus, a topological space E is lightly compact if and only if every star-countable open cover of E has a finite subfamily the union of which is dense.

A lightly compact space is necessarily pseudocompact (by (P4)). The converse is known to hold for completely regular spaces but not for completely Hausdorff spaces ([10]). We give an example of a regular non-lightly compact space which is pseudocompact. In fact, every continuous real function of this space is constant.

Suppose X is any regular space every continuous real function of which is constant (e.g. that of [8]). Let $E = X \times \mathbf{N}$ ($\mathbf{N}$ is the set of non-negative integers). As a product of two regular spaces, E is regular. Suppose a and b are two different fixed points of X (note that X is necessarily infinite). For each $n \in \mathbf{N}$ we identify the points $(a, 2n)$, $(a, 2n+1)$ as well as the points $(b, 2n+1)$, $(b, 2n+2)$ of E. Let E' be the resulting space with the identification topology. Suppose k is the canonical mapping from E onto E'. Now k is a closed mapping, since $k^{\leftarrow}(k(A))$ is closed for each closed A. Hence E is T_1. To prove that E' is T_3, we take a point $k((x, n)) \in E'$ and an open neighborhood V' of $k((x, n))$. We may assume that n is even. The case $x \notin \{a, b\}$ is straightforward. Suppose then $x = a$. Since E is regular, the point (a, n) has a closed neighborhood A_1 and the point $(a, n+1)$ a closed neighborhood A_2 so that $A_1 \cup A_2 \subset k^{\leftarrow}(V') - \{(b, n), (b, n+1)\}$. Obviously, $k(\mathrm{Int}(A_1) \cup \mathrm{Int}(A_2))$ is open. Hence $k(A_1 \cup A_2)$ is a closed neighborhood of $k((x, n))$ such that $k(A_1 \cup A_2) \subset V'$. Consequently, E' is regular. To prove that E' is not lightly compact, we choose an open non-void subset U of X such that $a \notin U$ and $b \notin U$. Then the infinite open family

$$\{k(U \times \{n\})\ n \in \mathbf{N}\}$$

of E' is locally finite and, consequently, E' is not lightly compact. We finally note that a continuous real function of E' is constant on each $k(X \times \{n\})$ and hence, by construction, constant on the whole of E'.

In the following proposition we present some simple properties of lightly compact spaces.

Proposition 4 *Let E be a topological space.*
(1) *If $A \subset E$ is lightly compact, so is $\bar{A}$,*
(2) *If E is T_2 and $A \subset E$ is lightly compact and Lindelöf, then A is closed,*
(3) *If E is lightly compact and T_3, then it is Baire.*

Proof: (1): Suppose $\{V_i \mid i < \omega\}$ is a countable open cover of $\bar{A}$. Since A is lightly compact, the open cover $\{V_i \cap A \mid i < \omega\}$ of A has a finite subfamily $\{V_i \cap A \mid i \leq n\}$ the union of which is dense in A. The union of the family $\{V_i \mid i \leq n\}$ is easily seen to be dense in $\bar{A}$. (2): Suppose there exists an $a \in \bar{A} - A$. For each $x \in A$, let U_x and V_x be open subsets of E such that $x \in U_x$, $a \in V_x$, and $U_x \cap V_x = \emptyset$. Since A is Lindelöf, there exists a countable $A' \subset A$ such that $A \subset \bigcup \{U_x \mid x \in A'\}$. By the light compactness of A, there is a finite $A'' \subset A'$ such that $A \subset \bigcup \{\overline{U_x} \mid x \in A''\}$. The set $V = \bigcap \{V_x \mid x \in A''\}$ is an open neighborhood of x, which does not meet A, since $V \cap U_x = \emptyset$ for each $x \in A''$. This contradiction shows that A is closed. (3): Suppose $U_0, U_1, \ldots$ is a sequence of dense open subsets of E. To prove that $\bigcap_{i<\omega} U_x$ is dense we consider an arbitrary non-void open subset U of E. Since $U \cap U_0 \neq \emptyset$ and E is T_3, there exists an open non-void A_0 such that $\overline{A_0} \subset U \cap U_0$. We assume, for an inductive construction, that $A_0, \ldots, A_n$ are open non-void subsets of E such that $\overline{A_{i+1}} \subset A_i \cap U \cap U_0 \cap \ldots \cap U_i$ for each $i < n$. Since $A_n \cap U \cap U_0 \cap \ldots \cap U_n \neq \emptyset$, there exists an open non-void A_{n+1} such that $\overline{A_{n+1}} \subset A_n \cap U \cap U_0 \cap \ldots \cap U_n$. The light compactness of E implies now $\emptyset \neq \bigcap_{i<\omega} \overline{A_i} \subset U \cap \bigcap_{i<\omega} U_i$. Consequently, $\bigcap_{i<\omega} U_i$ is dense. Thus we have proved the Baire property for E.

