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ON P-SOLVABLE FUNCTIONS FOR $\Delta u = Pu$ ON OPEN RIEMANN SURFACES

BY

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1. Introduction

1. Let R be a Riemann surface and P a *density*, that is, a C^1 function so that the elliptic partial differential equation

$$\Delta u = Pu \tag{1.1}$$

is invariantly defined on R. We suppose that P is *acceptable* which means that there exists a positive P-superelliptic function ω on R. The situation is introduced in [2]. Especially densities acceptable by the constant one are non-negative densities.

In the theory of harmonic functions there exists the class $N(R)$ of bounded continuous harmonizable functions. This class is used e.g. in the Wiener's compactification of R (Cf. [1] and [6]). Now we intend to construct a similar class with regard to the equation (1.1) on open Riemann surfaces. First we define the P-solvability, the counterpart of harmonizability. Then we present the class $N^P(\omega, R)$ of ω-bounded continuous P-solvable functions as well as its subclass $N^P_\Delta(\omega, R)$ and show their basic properties. Especially we examine their dependence on P in the case where densities are acceptable by the same function ω. The dependence appears to be related to the classification of densities made in [3]. In fact, $N^P(\omega, R)$ is equal to $N^Q(\omega, R)$ if and only if P and Q belong to the same density class. Finally we present a result concerning the dependence of the $N^P(\omega, R)$-compactification of R (Cf. [1]) on the density P.

2. First we present shortly some terms and results used here. The Riemann surface R will always be open. A density P is called *completely acceptable* if it is acceptable and has the Green's function $G_P(R)$ of (1.1) on R. A function u is said to be a *P-solution* in a region K if $u \in C^2(K)$ and it is a solution of (1.1) in K. By $I^P_K(f)$ is meant a P-solution in a region K, $\bar{K}$ being compact, with boundary values f. $\{R_n\}$ is the usual exhaustion of R with regular regions.

A continuous function v is said to be *P-subelliptic* in a region K if to any point $z_0 \in K$ there exists a parametric disc (V_0, z_0), $\bar{V}_0 \subset K$, such that in every disc (V, z_0), $V \subset V_0$, the first boundary value problem has a unique solution and $v(z_0) \leqq I^P_V(v, z_0)$. A function v is said to be

P-superelliptic if $-v$ is *P-subelliptic.* A family F of P-superelliptic functions is called a Perron family if $\{-v \mid v \in F\}$ is a Perron family of P-subelliptic functions (Cf. [2]).

All functions are real valued. A function f is called *ω-bounded* if $|f|/\omega$ is bounded. By $AP_\omega(K)$ is meant the set of ω-bounded P-solutions in a region K, $AP_\omega = AP_\omega(R)$. If P is acceptable by ω, then $\omega_P(K)$, the ω-measure of the region K, is the greatest P-solution in K which is not greater than ω, $\omega_P = \omega_P(R)$. A density P is called *ω-parabolic* on R if $\omega_P \equiv 0$, otherwise *ω-hyperbolic.* According to the usual convention, the density $P \equiv 0$ is considered ω-parabolic on parabolic surfaces and ω_0 is then defined to be zero.

A region L is *normal* if ∂L, the relative boundary of L, is either empty or regular for the Dirichlet problem of (1.1) for every acceptable density. If L is a normal region, P and Q acceptable densities and $u \in C^0(L)$ then we define the linear transformation $T^L_{PQ}u$,

$$(1.2)\quad T^L_{PQ}u(z_0) = u(z_0) + \frac{1}{2\pi} \iint_L (P(z) - Q(z))\, G_Q(L, z, z_0)\, u(z)\, dx dy ,$$

provided that $G_Q(L)$ exists and the integral converges (Cf. [2] and [4]). $T_{PQ}u = T^R_{PQ}u$. If u is a P-solution, then $T^L_{PQ}u$ is a Q-solution in L.

2. *P*-solvable functions

3. Let f be a real valued function on a Riemann surface R and L a normal region. We denote by $\bar{U}_P(L, f)$ the class of P-superelliptic functions s on L for which there exists a compact set $K_s \subset L$ with $s \geqq f$ on $L - K_s$. Similarly, $\underline{U}_P(L, f)$ is the class of P-subelliptic functions s on L for which there exists a compact set $K_s \subset L$ with $s \leqq f$ on $L - K_s$. These classes are either empty or Perron families. In the latter case the functions

$$\bar{u}_P(L, f, z) = \inf \{s(z) \mid s \in \bar{U}_P(L, f)\} ,$$
$$\underline{u}_P(L, f, z) = \sup \{s(z) \mid s \in \underline{U}_P(L, f)\} ,$$

are P-solutions in L with

$$\underline{u}_P(L, f) \leqq \bar{u}_P(L, f) .$$

If these functions coincide, the common function is denoted by $u_P(L, f)$.

