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Series A

I. MATHEMATICA

560

ON OPTIMIZING PARAMETERS OF THE POWER INEQUALITY FOR a_4 IN THE CLASS OF BOUNDED UNIVALENT FUNCTIONS

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1. Minimizing the right side

Consider univalent functions $f\colon U \to R^2$, $U = \{z \in C \,|\, |z| < 1)$, where

$$f(z) = \sum_{1}^{\infty} b_k z^k \left(a_n = \frac{b_n}{b_1}\right),$$

$$|f(z)| < 1 \quad \text{in} \quad U,$$

$$b_1 = \text{constant} \quad \in (0\,,\,1]\,.$$

The class of these bounded univalent functions is denoted by $S(b_1)$. The subclass $S_R(b_1) \subset S(b_1)$ consists of functions with all the coefficients b_n real.

In [4] an inequality, generalizing the Nehari inequality [1] for the $S(b_1)$ functions was derived. This was done by applying Green's formula to the generating function $g(w) = x_0 \log w + \sum_{k=-N}^{N} x_k w^k (k \neq 0)$ constructed by aid of powers of w. We will briefly call this generalized Nehari inequality the Power inequality of P-inequality. According to the index N, we may also speak about the P_N-inequality. Let us adopt the notations x_k, c_{hk}, y_k introduced in [4] and apply the P_3-inequality in the bilinear form

$$(1) \qquad -\operatorname{Re}\,(\sum_{1}^{3} k y_{-k} y_k + \sum_{1}^{3} k x_{-k} x_k) \leq \sum_{1}^{3} k |y_{-k}|^2 + \sum_{1}^{3} k |x_k|^2$$

given by the condition (58) of [4] for $x_0 = 0$. Here $x_k (k = \pm 1, \pm 2, \pm 3)$ are free parameters and y_k are linear combinations of these, constructed by aid of the power coefficients c_{hk} (cf. (5) and (9) in [4]).

Apply (1) to the function $\sqrt{f(z^2)}$. By aid of the expressions of y_{-1}, y_{-2}, y_{-3} replace x_{-1}, x_{-2}, x_{-3} by u_1, u_2, u_3 when choosing

$$(2) \qquad \begin{cases} y_{-3} = x_{-3} b_1^{-3/2} = -\dfrac{u_3}{3}\,, \\[2ex] y_{-2} = x_{-2} b_1^{-1} = -\dfrac{u_2}{2}\,, \\[2ex] y_{-1} = -\dfrac{3}{2} x_{-3} b_1^{-3/2} a_2 + x_{-1} b_1^{-1/2} = -u_1\,. \end{cases}$$

Conversely,

$$(3)\qquad \begin{cases} x_{-1} = -\, b_1^{1/2}\left(u_1 + \dfrac{a_2}{2}\, u_3\right), \\[2ex] x_{-2} = -\, \dfrac{b_1}{2}\, u_2\,, \\[2ex] x_{-3} = -\, \dfrac{b_1^{3/2}}{3}\, u_3\,. \end{cases}$$

The numbers

$$\begin{cases} y_1 = x_{-3}c_{-31} + x_{-1}\, c_{-11} + x_1 c_{11}\,, \\ y_2 = x_{-2}c_{-22} + x_2 c_{22}\,, \\ y_3 = x_{-3}c_{-33} + x_{-1}c_{-13} + x_1 c_{13} + x_3 c_{33}\,, \end{cases}$$

assume thus the expressions

$$(4)\qquad \begin{cases} y_1 = \dfrac{1}{2}\left(a_3 - \dfrac{3}{4}\, a_2^2\right) u_3 + \dfrac{a_2}{2}\, u_1 + x_1 b_1^{1/2}\,, \\[2ex] y_2 = \dfrac{1}{2}\, (a_3 - a_2^2) u_2 + b_1 x_2\,, \\[2ex] y_3 = \left(\dfrac{a_4}{2} - a_2 a_3 + \dfrac{13}{24}\, a_2^3\right) u_3 + \dfrac{1}{2}\left(a_3 - \dfrac{3}{4}\, a_2^2\right) u_1 + \dfrac{1}{2}\, b_1^{1/2} a_2 x_1 + b_1^{3/2} x_3\,. \end{cases}$$

Apply (2) and (4) and write (1) as follows:

$$\operatorname{Re}\{\sum_1^3 u_k y_k - \sum_1^3 k x_{-k} x_k\} \leqq \sum_1^3 \frac{|u_k|^2}{k} + \sum_1^3 k|x_k|^2\,;$$

$$(5)\qquad \begin{aligned} &\operatorname{Re}\{(a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3)u_3^2 + (a_3 - a_2^2)u_2^2 + a_2 u_1^2 + 2(a_3 - \tfrac{3}{4}a_2^2)u_1u_3 \\ &+ 4b_1^{1/2}u_1x_1 + 4b_1u_2x_2 + 4b_1^{3/2}u_3x_3 + 2b_1^{1/2}a_2u_3x_1\} \\ &\leqq 2\left[|u_1|^2 + \frac{|u_2|^2}{2} + \frac{|u_3|^2}{2} + |x_1|^2 + 2|x_2|^2 + 3|x_3|^2\right]. \end{aligned}$$

This is the P_3-inequality with six free parameters x_1, x_2, x_3; u_1, u_2, u_3. Normalize by rotation so that

$$a_4 = |a_4| = \operatorname{Re} a_4$$

and estimate this coefficient by taking $u_3 = 1$.

The parameters left will now be optimized subsequently so that the right side of the inequality for a_4, given by (5), is minimized. As an example, take the expression depending on x_3:

$$6\,|x_3|^2 - 4b_1^{3/2}\,\mathrm{Re}\,x_3 \geqq 2\,[3(\mathrm{Re}\,x_3)^2 - 2b_1^{3/2}\,\mathrm{Re}\,x_3]$$
$$= 2\left[(\mathrm{Re}\,x_3 - \tfrac{1}{3}b_1^{3/2})^2 - \frac{b_1^3}{3}\right] \geqq -\tfrac{2}{3}b_1^3\,.$$

Equality is reached by taking

$$x_3 = \mathrm{Re}\,x_3 = \tfrac{1}{3}b_1^{3/2}\,. \tag{6}$$

Similarly, the following optimal choices hold:

$$\begin{cases} x_2 = \dfrac{b_1}{2}\,\bar{u}_2\,, \\ x_1 = b_1^{1/2}(\tfrac{1}{2}\bar{a}_2 + \bar{u}_1)\,, \\ u_2 = 0\,. \end{cases} \tag{7}$$

The inequality left has u_1 as a free parameter

$$\begin{aligned} &\mathrm{Re}\,\{a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3\} - \tfrac{2}{3}(1 - b_1^3) + \frac{b_1}{2}\,|a_2|^2 \\ &\leqq -\,\mathrm{Re}\,\{a_2u_1^2 + 2(a_3 - \tfrac{3}{4}a_2^2)u_1\} + 2(1 - b_1)|u_1|^2 - 2b_1\,\mathrm{Re}\,(\bar{a}_2u_1)\,. \end{aligned} \tag{8}$$