Arens and Dugundji ([2]) have proved that for a T_1-space each of the following conditions is equivalent to countable compactness:

(A) Every infinite open cover of the space has a proper subcover.

(F) Every infinite subset of the space has an accumulation point.

The T_1-assumption is essential as is seen from the following examples. At first, let $E = \mathbf{N}$ with the topology $\{\{0, 1, \ldots, n\} \mid n \in \mathbf{N}\}$. Obviously, E is T_0. E is also T_4, since no two non-void closed subsets are disjoint. It is easily verified that E satisfies (A) but is not countably compact. It may also be noted that E is lightly compact, since every non-void open subset of E is dense. Thus a lightly compact T_4-space need not be countably compact. The topological sum of an infinite collection of copies of E is a T_0-space satisfying (F) but not (A). On the other, it is known that countable compactness implies (A) and (A) implies (F).

The following lemma will be needed later.

Lemma 5 *A* T_3*-space satisfying* (A) *is countably compact.*

Proof: Suppose E is a T_3-space satisfying (A). Let E_0 be the T_0-space associated with E. It is easily seen that E_0 is still a T_3-space satisfying

(A). But being T_0, E_0 is T_1 and, therefore, countably compact. It follows that also E is countably compact.

The condition (F) can be strengthened to a form equivalent to (A). To see this we introduce a new concept. A point p of a topological space E is an *n-accumulation point of* $A \subset E$ if $\operatorname{card}(U \cap A) \geq n$ for each neighborhood U of p.

Proposition 6 *Let E be a topological space. Each of the following conditions is equivalent to* (A):

(A1)$_n$ *Every infinite subset of E has an n-accumulation point ($n \geq 2$ is a fixed integer),*

(A2) *Every infinite subset of E has an n-accumulation point for each integer $n \geq 2$,*

(A3) *Every discrete family of subsets of E is finite,*

(A4) *Every discrete family of closed subsets of E is finite.*

Proof: We at first prove (A) → (A4) and, therefore, assume (A). Suppose $\mathcal{A} = \{A_i \mid i < \eta\}$ is an infinite discrete collection of closed non-void subsets of E (indexed without repetitions). We define for each $i < \eta$

$$V_i = E - \bigcup_{j \neq i} A_j \,.$$

Since $\mathcal{A}$ is locally finite, the set $\bigcup_{j\neq i} A_j$ is closed and hence V_i is open. It is easily verified that $\{V_i \mid i < \eta\}$ is an open cover of E having no proper subcover, a contradiction. The implication (A4) → (A3) follows from the fact that the closures of the members of a discrete family form a discrete family. To prove (A3) → (A1)$_n$ we assume (A3) and use induction to prove (A1)$_n$. If A were an infinite subset of E having no 2-accumulation points, then the family $\{\{a\} \mid a \in A\}$ would be discrete, contradicting (A3). Hence we have (A1)$_2$. As an induction hypothesis we assume (A1)$_{k-1}$ ($k > 2$). For the reductio ad absurdum, we take an infinite set $A \subset E$ which has no k-accumulation points. Using the induction hypothesis we can select a (k-1)-accumulation point b_0 of A and an open neighborhood V_0 of b_0 such that $\operatorname{card}(V_0 \cap A) = k-1$. Now, the induction hypothesis can be applied to $A - V_0$. Suppose b_1 is a $(k-1)$-accumulation point of $A - V_0$ and V_1 is an open neighborhood of b_1 such that $\operatorname{card}(V_1 \cap (A - V_0)) = k$-1. Inductively, we can construct an infinite sequence $b_0, b_1, \ldots$ of (k-1)-accumulation points of A such that for each $i < \omega$, $b_i \in V_i$ and