A function f on R is called *P-solvable* if $\bar{U}_P(L, f)$ and $\underline{U}_P(L, f)$ are not empty and $\bar{u}_P(L, f) = \underline{u}_P(L, f)$ for every normal region L. If especially $P \equiv 0$, we suppose in addition that L is not parabolic.

The concept of P-solvability is essential for our purposes. Therefore, we consider only open Riemann surfaces. In fact, let us suppose that P is acceptable by ω on a closed Riemann surface. The only existing P-solutions are the constant zero and, if ω is a P-solution, the multiples of ω (Cf. [3]). Therefore the P-solvability of a function is either trivial or not meaningful. That is why the closed Riemann surfaces are not interesting in this connection.

4. We start the examination of P-solvability with a couple of auxiliary results.

Proposition 2.1. *Let f and g be functions on R and L a normal region. If the classes $\bar{U}_P$ and $\underline{U}_P$ in question are not empty,*

$$\underline{u}_P(L\,,g) \leqq \bar{u}_P(L\,,f+g) - \bar{u}_P(L\,,f) \leqq \bar{u}_P(L\,,g)\,,$$
$$\underline{u}_P(L\,,g) \leqq \underline{u}_P(L\,,f+g) - \underline{u}_P(L\,,f) \leqq \bar{u}_P(L\,,g)\,.$$

Proof: Let $s \in \bar{U}_P(L\,,f)$ and $s' \in \bar{U}_P(L\,,g)$. Then $s + s' \in \bar{U}_P(L\,,f+g)$. This implies

$$\bar{u}_P(L\,,f+g) \leqq \bar{u}_P(L\,,f) + \bar{u}_P(L\,,g)\,. \tag{2.1}$$

If then $s \in \bar{U}_P(L\,,f+g)$ and $s' \in \underline{U}_P(L\,,g)$, $f - s' \leqq f + g \leqq s$ in $L - K_s \cup K_{s'}$. Thus $s - s' \in \bar{U}_P(L\,,f)$ and

$$\bar{u}_P(L\,,f) \leqq \bar{u}_P(L\,,f+g) - \underline{u}_P(L\,,g)\,. \tag{2.2}$$

Together (2.1) and (2.2) imply the first statement. The second is proved similarly.

Proposition 2.2. *Let f be a non-negative continuous function on R having a P-superelliptic majorant. If $\bar{u}_P(R\,,f) = 0$, then f is P-solvable.*

Proof: Because $f \geqq 0$, $\underline{u}_P(R\,,f)$ exists and is equal to zero. If $P \equiv 0$ and R is parabolic, then $\bar{u}_P(R\,,f)$ is defined to be zero. In this case every bounded continuous function is harmonizable i.e. 0-solvable (Cf. [6] p. 224). So we exclude this possibility. Let $\{R_n\}$ be an exhaustion of R and L a normal region. It is sufficient to consider the case where $L \cup \partial L$ is noncompact. Let u_n be a continuous P-subelliptic function on $L \cup \partial L - (\partial L) \cap \partial R_n$ so that u_n is a P-solution in $L \cap R_n$ and

$$u_n = \begin{cases} f & \text{on } R_n \cap \partial L\,, \\ 0 & \text{in } L - \bar{R}_n\,. \end{cases}$$

Then $\{u_n\}$ is a non-decreasing non-negative bounded sequence and there exists a P-solution $u = \lim_{n\to\infty} u_n$ with $u = f$ on ∂L.

If now $s \in \bar{U}_P(R, f), s \not\equiv 0$, and if $K_s \subset R_n$, then $u_n - s < f$ on $R_n \cap \partial L$ and $u_n - s \leqq f$ in $L - R_n$. So $u_n - s \in \underline{U}_P(L, f)$ and

$$(2.3) \qquad u = \lim_{n\to\infty} u_n - \bar{u}_P(R, f) \leqq \underline{u}_P(L, f) .$$

On the other hand, $s + u > f$ on ∂L and $s + u \geqq f$ in $L - K_s$. So $s + u \in \bar{U}_P(L, f)$ and

$$(2.4) \qquad u = u + \bar{u}_P(R, f) \geqq \bar{u}_P(L, f) .$$

Because $\underline{u}_P(L, f) \leqq \bar{u}_P(L, f)$, (2.3) and (2.4) show that $u_P(L, f)$ exists and is equal to u. So we have the statement.