Observe that this is the Nehari inequality which we have utilized in [3] and which can be derived also without the power coefficients, by aid of Faber polynomials. As a matter of fact, (6) and (7) imply the corresponding symmetric choice of parameters,

$$x_{-k} = -\,\bar{x}_k \qquad (k = 1\,,2\,,3)\,, \tag{9}$$

because (6) and (7) give by aid of (3)

$$\begin{cases} x_1 = b_1^{1/2}(\bar{u}_1 + \tfrac{1}{2}\bar{a}_2) = -\,\bar{x}_{-1}\,, \\ x_2 = \tfrac{1}{2}b_1\bar{u}_2 = -\,\bar{x}_{-2}\,, \\ x_3 = \tfrac{1}{3}b_1^{3/2}\bar{u}_3 = -\,x_{-3} = -\,\bar{x}_{-3}\,. \end{cases} \tag{10}$$

Observe that the choice $x_{-k} = -\,\bar{x}_k$ is automatically connected with the equality of the general P_N-inequality, as can be seen by considering the equality condition (39) of [4].

The choice (2) leads us to the following practical observation: P_N-inequality with $x_{-k} = -\,\bar{x}_k$ and $y_{-k} = -\,\dfrac{u_k}{k}$ $(k = 1\,,\ldots,N)$ gives the

Nehari condition in the form obtained by aid of Faber's polynomials. The use of power coefficients shortens calculations considerably.

In (8) we have one free parameter, u_1, available. In our previous considerations in [3] and [6] we have overlooked some of the power u_1 includes. Therefore, we have to optimize u_1, too.

The part on the right side of (8), depending on u_1, is

$$\begin{aligned} (11) \qquad H &= -\operatorname{Re}(a_2 u_1^2) - 2\operatorname{Re}\{(a_3 - \tfrac{3}{4}a_2^2 + b_1\bar{a}_2)u_1\} + 2(1-b_1)|u_1|^2 \\ &= \tilde{A}u^2 + \tilde{B}v^2 + 2buv - 2\alpha u + 2\beta v \end{aligned}$$

where

$$(12) \qquad \begin{cases} a_2 = a + ib\,, \\ \nu\ = a_3 - \tfrac{3}{4}a_2^2 + b_1\bar{a}_2 = \alpha + i\beta\,, \\ u_1 = u + iv\,, \\ \tilde{A} = 2(1-b_1) - a\,, \\ \tilde{B} = 2(1-b_1) + a\,. \end{cases}$$

Because

$$\Delta = \tilde{A}\tilde{B} - b^2 = [2(1-b_1)]^2 - |a_2|^2 > 0$$

for non-radial-slit mappings, which can be excluded as a trivial case, H is a definite form, minimized for

$$\frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0\,; \quad \begin{cases} u = \dfrac{\alpha\tilde{B} + \beta b}{\Delta}\,, \\[2ex] v = -\dfrac{\beta\tilde{A} + b\alpha}{\Delta}\,. \end{cases}$$

Thus, the optimal choice of u_1 is

$$u_1 = u + iv = \frac{\alpha(\tilde{B} - ib) + \beta(b - i\tilde{A})}{\Delta}\,;$$

$$u_1 = \frac{\bar{a}_2\nu + 2(1-b_1)\bar{\nu}}{\Delta}\,.$$

The corresponding minimum is

$$\begin{aligned} H = -\alpha u + \beta v &= -\frac{\tilde{A}\beta^2 + \tilde{B}\alpha^2 + 2b\alpha\beta}{\Delta} \\ &= -\frac{2(1-b_1)|\nu|^2 + \operatorname{Re}(\bar{a}_2\nu^2)}{\Delta}\,. \end{aligned}$$

Thus, the optimized a_4-inequality reads

$$\operatorname{Re}\{a_4 - 2a_2a_3 + \tfrac{13}{12}a_2^3\} - \tfrac{2}{3}(1 - b_1^3) + \frac{b_1}{2}|a_2|^2 \tag{13}$$

$$\leqq - \frac{2(1-b_1)|\nu|^2 + \operatorname{Re}(\bar{a}_2\nu^2)}{[2(1-b_1)]^2 - |a_2|^2},$$

where

$$\begin{cases} u_1 = \dfrac{\bar{a}_2\nu + 2(1-b_1)\bar{\nu}}{[2(1-b_1)]^2 - |a_2|^2}; \\ \nu \;\; = a_3 - \tfrac{3}{4}a_2^2 + b_1\bar{a}_2 . \end{cases} \tag{14}$$

2. Utilizing the range of ν

The number ν defined in (14) can be restricted. Apply the P_3-inequality by choosing all $x_\nu = 0$ except x_{-1}, x_1:

$$\sum_1^\infty k|y_k|^2 \leqq |y_{-1}|^2 + |x_1|^2 - |x_{-1}|^2 . \tag{15}$$

Here

$$y_k = x_{-1}b_1^{-1}\alpha_k + x_1 b_k$$

where the coefficients α_k are defined by the development

$$f(z)^{-1} = b_1^{-1}(z^{-1} + \sum_0^\infty \alpha_k z^k) .$$

(15) implies

$$\sum_1^3 k|x_{-1}\alpha_k + x_1 b_1 b_k|^2 \leqq (1 - b_1^2)\,|x_{-1}|^2 + |x_1|^2 b_1^2 .$$

When applied to the function $\sqrt{f(z^2)}$ this gives

$$\tfrac{3}{4}|x_{-1}(a_3 - \tfrac{3}{4}a_2^2) - x_1 b_1 a_2|^2$$

$$\leqq - \left| x_{-1}\frac{a_2}{2} - x_1 b_1 \right|^2 + (1 - b_1)|x_{-1}|^2 + |x_1|^2 b_1 .$$

Take here

$$x_{-1} = 1\,,\;\; x_1 = -\frac{\bar{a}_2}{a_2} \quad (a_2 \neq 0)\,;$$

$$\tfrac{3}{4}|a_3 - \tfrac{3}{4}a_2^2 + b_1\bar{a}_2|^2 \leqq 1 - \left|\frac{a_2}{2} + b_1\frac{\bar{a}_2}{a_2}\right|^2 . \tag{16}$$

This inequality is sharp for the radial slit mapping. The range for ν is thus

$$(17) \qquad |\nu| \leqq \frac{2}{\sqrt{3}} \sqrt{1 - \left| \frac{a_2}{2} + b_1 \frac{\bar{a}_2}{a_2} \right|^2} = r(a_2) = r \, .$$

Together with the estimate (13) this is able to improve some of our previous results for a_4.