$$\operatorname{card}(V_i \cap (A - \bigcup_{j<i} V_j)) = k\text{-}1.$$

Suppose b is a 2-accumulation point of the infinite set $\{b_i \mid i < \omega\}$. The point b is not a k-accumulation point of A and, therefore, has an open neighborhood V such that

$$\text{card } (V \cap A) \leq k-1.$$

Using the definition of b we can select $b_i \in V$ and $b_j \in V$ such that e.g. $j < i$. But the definition of b_i and b_j implies card $(V_i \cap V \cap (A - V_j)) \geq k-1$ and card $(V_j \cap V \cap A) \geq k-1$, whence card $(V \cap A) \geq$ card $(V_i \cap V \cap (A - V_j))$ + card $(V_j \cap V \cap A) \geq k-1+k-1 > k$, a contradiction. We next prove $(\text{A1})_2 \rightarrow$ (A). Suppose $\{V_i \mid i < \eta\}$ is an infinite open cover without a proper subcover. The sets

$$A_i = E - \bigcup_{j \neq i} V_j$$

are non-void and pairwise disjoint. Consequently, if we pick out one point a_i from each A_i, the resulting set $A = \{a_i \mid i < \eta\}$ is infinite and, therefore, has a 2-accumulation point a. Since $\{V_i \mid i < \eta\}$ is a cover, there is an i such that $a \in V_i$. But card $(V_i \cap A) = 1$, a contradiction. The final equivalence (A) $\leftrightarrow$ (A2) is now obvious.

Arens and Dugundji ([2]) have also shown that each point-finite open cover of a space satisfying (A) has a finite subcover. Hence (A) implies the following condition:

(L) Every locally finite open cover of the space has a finite subcover.

The condition (L) is, by definition, weaker than light compactness and is equivalent to it in T_1- (or T_3-) spaces. An example of a non-lightly compact T_0-space satisfying (L) will be given on page 10. Since (L) implies (P4), we conclude that (A) implies pseudocompactness. We are going to prove the converse for T_4-spaces. For this purpose we need the following simple lemma, which also demonstrates the superfluousness of a notion of a countably collectionwise T_4-space.

Lemma 7 *A topological space E is T_4 if and only if it satisfies the condition:*

For every countable discrete family $\{A_i \mid i < \omega\}$ of closed subsets of E there is a family $\{U_i \mid i < \omega\}$ of pairwise disjoint open subsets such that $A_i \subset U_i$ for each $i < \omega$.

Proof: It suffices to prove the necessity. Suppose $\{A_i \mid i < \omega\}$ is a countable discrete family of closed subsets of E. The sets $\bigcup_{j \leq i} A_j$ and $\bigcup_{j > i} A_j$ are closed and disjoint. Hence they can be separated by disjoint open sets $V_i \supset \bigcup_{j \leq i} A_j$ and $W_i \supset \bigcup_{j > i} A_j$. Let $U_0 = V_0$ and $U_{n+1} = V_{n+1} \cap \bigcap_{j \leq n} W_j$. Then $\{U_i \mid i < \omega\}$ is the required family.

Proposition 8 *A pseudocompact* T_4*-space satisfies* (A).