5. Every P-solution u on R is P-solvable and $u_P(L, u) = u$ for every normal region L. There also exist other naturally P-solvable functions.

Proposition 2.3. *Every P-superelliptic function on R having a P-subelliptic minorant is P-solvable.*

Proof: Let s be P-superelliptic on R, v its P-subelliptic minorant and L a normal region. Then $s \in \bar{U}_P(L, s)$, $v \in \underline{U}_P(L, s)$ and

$$v \leqq \underline{u}_P(L, s) \leqq \bar{u}_P(L, s) \leqq s .$$

Therefore $\bar{u}_P(L, s) \in \underline{U}_P(L, s)$ and it must be equal to $\underline{u}_P(L, s)$. The proposition is proved.

In the same context we see that every P-subelliptic function on R having a P-superelliptic majorant is P-solvable as well as every ω-bounded P-sub- or P-superelliptic function. Especially the accepting function ω is P-solvable and

$$(2.5) \qquad u_P(L, \omega) = \omega_P(L)$$

for every normal region L.

6. A useful feature is that P-solvable functions form a vector space.

Proposition 2.4. *Let f, g be P-solvable functions and α, β real numbers. Then $\alpha f + \beta g$ is P-solvable. If L is a normal region, then*

$$u_P(L, \alpha f + \beta g) = \alpha u_P(L, f) + \beta u_P(L, g) .$$

Proof: From proposition 2.1 we see that $f + g$ is P-solvable and

$$u_P(L, f + g) = u_P(L, f) + u_P(L, g) .$$

If $\alpha > 0$, then $\alpha s \in \bar{U}_P(L, \alpha f)$ for every $s \in \bar{U}_P(L, f)$ and vice versa. So $u_P(L, \alpha f)$ exists and is equal to $\alpha u_P(L, f)$. Because $u_P(L, -f) = -u_P(L, f)$ the result remains valid for $\alpha < 0$ and it is trivially valid for $\alpha = 0$. These two facts give the proposition.

3. The Banach lattice $N^P(\omega, R)$

7. The class of all P-solvable functions is too large for our purposes. We introduce a smaller subclass of it which has more regularity properties. Let $B(\omega, R)$ be the family of those continuous functions on R which are ω-bounded. We define

$$N^P(\omega, R) = \{f \in B(\omega, R) \mid f \; P\text{-solvable}\} .$$

This class of P-solvable functions is thus a subclass of $B(\omega, R)$. It appears that we can say exactly when these classes coincide.

Theorem 3.1. *$N^P(\omega, R) = B(\omega, R)$ if and only if P is ω-parabolic on R.*

Proof: Let first P be ω-hyperbolic and $\{R_n\}$ an exhaustion of R. Let $f \in B(\omega, R)$, $0 \leqq f \leqq \omega$,

$$f = \begin{cases} \omega \text{ in } \bigcup_{n=1}^{\infty} (\bar{R}_{4n-2} - R_{4n-3}) \\ 0 \text{ in } \bigcup_{n=1}^{\infty} (\bar{R}_{4n} - R_{4n-1}) . \end{cases}$$

If $s \in \bar{U}_P(R, f)$, then $s \geqq f$ in $\bar{R}_{4(n+m)-2} - R_{4n-2}$ for some n and every $m > 0$. Since $s \geqq f = \omega$ on $\partial R_{4(n+m)-2}$ and $\omega \geqq \omega_P(R)$, $s \geqq \omega_P(R)$ in $R_{4(n+m)-2}$ for every $m > 0$. We conclude that $s \geqq \omega_P(R)$ on R. For an $s \in \underline{U}_P(R, f)$ we similarly conclude by considering $\bar{R}_{4(n+m)} - R_{4n}$ that $s \leqq 0$ on R. Thus.

$$\underline{u}_P(R, f) \leqq 0 < \omega_P(R) \leqq \bar{u}_P(R, f)$$

and $f \notin N^P(\omega, R)$.