Write (13) in the form

$$(18) \qquad a_4 \leqq \tfrac{2}{3}(1 - b_1^3) + \tfrac{5}{12} \operatorname{Re}(a_2^3) - \tfrac{5}{2} b_1 |a_2|^2 + G \, ;$$

$$(19) \qquad G = 2 \operatorname{Re}(a_2 \nu) - \frac{2(1 - b_1)|\nu|^2 + \operatorname{Re}(\bar{a}_2 \nu^2)}{[2(1 - b_1)]^2 - |a_2|^2} \, ,$$

and express (19) in a, b; α, β defined by (12):

$$(20) \qquad G = A\alpha^2 + B\beta^2 + 2C\alpha\beta + 2D\alpha + 2E\beta \, ;$$

$$(21) \qquad \begin{cases} A = -\dfrac{2(1 - b_1) + a}{\Delta} \, , \\[2ex] B = -\dfrac{2(1 - b_1) - a}{\Delta} \, , \\[2ex] C = -\dfrac{b}{\Delta} \, , \\[1ex] D = a \, , \\ E = -b \, , \\ \Delta = [2(1 - b_1)]^2 - |a_2|^2 \, . \end{cases}$$

In order to express G in complete squares, rotate the coordinate system $O\alpha\beta$ by a proper angle φ to the system $Ox'y'$ (Figure 1).

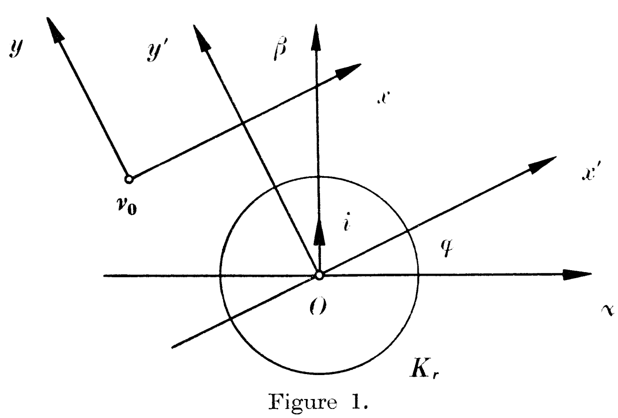


Figure 1.

The corresponding connections

$$
(22)\qquad \begin{cases} x' = l\alpha + m\beta\,, & \alpha = \mu x' - my'\,, \\ y' = \lambda\alpha + \mu\beta\,, & \beta = -\lambda x' + ly'\,, \\ l = \cos\varphi\,, & m = \sin\varphi\,, \\ \lambda = -\sin\varphi\,, & \mu = \cos\varphi\,, \end{cases}
$$

give

$$
(23)\qquad \begin{cases} G = Kx'^2 + Ly'^2 + 2C'x'y' + 2Mx' + 2Ny'\,; \\ K = A\mu^2 + B\lambda^2 - 2C\lambda\mu\,, \\ L = Am^2 + Bl^2 - 2Clm\,, \\ C' = -A\mu m - B\lambda l + C(l\mu + \lambda m)\,, \\ M = D\mu - E\lambda\,, \\ N = -Dm + El\,. \end{cases}
$$

Define φ from the condition $C' = 0$:

$$
\tan 2\varphi = \frac{b}{a}\,;
$$

$$
(24)\qquad e^{i2\varphi} = \frac{a_2}{|a_2|}\,.
$$

This gives for the remaining coefficients of G the expressions

$$
(25)\qquad \begin{cases} K = -\dfrac{2(1-b_1)+|a_2|}{\Delta} = -\dfrac{1}{2(1-b_1)-|a_2|}\,, \\ L = -\dfrac{2(1-b_1)-|a_2|}{\Delta} = -\dfrac{1}{2(1-b_1)+|a_2|}\,, \\ M = |a_2|\cos 3\varphi\,, \\ N = -|a_2|\sin 3\varphi\,. \end{cases}
$$

In these coefficients G is rewritten as follows:

$$
(26)\qquad \begin{aligned} G &= Kx'^2 + Ly'^2 + 2Mx' + 2Ny' \\ &= K\left(x' + \frac{M}{K}\right)^2 + L\left(y' + \frac{N}{L}\right)^2 - \left(\frac{M^2}{K} + \frac{N^2}{L}\right). \end{aligned}
$$

The non-positive constant assumes the following expression in a_2:

$$(27) \qquad -\left(\frac{M^2}{K} + \frac{N^2}{L}\right) = -\Delta(M^2L + N^2K)$$
$$= 2(1-b_1)(M^2+N^2) + |a_2|(N^2-M^2) = 2(1-b_1)|a_2|^2 - \mathrm{Re}\,(a_2^3)\,.$$

Denote

$$(28) \qquad -\frac{M}{K} = x_0', \quad -\frac{N}{L} = y_0'\,.$$

These are the $x'y'$-coordinates of a point P_0. The corresponding $\alpha\beta$-coordinates are denoted by α_0, β_0. Because

$$\begin{cases} \alpha_0 = x_0' \cos\varphi - y_0' \sin\varphi\,, \\ \beta_0 = x_0' \sin\varphi + y_0' \cos\varphi\,, \end{cases}$$

we have

$$\nu_0 = \alpha_0 + i\beta_0 = e^{i\varphi}(x_0' + iy_0') = e^{-i\varphi}\left(\frac{M}{K} + i\,\frac{N}{L}\right)$$
$$= -\,e^{i\varphi}(ML + iNK)\Delta = e^{i\varphi}|a_2|\,[2(1-b_1)e^{-i3\varphi} - |a_2|e^{i3\varphi}]\,;$$

$$(29) \qquad \nu_0 = 2(1-b_1)\bar{a}_2 - a_2^2\,.$$

Express the number

$$R = K\left(x' + \frac{M}{L}\right)^2 + L\left(y' + \frac{N}{L}\right)^2$$

by aid of ν_0 and

$$(30) \qquad \nu = \alpha + i\beta = e^{i\varphi}(x' + iy')\,.$$

From (29) and (30)

$$\begin{cases} x' = \mathrm{Re}\,(e^{-i\varphi}\nu)\,, \\ y' = \mathrm{Im}\,(e^{-i\varphi}\nu)\,; \end{cases} \qquad \begin{cases} x_0' = \mathrm{Re}\,(e^{-i\varphi}\nu_0)\,, \\ y_0' = \mathrm{Im}\,(e^{-i\varphi}\nu_0)\,. \end{cases}$$

Thus

$$\begin{aligned} R &= K(x' - x_0')^2 + L(y' - y_0')^2 \\ &= K\,\mathrm{Re}^2\,\{e^{-i\varphi}(\nu - \nu_0)\} + L\,\mathrm{Im}^2\,\{e^{-i\varphi}(\nu-\nu_0)\} \\ &= \tfrac{1}{2}\,(K - L)\,\mathrm{Re}\,\{e^{-i2\varphi}(\nu-\nu_0)^2\} + \tfrac{1}{2}\,(K+L)\,|\nu-\nu_0|^2 \\ &= -\,\frac{\mathrm{Re}\,\{\bar{a}_2(\nu-\nu_0)^2\} + 2(1-b_1)\,|\nu-\nu_0|^2}{\Delta}\,. \end{aligned}$$