Proof: We show that the condition (A4) is fulfilled. Let $\{A_i \mid i < \omega\}$ be a countably infinite discrete family of non-void closed subsets of a pseudocompact T_4-space E. Lemma 7 can be used to establish a collection $\{U_i \mid i < \omega\}$ of pairwise disjoint open subsets of E such that $A_i \subset U_i$ for each $i < \omega$. The family

$$\mathcal{U} = \{U_i \mid i < \omega\} \cup \{E - \bigcup_{i<\omega} A_i\}$$

is clearly a point-finite open cover of E. It is well known that a point-finite open cover of a T_4-space is shrinkable, i.e., we are able to select an open cover $\{V_i \mid i \leq \omega\}$ of E with the property that $\overline{V_i} \subset U_i$ for each $i < \omega$, and $\overline{V_\omega} \subset E - \bigcup_{i<\omega} A_i$. Again by the T_4-property, the sets $V_i (i \leq \omega)$ can be extended to cozero-sets N_i $(i \leq \omega)$ in such a way that the family $\mathcal{N} = \{N_i \mid i \leq \omega\}$ is still a refinement of $\mathcal{U}$. But, being a countable cozero-set cover of the pseudocompact space E, $\mathcal{N}$ has a finite subcover, which is, obviously, impossible.

From the above proposition it can be concluded that for T_4-spaces the conditions (A) and (L) are both equivalent to pseudocompactness. Yet a pseudocompact T_4-space need not be lightly compact (and hence not countably compact), which is seen as follows. Let $E = \mathbf{N} \times \{0, 1\}$. The points $(n, 1)$ are defined to be isolated, and a neighborhood of a point $(n, 0)$ is defined to be any subset of E containing $(\bigcup_{i \leq n} \{i\}) \times \{0, 1\}$. We at first note that E is T_4, since no two non-void closed subsets are disjoint. Clearly E is T_0. E is not lightly compact, since the infinite open family $\{\{(n, 1)\} \mid n \in \mathbf{N}\}$ is, obviously, locally finite. To see that E satisfies (A), we take an arbitrary infinite open cover $\mathcal{V}$ of E. Let $V \in \mathcal{V}$. If $\{V\}$ is a cover, there is nothing to prove. Suppose then $V \neq E$. Since V is open, there is an $n \in \mathbf{N}$ such that $(n, 0) \notin V$. For each $m \geq n$ we choose a $V_m \in \mathcal{V}$ such that $(m, 0) \in V_m$. The family $\{V_m \mid m \geq n\}$ is a proper subcover of $\mathcal{V}$.

We now have the following relations: A countably compact space satisfies (A) and the converse holds for T_3- (Lemma 5) but not for T_4-spaces. (A) implies pseudocompactness and the converse holds for T_4- but not (by Lemma 5) for T_3-spaces. Especially, a pseudocompact T_4- and T_3-space is countably compact, but neither T_4- nor T_3-assumption can be omitted. Furthermore, we have shown that the class of pseudocompact T_4-spaces properly includes the class of lightly compact T_4-spaces, which in turn properly includes the class of countably compact T_4-spaces.

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References

[1] Alo, R. A. and Shapiro, H. L.: Extensions of totally bounded pseudometrics. - Proc. Amer. Math. Soc. 19 (1968), 877–884.

[2] Arens, R. F. and Dugundji, J.: Remark on the concept of compactness. - Portugaliae Math. 9 (1950), 141–143.

[3] Doss, R.: On continuous functions in uniform spaces. - Ann. Math. 48 (1947), 843–844.

[4] Gillman, L. and Jerison, M.: Rings of continuous functions. - Van Nostrand, Princeton–Toronto–London–New York (1960).

[5] Frolik, Z.: Generalizations of compact and Lindelöf spaces. - Czech. Math. J. 9 (1959), 172–217 (Russian, English summary).

[6] Kerstan, J.: Zur Characterisierung der pseudokompakten Räume. - Math. Nachr. 16 (1957), 289–293.

[7] Mardešić, S. and Papic, P.: Sur les espaces dont toute transformation réelle est bornée. - Glasnik Mat. Fiz. Astr. 10 (1955), 225–232.

[8] Novak, J.: Regular space, on which every continuous function is constant. - Časopis Pěst Mat. Fys. 73 (1948), 58–68.

[9] Shapiro, H. L.: Extensions of pseudometrics. - Canad. J. Math. 18 (1966), 981–998.

[10] Stephenson, R. M., jr.: Pseudocompact spaces. - Trans. Amer. Math. Soc. 134 (1968), 437–448.

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