Suppose then that P is ω-parabolic and take an $f \in B(\omega, R)$, $|f| \leqq M\omega$. We use the standard decomposition $f = f^+ - f^-$; $f^+, f^- \geqq 0$. Then $M\omega \in \bar{U}_P(R, f^+)$ and so $0 \leqq \bar{u}_P(R, f^+) \leqq M\omega$. This implies

$$0 \leqq \bar{u}_P(R, f^+) \leqq M\omega_P = 0 ,$$

unless $P \equiv 0$ in which case it is defined $\bar{u}_P(R, f^+) = 0$. Similarly we get that $\bar{u}_P(R, f^-) = 0$. Then f^+ and f^- are P-solvable by proposition 2.2 and so f, too. Thus $f \in N^P(\omega, R)$.

The theorem is proved.

8. We turn to the regularity properties of $N^P(\omega, R)$. First we introduce some notations. Let f and g be real valued functions and u, v P-solutions. Then we denote

$$f \cup g = \max(f, g), \quad f \cap g = \min(f, g),$$

$$u \vee v = \inf\{s \mid s \ P\text{-solution}, \ s \geqq u \cup v\},$$

$$u \wedge v = \sup\{s \mid s \ P\text{-solution}, \ s \leqq u \cap v\},$$

provided that $u \vee v$ and $u \wedge v$ are P-solutions. So $u \vee v$ is the least P-solution which is a majorant of $u \cup v$ and $u \wedge v$ is the greatest P-solution which is a minorant of $u \cap v$.

A vector space is called a vector lattice if it is closed under the operations $\cup$ and $\cap$ mentioned above. If the space is moreover complete, it is called a Banach lattice.

Theorem 3.2. *$N^P(\omega, R)$ is a Banach lattice under the ω-norm*

$$\|f\|_\omega = \sup_R \frac{|f|}{\omega}.$$

Moreover, if f and g belong to $N^P(\omega, R)$ and L is a normal region,

$$u_P(L, f \cup g) = u_P(L, f) \vee u_P(L, g). \tag{3.1}$$

$$u_P(L, f \cap g) = u_P(L, f) \wedge u_P(L, g). \tag{3.2}$$

Proof: Evidently $N^P(\omega, R)$ is a vector space. Let $f, g \in N^P(\omega, R)$ and let L be a normal region. We show first that $f \cap g \in N^P(\omega, R)$. Because $f \cap g$ is ω-bounded and a minorant of both f and g,

$$\bar{u}_P(L, f \cap g) \leqq \bar{u}_P(L, f) \cap \bar{u}_P(L, g) = u_P(L, f) \cap u_P(L, g).$$

The function $s = u_P(L, f) \cap u_P(L, g)$ is P-superelliptic and has a P-subelliptic minorant. Thus s is P-solvable. According to the definition, $u_P(L, s)$ must be equal to $u_P(L, f) \wedge u_P(L, g)$. So we have

$$\bar{u}_P(L, f \cap g) \leqq u_P(L, f) \wedge u_P(L, g). \tag{3.3}$$

Let then $\varepsilon > 0$ and $z_0 \in R$. We can find an $s \in \underline{U}_P(L, f)$ and an $s' \in \underline{U}_P(L, g)$ such that

$$-\varepsilon < s(z_0) - u_P(L, f, z_0) \leqq 0 ,$$
$$-\varepsilon < s'(z_0) - u_P(L, g, z_0) \leqq 0 .$$

Because the P-subelliptic function

$$(s - u_P(L, f)) + (s' - u_P(L, g)) + u_P(L, f) \wedge u_P(L, g)$$

is a minorant of both s and s', it belongs to $\underline{U}_P(L, f \cap g)$. At the point z_0 is valid

$$-2\varepsilon + u_P(L, f) \wedge u_P(L, g)(z_0) \leqq \underline{u}_P(L, f \cap g, z_0) .$$

Since $\varepsilon > 0$ and z_0 were arbitrary we must have

$$u_P(L, f) \wedge u_P(L, g) \leqq \underline{u}_P(L, f \cap g) . \tag{3.4}$$

Together (3.3) and (3.4) imply that $f \cap g \in N^P(\omega, R)$ and (3.2) is valid.

Similarly we can prove that $f \cup g \in N^P(\omega, R)$ and (3.1) is valid. So $N^P(\omega, R)$ is a vector lattice. It remains to prove the completeness.