By aid of

$$G = R + 2(1 - b_1)\,|a_2|^2 - \mathrm{Re}\,(a_2^3)$$

we now rewrite (18):

$$a_4 \leqq \tfrac{2}{3}(1 - b_1^3) - \tfrac{7}{12}\,\mathrm{Re}\,(a_2^3) + \tfrac{1}{2}(4 - 9b_1)\,|a_2|^2 + R\,; \tag{31}$$

$$R = -\frac{\mathrm{Re}\,\{\bar{a}_2(\nu - \nu_0)^2\} + 2(1 - b_1)|\nu - \nu_0|^2}{[2(1 - b_1)]^2 - |a_2|^2}\,. \tag{32}$$

Next we estimate R by regarding ν a free variable, independent on a_2, restricted only by the condition (17). Denoting

$$\overline{K_r(O)} = \{(\alpha\,,\beta) \in O\alpha\beta \mid \alpha^2 + \beta^2 \leqq r^2\}\,,$$

we have $\nu \in \overline{K_r(O)}$.

To simplify notations, shift the coordinate system $Ox'y'$ to a parallel system P_0xy having the origin at $P_0 = (x_0'\,, y_0')$. The connections

$$\begin{cases} x = x' - x_0'\,, \\ y = y' - y_0'\,, \end{cases}$$

thus give

$$\begin{cases} R = Kx^2 + Ly^2\,, \\ \overline{K_r(O)} \Rightarrow \overline{K_r(Q_0)} = \{(x\,,y) \in P_0xy|(x + x_0')^2 + (y + y_0')^2 \leqq r^2\}\,. \end{cases} \tag{33}$$

Clearly, the free maximum point for R is $P_0 = (0, 0)$. The important cases are those where P_0 lies outside $\overline{K_r(Q_0)}$. In these R is maximized at the boundary ∂K_r.

Using Lagrange's multiplier λ write

$$\Phi(x\,,y) = Kx^2 + Ly^2 - \lambda[(x + x_0')^2 + (y + y_0')^2 - r^2]$$

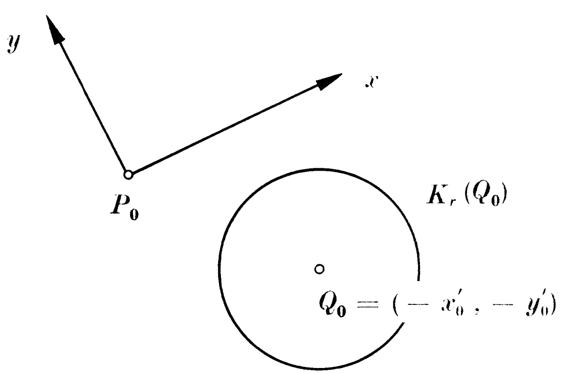


Figure 2.

with the necessary extremum conditions

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial \Phi}{\partial x} = Kx - \lambda(x + x_0') = 0\,, \\ \dfrac{1}{2}\dfrac{\partial \Phi}{\partial y} = Ly - \lambda(y + y_0') = 0\,. \end{cases}$$

These give the solution to our extremum problem. For $\nu_0 \in -\bar{K}_r$

$$\text{Max}\, R = \text{Max}\left[-\lambda\left(\frac{Kx_0'^2}{K-\lambda} + \frac{Ly_0'^2}{L-\lambda} - r^2\right)\right] < 0\,, \tag{34}$$

where λ is the maximizing root of the real roots of the equation

$$\frac{K^2 x_0'^2}{(K-\lambda)^2} + \frac{L^2 y_0'^2}{(L-\lambda)^2} = r^2\,. \tag{35}$$

For $\nu_0 \in \bar{K}_r$, $\text{Max}\, R = 0$.

Rewrite u_1 from (14) using ν_0 defined in (29):

$$u_1 = a_2 + \frac{\bar{a}_2(\nu - \nu_0) + 2(1 - b_1)\,\overline{(\nu - \nu_0)}}{\Delta}\,. \tag{36}$$

In [3] the coefficient a_4 was maximized by aid of Nehari inequality by use of the value

$$u_1 = a_2\,. \tag{37}$$

Thus, our previous choice of u_1 has been optimal only for $\nu_0 \in \bar{K}_r$.

The equation (35) is of fourth degree. Clearly, no further simplifications in the general case can be achieved. Therefore, let us consider the simpler case, the subclass $S_R(b_1)$.

3. The subclass $S_R(b_1)$

To test our formulas with respect to some previous results, consider the special case $S_R(b_1) \subset S(b_1)$, where all the coefficients are real. In this case (24) reduces to the form

$$e^{i2\varphi} = \pm 1$$

where the upper sign belongs to $a_2 > 0$ and the lower sign to $a_2 < 0$. In the following we will concentrate mainly on the later case, where

$$\varphi = \frac{\pi}{2}\,,\ \alpha_0 = \nu_0\,,\ \beta_0 = 0\,,\ x_0' = 0\,,\ y_0' = -\nu_0\,.$$

(35) reduces to the form

$$\frac{L^2|v_0|^2}{(L-\lambda)^2} = r^2 ;$$

$$\lambda - L = \pm \frac{L}{r} |v_0| , \ \lambda = \frac{L}{r} (r \pm |v_0|) ;$$

$$- \lambda\left(\frac{Kx_0'^2}{K-\lambda} + \frac{Ly_0'^2}{L-\lambda} - r^2\right) = - \lambda\left(\frac{|v_0|^2}{L-\lambda} - r^2\right)$$

$$= - L \frac{r \pm |v_0|}{r}\left(\frac{L|v_0|^2}{L-r} - r^2\right) = L(|v_0| \pm r)^2 .$$

Hence, for $a_2 < 0$,

$$\text{(38)} \qquad \operatorname{Max} \mathrm{R} = L(|v_0| - r)^2 , \ L = - \frac{1}{2(1-b_1) - a_2} .$$

On the other hand, for $S_R(b_1)$ (31), (32), (17) and (29) give directly, if $|v_0(a_2)| > r(a_2)$:

$$\text{(39)} \qquad a_4 \leqq \tfrac{2}{3}(1 - b_1^3) - \tfrac{7}{12} a_2^3 + \tfrac{1}{2}(4 - 9b_1)a_2^2 + R = M(a_2) ,$$

$$\text{(40)} \qquad R = R(a_2) = - \frac{(v - v_0)^2}{2(1-b_1) - a_2} \leqq - \frac{[|v_0(a_2)| - r(a_2)]^2}{2(1-b_1) - a_2} ,$$

$$\text{(41)} \qquad r = r(a_2) = \sqrt{\tfrac{1}{3}[2(1-b_1) - a_2][2(1+b_1) + a_2]} ,$$

$$\text{(42)} \qquad v_0 = v_0(a_2) = [2(1-b_1) - a_2]a_2 .$$

This confirms the result (34) in this special case,

$$f \in S_R(b_1) , \quad |v_0(a_2)| > r(a_2) ,$$

and gives an estimate for a_4 which is stronger than those utilized formerly, in [5] and [2].

In Figure 3 there is the graph of the main part

$$\text{(43)} \qquad Q(a_2) = - \tfrac{7}{12} a_2^3 + \tfrac{1}{2}(4 - 9b_1) a_2^2$$

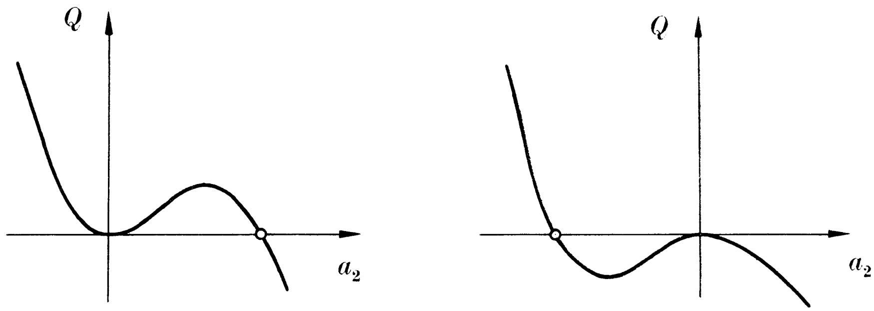
of the right side of (39) in two different cases:

$$1^\circ . \ 4 - 9b_1 > 0 ; \ \ 0 < b_1 < \tfrac{4}{9} ,$$

$$2^\circ . \ 4 - 9b_1 < 0 ; \ \ \tfrac{4}{9} < b_1 < 1 .$$

For $a_2 \in [-2(1-b_1) , 0]$ the correction term R alters the form of the graph according to Figure 4.

In Table 1 and 2 there are numerical values connected with the graphs for such limit values of b_1 which still give results desired. I am indebted to Mr A. Herva for evaluating the functions involved on a digital computer for various values of b_1.



$1° : \frac{6}{7}(4 - 9\,b_1) > 0$ $2° : \frac{6}{7}(4 - 9\,b_1) < 0$

Figure 3.

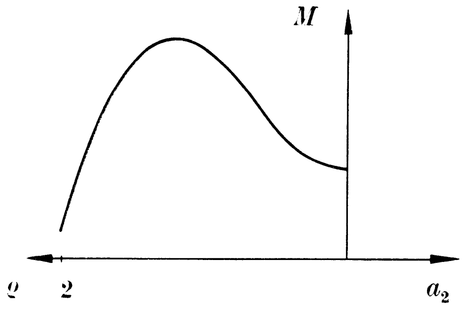
Table 1

$b_1 = 0.15969$

$a_2 = -\,\varrho(1 - b_1)$

$\frac{2}{3}(1 - b^3) + Q(\frac{4}{7}(4 - 9b_1)) = 1.579986$

ϱ	$\lvert v_0(a_2)\rvert$	$r(a_2)$	$M(a_2)$
2.0	5.648967	0.845976	0.189087
1.9	5.232356	0.888583	0.546986
1.8	4.829867	0.926701	0.843165
1.7	4.441500	0.960866	1.082896
1.6	4.067256	0.991487	1.270755
1.5	3.707135	1.018882	1.410819
1.4	3.361135	1.043305	1.506795
1.3	3.029259	1.064963	1.562111
1.2	2.711504	1.084019	1.579980
1.1	2.407872	1.100609	1.563447
1.0	2.118363	1.114844	1.515423
0.9	1.842976	1.126813	1.438714
0.8	1.581711	1.136586	1.336043
0.7	1.334568	1.144221	1.210066
0.6	1.101549	1.149760	1.064450
0.5	0.882651	1.153234	0.933423
0.4	0.677876	1.154660	0.830875
0.3	0.487223	1.154047	0.754731
0.2	0.310693	1.151391	0.702914
0.1	0.148285	1.146678	0.673346
0.0	0.000000	1.139883	0.663952



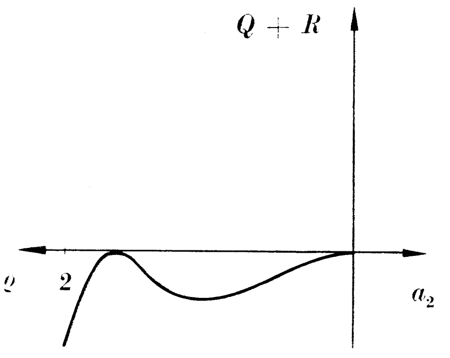


Figure 4.

Table 2

$$b_1 = 0.53857$$

$$a_2 = -\varrho(1 - b_1)$$

ϱ	$\lvert v_0(a_2)\rvert$	$r(a_2)$	$-(Q + R)$
2.0	1.703341	1.151260	0.067389
1.9	1.577720	1.148888	0.034662
1.8	1.456357	1.145892	0.012934
1.7	1.339252	1.142267	0.001794
1.6	1.226406	1.138007	0.000833
1.5	1.117818	1.133105	0.009492
1.4	1.013488	1.127552	0.019502
1.3	0.913417	1.121339	0.026501
1.2	0.817604	1.114455	0.030833
1.1	0.726049	1.106887	0.032842
1.0	0.638753	1.098621	0.032874
0.9	0.555715	1.089641	0.031270
0.8	0.476936	1.079929	0.028375
0.7	0.402414	1.069466	0.024533
0.6	0.332152	1.058229	0.020087
0.5	0.266147	1.046194	0.015382
0.4	0.204401	1.033332	0.010762
0.3	0.146913	1.019611	0.006569
0.2	0.093684	1.004998	0.003149
0.1	0.044713	0.989452	0.000845
0.0	0.000000	0.972928	0.000000

Table 2 and the corresponding graph indicate that $Q + R < 0$ for $a_2 \in [-2(1-b_1), 2(1-b_1)]$ with the equality exactly for $a_2 = 0$. Thus

$$a_4 \leqq \tfrac{2}{3}(1 - b_1^3) \quad \text{for} \quad b_1 \in [0.53857, 1]. \tag{44}$$

In the case 1° we compare the estimates to the value

$$\begin{aligned} &\tfrac{2}{3}(1-b_1^3) + \max_{a_2 \in [0,\, 2(1-b_1)]} Q(a_2) \\ &= \tfrac{2}{3}(1-b_1^3) + Q(\tfrac{4}{7}(4-9b_1)) \\ &= \tfrac{2}{3}(1-b_1^3) + \tfrac{8}{147}(4-9b_1)^3 > 1.579985 \end{aligned}$$

for $b_1 = 0.15969$. Connecting this to our former results in [2] we get:

$$a_4 \leqq \begin{cases} 4 - 20b_1 + 30b_1^2 - 14b_1^3, & \text{for } b_1 \in (0, \tfrac{1}{11}], \\ \tfrac{2}{3}(1-b_1^3) + \tfrac{8}{147}(4-9b_1)^3 & \text{for } b_1 \in [\tfrac{1}{11}, 0.15969]. \end{cases} \tag{45}$$

Thus, the interval of b_1 has been somewhat extended from 0.12, reached in [2] by aid of the inequality $|a_4 - a_2| \leqq 2$, which was able to exclude negative a_2 so far as $a_4 \geqq 2$. Also in the case 2° the interval of b_1 is extended from $\frac{19}{34}$ to 0.53857.