Let $\{f_n\} \subset N^P(\omega, R)$ be a Cauchy sequence under the ω-norm. Then there exists $\lim\limits_{n\to\infty} f_n = f$ such that $\lim\limits_{n\to\infty} \|f - f_n\|_\omega = 0$ and $f \in B(\omega, R)$. If now $s \in \underline{U}_P(L, f_n)$, then $s \leqq f_n$ in $L - K_s$ and

$$s - \|f - f_n\|_\omega \omega \leqq \underline{u}_P(L, f)$$

in $L - K_s$, too. Therefore

$$u_P(L, f_n) - \|f - f_n\|_\omega \omega \leqq \underline{u}_P(L, f) . \tag{3.5}$$

In the same way we see that

$$u_P(L, f_n) + \|f - f_n\|_\omega \omega \geqq \bar{u}_P(L, f) . \tag{3.6}$$

Together the inequalities (3.5) and (3.6) imply when n goes to infinity that f is P-solvable and

$$u_P(L, f) = \lim_{n\to\infty} u_P(L, f_n) . \tag{3.7}$$

So $f \in N^P(\omega, R)$ and the theorem is proved.

Corollary 3.3. *Let $\{f_n\} \subset N^P(\omega, R)$ and $\lim\limits_{n\to\infty} f_n = f$ with regard to the ω-norm. Then $f \in N^P(\omega, R)$ and (3.7) is valid for every normal region L.*

9. Together with $N^P(\omega, R)$ we need a subclass of it. We define

$$N^P_\Delta(\omega, R) = \{f \in N^P(\omega, R) \mid u_P(R, f) = 0\} .$$

This class has similar properties as $N^P(\omega, R)$. In fact, from proposition

2.4 and formulas (3.1), (3.2) we see that it is a vector lattice and from (3.7) that it is complete. So we have the following result.

Theorem 3.4. $N^P_\Delta(\omega, R)$ *is a Banach lattice.*

An interesting fact is that $N^P_\Delta(\omega, R)$ coincides with $N^P(\omega, R)$ if and only if the latter coincides with $B(\omega, R)$.

Theorem 3.5. $N^P_\Delta(\omega, R) = N^P(\omega, R)$ *if and only if* P *is* ω*-parabolic.*

Proof: Let P be ω-parabolic and $f \in N^P(\omega, R)$, $|f| \leqq M\omega$. If $P \equiv 0$, then $f \in N^P_\Delta(\omega, R)$ by definition, otherwise it is seen from the inequality $|u_P(R, f)| \leqq M\omega_P = 0$. Because $N^P_\Delta(\omega, R)$ is a subclass of $N^P(\omega, R)$ they must be equal.

Let then P be ω-hyperbolic. Then ω_P, the ω-measure of R, is positive. So it belongs to $N^P(\omega, R)$ but not to $N^P_\Delta(\omega, R)$. Thus these classes are not equal.

10. Finally we remark that $N^P(\omega, R)$ can be presented as a direct sum of $AP_\omega(R)$ and $N^P_\Delta(\omega, R)$.

Theorem 3.6. $N^P(\omega, R) = AP_\omega(R) + N^P_\Delta(\omega, R)$.

Proof: Let $f \in N^P(\omega, R)$ and denote $g = f - u_P(R, f)$. The P-solution $u_P(R, f)$ is ω-bounded and thus belongs both to $AP_\omega(R)$ and $N^P(\omega, R)$. This implies that $g \in N^P(\omega, R)$. Because

$$u_P(R, g) = u_P(R, f) - u_P(R, f) = 0 ,$$

g belongs in fact to $N^P_\Delta(\omega, R)$. So we have a decomposition

$$f = u_P(R, f) - g \tag{3.8}$$

where $u_P(R, f) \in AP_\omega(R)$ and $g \in N^P_\Delta(\omega, R)$. The only common function of $AP_\omega(R)$ and $N^P_\Delta(\omega, R)$ is the constant zero by the definition of $N^P_\Delta(\omega, R)$. Thus the decomposition (3.8) is unique.

4. The dependence of $N^P(\omega, R)$ on P

11. Let Ω be the class of densities which are acceptable by the same function ω. We suppose moreover that Ω has a smallest member W so that ω is a W-solution. This situation emerges e.g. when $\omega \in C^3$ in

which case $W = \Delta\omega/\omega$. If we especially choose $\omega \equiv 1$, then $W \equiv 0$ and Ω is the class of non-negative densities.