The considerations of [7] for the extremum function continue to hold in the case 2°, also on the extended interval. In the case 1° for $b_1 \in (0, \frac{1}{11}]$ the left radial slit mapping is the only possible extremum case. The remaining interval $(\frac{1}{11}, 0.15969]$ is to be treated separately.

Let g be the generating function of P_3-inequality (cf. [4]) and f extremal in the above sence. The development of $g(f(z))$, according to part 1, necessarily obtains the form

$$g(f(z)) = \sum_{-3}^{3} x_k f^{k/2}(z) = \sum_{-3}^{3} y_k z^{k/2}. \tag{46}$$

Further, the results of part 1 have shown that $x_{-k} = -\bar{x}_k$ and, especially,

$$\begin{cases} x_0 = x_{-2} = x_2 = 0, \\ x_3 = \tfrac{1}{3} b_1^{3/2}, \\ x_1 = b_1^{1/2}\left(\bar{u}_1 + \dfrac{\bar{a}_2}{2}\right). \end{cases}$$

Because equality in (45) is attained for $a_2 = \frac{4}{7}(4 - 9b_1) > 0$, $\nu = \nu_0$, we have from (36)

$$u_1 = a_2 \tag{47}$$

and therefore

$$x_1 = \tfrac{3}{2}\, b_1^{1/2}\, a_2 \,.$$

From the condition $\nu = \nu_0$ we obtain further

$$a_3 - \tfrac{3}{4}\, a_2^2 + b_1 a_2 = 2(1 - b_1)a_2 - a_2^2\,;$$

$$a_3 = - \tfrac{1}{4}\, a_2^2 + (2 - 3b_1)a_2 \,.$$

Collect the numbers governing the necessary extremum condition (46), resulting from the above formulaes:

$$\begin{cases} a_2 = \tfrac{4}{7}\,(4 - 9b_1)\,, \\ a_3 = \tfrac{8}{49}\,(4 - 9b_1)\,(5 - 6\,b_1)\,, \\ a_4 = \tfrac{2}{3}\,(1 - b_1^3) + \tfrac{8}{147}\,(4 - 9b_1)^3\,; \end{cases}$$

$$\begin{cases} x_1 = \tfrac{6}{7}\, b_1^{1/2}\,(4 - 9b_1)\,, \\ x_2 = 0\,, \\ x_3 = \tfrac{1}{3}\, b_1^{3/2}\,; \end{cases}$$

$$\begin{cases} u_1 = a_2\,, \\ u_2 = 0\,, \\ u_3 = 1\,. \end{cases}$$

From these formulaes we decide further in view of part 1 and [4]:

$$\begin{cases} y_{-3} = - \tfrac{1}{3}\,, \\ y_{-2} = 0\,, \\ y_{-1} = - a_2 = - \tfrac{4}{7}\,(4 - 9b_1)\,, \\ y_0 = 0\,; \end{cases}$$

$$\begin{cases} y_1 = \tfrac{1}{2}\,(a_3 - \tfrac{3}{4}\, a_2^2) + \tfrac{1}{2}\, a_2^2 + \tfrac{6}{7}\,(4 - 9b_1) \\ \quad = \tfrac{4}{7}\,(4 - 9b_1)\,, \\ y_2 = 0\,, \\ y_3 = \tfrac{1}{2}\, a_4 - a_2 a_3 + \tfrac{13}{24}\, a_2^3 + \tfrac{1}{2}\,(a_2 a_3 - \tfrac{3}{4}\, a_2^3) \\ \quad + \tfrac{3}{7}\, b_1\,(4 - 9b_1)a_2 + \tfrac{1}{2}\, b_1^3 = \tfrac{1}{3}\,. \end{cases}$$

Hence, (46) gives for f

$$x_3 f^{3/2} + x_1 f^{1/2} + x_{-1} f^{-1/2} + x_{-3} f^{-3/2} = y_3 z^{3/2} + y_1 z^{1/2} + y_{-1} z^{-1/2} + y_{-3} z^{-3/2}\,;$$

$$(48)\quad \begin{cases} b_1^{3/2}\,\dfrac{(f-1)\,[f^2+(4-3R)f+1]}{f^{3/2}} = \dfrac{(z-1)\,[z^2+(4-3\tau)z+1]}{z^{3/2}}\,, \\ R = \dfrac{61b_1-24}{7b_1}\,,\quad \tau = \frac{9}{7}\,(4b_1-1)\,. \end{cases}$$

Here $R = r + 1/r$, $r \in [-1, 0)$, $\tau = 2\cos\varphi$, where

$$f(e^{\pm i\varphi}) = r$$

is the branch point of the forked slit in $f(U)$.

In order to determine $f(U)$ we will split the mapping into two parts. First, consider the two-parametric family

$$(49)\qquad a^{3/2}\,\frac{(f-1)\,(f^2+10f+1)}{f^{3/2}} = \frac{(z-1)\,(z^2+kz+1)}{z^{3/2}}\,,$$

where k and a are real, $a \in (0,1]$. If here $f(e^{i\varphi}) = Me^{i\Phi}$, we get for the parametric presentation

$$(50)\qquad \begin{cases} \Phi = \Phi(\varphi)\,, \\ M = M(\varphi)\,, \end{cases}$$

the conditions

(51)

$$\begin{cases} \left(\sqrt{M}-\dfrac{1}{\sqrt{M}}\right)\cos\dfrac{\Phi}{2}\left[M+\dfrac{1}{M}+10-4\left(M+\dfrac{1}{M}+1\right)\sin^2\dfrac{\Phi}{2}\right] = 0\,, \\ a^{3/2}\left(\sqrt{M}+\dfrac{1}{\sqrt{M}}\right)\sin\dfrac{\Phi}{2}\left[-M-\dfrac{1}{M}+10+4\left(M+\dfrac{1}{M}-1\right)\cos^2\dfrac{\Phi}{2}\right] \\ = 2\sin\dfrac{\varphi}{2}\,(2\cos\varphi+k)\,. \end{cases}$$

The first condition implies two alternatives.

$$1)\qquad \left(\sqrt{M}-\frac{1}{\sqrt{M}}\right)\cos\frac{\Phi}{2} = 0\,;$$

$$M \equiv 1 \quad\text{or}\quad \Phi \equiv \pi\,.$$

The part of the boundary $\partial f(U)$ got from this is a radial-slit-figure. At the end point of the radial slit

$$(52)\qquad \begin{cases} \varphi = \pi\,, \\ a^{3/2}\left(\sqrt{M}+\dfrac{1}{\sqrt{M}}\right)\left(10-M-\dfrac{1}{M}\right) = 2(k-2)\,. \end{cases}$$

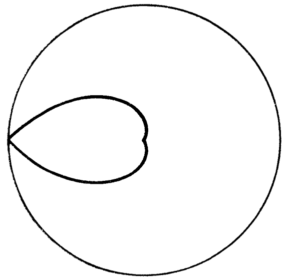


Figure 5.

2)

$$\sin^2 \frac{\Phi}{2} = \frac{1}{4} \frac{M^2 + 10M + 1}{M^2 + M + 1} . \tag{53}$$

In Figure 5 there is the graph of (53).

The conditions (51) determine the parametric presentation (50) of the slits. The form of these conditions shows that at the end points $f(e^{i\varphi})$ of the slits there holds

$$\frac{d}{d\varphi}\left[2 \sin \frac{\varphi}{2} (2 \cos \varphi + k)\right] = 0 ;$$

$$\cos \frac{\varphi}{2}\left(1 + \frac{k}{2} - 6 \sin^2 \frac{\varphi}{2}\right) = 0.$$

For the radial slit $\varphi = \pi$ and for the curved slits

$$\sin \frac{\varphi}{2} = \pm \sqrt{\frac{k+2}{12}} \quad (-2 \leqq k < 10) . \tag{54}$$

The boundary of the image domain belonging to (49) is finally of the type presented in Figure 6.

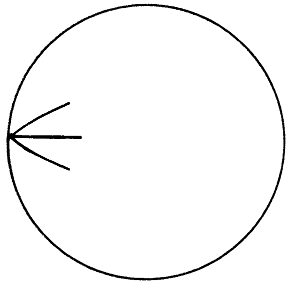


Figure 6.

We can now show that every mapping (48) is obtained by combining a defined mapping of the family (49) to a left radial-slit mapping. For fixed b_1 choose

$$k = 4 - 3\tau\,, \quad -2 < \tau \leqq 2\,, \quad -2 \leqq k < 10\,.$$

Take a as a new parameter and apply the radial-slit mapping

$$\frac{b_1}{a}\left(f + \frac{1}{f} - 2\right) = \tilde{f} + \frac{1}{\tilde{f}} - 2\,, \quad \frac{b_1}{a} \leqq 1\,. \tag{55}$$

Now, eliminate $\tilde{f} + \dfrac{1}{\tilde{f}}$ from (55) and

$$a^{3/2}\,\frac{(\tilde{f} - 1)(\tilde{f}^2 + 10\tilde{f} + 1)}{\tilde{f}^{3/2}} = \frac{(z-1)[z^2 + (4 - 3\tau)z + 1]}{z^{3/2}}\,. \tag{56}$$

The result is

$$b_1^{3/2}\,\frac{(f-1)\left[f^2 + \left(12\,\dfrac{a}{b_1} - 2\right)f + 1\right]}{f^{3/2}} = \frac{(z-1)[z^2 + (4 - 3\tau)\,z + 1]}{z^{3/2}}\,.$$

This is to be identified with (48):

$$12\,\frac{a}{b_1} - 2 = 4 - 3R\,;$$

$$\frac{b_1}{a} = \frac{4}{2 - R} \leqq 1 \quad \text{for} \quad R \leqq -2\,.$$

We thus proved: The mapping belonging to (48) can be constructed as follows.

1) The starting function $\tilde{f}$ is determined by taking in the family

$$a^{3/2}\,\frac{(\tilde{f} - 1)(\tilde{f}^2 + 10\tilde{f} + 1)}{\tilde{f}^{3/2}} = \frac{(z-1)(z^2 + kz + 1)}{z^{3/2}} \tag{57}$$

$$\begin{cases} k = 4 - 3\tau\,, \quad a = \dfrac{2 - R}{4}\,b_1\,, \\[2ex] \tau = \frac{9}{7}(4b_1 - 1)\,, \quad R = \dfrac{61b_1 - 24}{7b_1}\,. \end{cases} \tag{58}$$

2) The function f sought is obtained by aid of the left radial-slit mapping

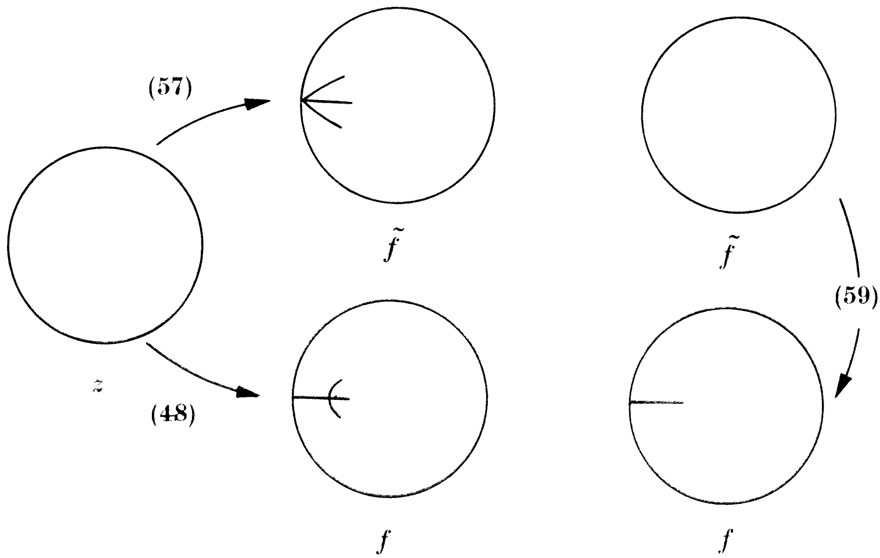


Figure 7.

$$\frac{b_1}{a}\left(f+\frac{1}{f}-2\right)=\tilde{f}+\frac{1}{\tilde{f}}-2\,. \tag{59}$$

The result is a three-fork mapping (Figure 7).

b_1 diminishing, the curved parts of the fork shrink towards a point. We will determine the end situation by requiring that the end points of the curved slits lie at $\tilde{f}=-1$. (54) and (58) give for $e^{i\varphi}$

$$e^{i\frac{\varphi}{2}}-e^{-i\frac{\varphi}{2}}=2i\sin\frac{\varphi}{2}\,,$$

$$e^{i\varphi}-e^{-i\varphi}=-\frac{k-4}{3}\,;$$

$$e^{i\varphi}=\frac{\tau}{2}\pm i\sqrt{1-\frac{\tau^2}{4}}\,. \tag{60}$$

Write (57) in the form

$$a^3\left(\tilde{f}+\frac{1}{\tilde{f}}-2\right)\left(\tilde{f}+\frac{1}{\tilde{f}}+10\right)^2=\left(z+\frac{1}{z}-2\right)\left(z+\frac{1}{z}+k\right)^2$$

and substitute in it $z=e^{i\varphi}$ from (60) and $\tilde{f}=-1$. This gives

$$(\tau-2)^3+8^2a^3=0\,. \tag{61}$$

Express a and τ in b_1 according to (58) and substitute in (61). This gives for b_1

$$8^2\left(\frac{24-47b_1}{28}\right)^3 = \left(\frac{23-36b_1}{7}\right)^3 ;$$

$$b_1 = \frac{1}{11} .$$

Hence: At the point $b_1 = \frac{1}{11}$ the three-fork mapping is reduced to the radial-slit mapping.

The three-fork mapping is limited to the other direction by the condition

$$R = \frac{61b_1 - 24}{7b_1} \leqq -2 ;$$

$$b_1 \leqq \frac{8}{25} = 0{,}32 .$$

The limit case $b_1 = \frac{8}{25}$ gives actually the mapping (57) for $a = b_1$, $k = \frac{73}{25}$. Our method allows utilizing the three-fork mapping as an extremal case up to $b_1 = 0.15969$.

4. Comparison to Schiffer's equation

Observe that the equation (48) is got by integrating Schiffer's differential equation for the three-slit case, called 1:3 in [2]. The general form of Schiffer's differential equation for functions $f \in S(b_1)$, maximizing $a_4 = |a_4| = \mathrm{Re}\, a_4$, utilized in [2] for $S_R(b_1)$-functions, is

$$\left(z\,\frac{f'(z)}{f(z)}\right)^2 [\;]_l = [\;]_r , \tag{62}$$

$$[\;]_l = \frac{b_1^3}{f^3} + \frac{3b_1b_2}{f^2} + \frac{2b_3 + b_1a_2^2}{f} + a_4 + \lambda$$
$$+ (2\bar b_3 + b_1\bar a_2^2)f + 3b_1\bar b_2 f^2 + b_1^3 f^3 ,$$

$$[\;]_r = \frac{1}{z^3} + \frac{2a_2}{z^2} + \frac{3a_3}{z} + 4a_4 + \lambda + 3\bar a_3 z + 2\bar a_2 z^2 + z^3 ,$$

$\lambda \in R$.

It is useful to consider more closely the similarities which hold between Schiffer's equation and the necessary condition got from P_3-inequality by requiring that equality necessarily holds for the a_4-conditions.

The P_3-inequality is derived by applying Green's identity to the generating function (Σ' indicates that the index omits the number 0)

$$f(w) = x_0 \log w + \sum_{-N}^{N}{}' x_k w^k \tag{63}$$

giving the development

$$g(f(z)) = x_0 \log z + \sum_{-N}^{\infty} y_k z^k$$

(cf. [4]). Equality in the a_4-condition is possible only if $y_4 = y_5 = \ldots = 0$. Thus, *if* the P_3-inequality is able to give an exact upper bound for a_4, the extremum function f satisfies necessarily the condition

$$x_0 \log \sqrt{f(z^2)} + \sum_{-3}^{3}{}' x_k \sqrt{f(z^2)} = \sum_{-3}^{3} y_k z^k$$

or

$$\frac{x_0}{2} \log f(z) + \sum_{-3}^{3}{}' x_k f(z)^{k/2} = \sum_{-3}^{3} y_k z^{k/2} .$$

Differentiate with respect to z:

$$\left[\frac{x_0}{2} f^{-1} + \sum_{-3}^{3} \frac{k}{2} x_k f^{k/2-1}\right] f' = \sum_{-3}^{3} \frac{k}{2} y_k z^{k/2-1} ;$$

$$z \frac{f'(z)}{f(z)} \left[x_0 + \sum_{-3}^{3} k x_k f^{k/2}\right]_1 = \left[\sum_{-3}^{3} k y_k z^{k/2}\right]_2 ; \tag{64}$$

$$\left(z \frac{f'(z)}{f(z)}\right)^2 \left[x_0 + \sum_{-3}^{3} k x_k f^{k/2}\right]_1^2 = \left[\sum_{-3}^{3} k y_k z^{k/2}\right]_2^2 . \tag{65}$$

The form of the equation (65) is comparable to the condition (62) if

$$x_0 = x_{-2} = x_2 = 0 .$$

Further, compare the expressions $[\ \ ]_1$ and

$$[\ \ ]_1^2 = 9x_{-3}^2 f^{-3} + 6x_{-3}x_{-1} f^{-2} + (x_{-1}^2 - 6x_{-3}x_{-1}) f^{-1}$$
$$- (2x_{-1}x_1 + 18x_{-3}x_3) + (x_1^2 - 6x_{-1}x_3) f + 6x_1 x_3 f^2 + 9x_3^2 f^3 .$$

Similarity requires first that $9x_{-3}^2 = 9x_3^2 = b_1^3$;

$$x_{-3} = - x_3 = - \bar{x}_3 = - \tfrac{1}{3} b_1^{3/2} \tag{66}$$

and secondly that $6x_{-3}x_{-1} = 3b_1 b_2$, $6x_1 x_3 = 3b_1 \bar{b}_2$ i.e.

(67) $$x_{-1} = -\bar{x}_1 = -\tfrac{3}{2} b_1^{1/2} a_2 \,.$$

Observe that this direct comparison gives for x_{-2}, x_2; x_{-3}, x_3 exactly the values true for optimized P_3-inequality (cf. 10)) and imply the symmetric choice (9). The freedom (36) of u_1 does not come out from the comparison. Finally, in the maximum case $u_1 = a_2$ and the first formula (10) is reduced to (67) in this case.

The above example shows that in order to make an inequality of Grunsky type successful we have 1) to choose the generating function g so that it agrees with the integrated left side of Schiffer's differential equation; 2) to optimize the parameters x_k by minimizing the right side of the inequality.

This leads further to the following conclusions:

1) *a_{2n} and b_1 close to* 0. The P-inequality has every opportunity of being successful, provided that the numbers corresponding to the above $\boldsymbol{v}$ are governed by aid of lower P-conditions. In this connection the unsymmetric choice of x_k:s might be useful (cf. derivation of (16)). — The same hold in general for a_n and b_1 close to 1.

2) *For all* a_n $(n \geqq 4)$ there is an interval of b_1 between 0 and 1 where we have no proper inequality with g fitting with Schiffer's differential equation. — The same holds for a_{2n+1} and b_1 close to 0 $(n = 2, 3, \ldots)$.

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