In Ω we define a classification of its members by introducing to every $P \in \Omega$ a W-solution ν_{WP}, the smallest W-majorant of $\omega_P(R)$ and by defining

$$N(P) = \{Q \in \Omega \mid \nu_{WQ} = \nu_{WP}\} .$$

All ω-parabolic densities belong to the same class but ω-hyperbolic may be divided into several classes. This classification is connected with the comparison of solution spaces AP_ω. If P and Q belong to Ω, then AP_ω and AQ_ω are strongly isometric if and only if $P \in N(Q)$ (Cf. [3]).

Now we examine in Ω the dependence of Banach lattices $N^P(\omega , R)$ and $N^P_\Delta(\omega , R)$ on P. In the class of ω-parabolic densities this dependence is clear, $N^P_\Delta(\omega , R) = N^P(\omega , R) = B(\omega , R)$ for every density P in this class. Therefore we suppose in the sequel that Ω also contains ω-hyperbolic densities $P , P \not\equiv W$. This condition implies at the same time that every $P \in \Omega$ is completely acceptable (Cf. lemmas 3.3. and 3.5 in [3]).

12. First we prove some auxiliary results in which we do not presume that densities belong to Ω.

Proposition 4.1. *Let P and Q be acceptable densities with $P \leqq Q$ and L a normal region. If f is a non-negative function having a P-superelliptic majorant on L, then*

$$\underline{u}_Q(L , f) \leqq \underline{u}_P(L , f) \ \textit{and} \ \bar{u}_Q(L , f) \leqq \bar{u}_P(L , f) .$$

Proof: For any $s \in \underline{U}_Q(L , f) , s \cup 0$ is P-subelliptic whence it belongs to $\underline{U}_P(L , f)$ and

$$\underline{u}_Q(L , f) \leqq \underline{u}_P(L , f) .$$

For any $s \in \bar{U}_P(L , f) , s \geqq 0$ in L because $f \geqq 0$. So s is also Q-superelliptic and belongs to $\bar{U}_Q(L , f)$. Thus

$$\bar{u}_Q(L , f) \leqq \bar{u}_P(L , f) .$$

In the next result we need the mapping T^L_{PQ} and so we suppose that Q is completely acceptable.

Proposition 4.2. *Let P and Q be acceptable densities so that $P \leqq Q$ and Q is completely acceptable. Let L be a normal region and f a non-negative function so that $\underline{u}_P(L , f)$ is finite. Then the greatest Q-minorant of $\underline{u}_P(L , f)$ is*

$$\underline{u}_Q(L\,,f) = T^L_{PQ}\underline{u}_P(L\,,f)\,.$$

Proof: By proposition 4.1 we have

$$0 \leqq \underline{u}_Q(L\,,f) \leqq \underline{u}_P(L\,,f)\,.$$

This implies by lemma 3.4 in [3] that the greatest Q-minorant of $\underline{u}_P(L\,,f)$ exists and is equal to $T^L_{PQ}\underline{u}_P(L\,,f)$. Because $P \leqq Q$,

$$\underline{u}_Q(L\,,f) \leqq T^L_{PQ}\underline{u}_P(L\,,f) \leqq \underline{u}_P(L\,,f)\,. \tag{4.1}$$

Let then $\varepsilon > 0$ and $z_0 \in L$. There exists an $s \in \underline{U}_P(L\,,f)$, $s \geqq 0$, so that

$$-\varepsilon < s(z_0) - \underline{u}_P(L\,,f\,,z_0) \leqq 0\,.$$

The function

$$s - \underline{u}_P(L\,,f) + T^L_{PQ}\underline{u}_P(L\,,f)$$

is a minorant of s and Q-subelliptic. Therefore, it belongs to $\underline{U}_Q(L\,,f)$ and we have

$$-\varepsilon + T^L_{PQ}\underline{u}_P(L\,,f)(z_0) \leqq \underline{u}_Q(L\,,f\,,z_0)\,.$$

Since $\varepsilon > 0$ and z_0 were arbitrary,

$$T^L_{PQ}\underline{u}_P(L\,,f) \leqq \underline{u}_Q(L\,,f)\,. \tag{4.2}$$

Together (4.1) and (4.2) imply that $\underline{u}_Q(L\,,f)$ is equal to $T^L_{PQ}\underline{u}_P(L\,,f)$. This proves the proposition.

The requirement of non-negativity of the function f can be replaced with the ω-boundness. Then we also get a similar result to the functions $\bar{u}_Q(L\,,f)$ and $\bar{u}_P(L\,,f)$.

Proposition 4.3. *Let P and Q be densities acceptable by ω so that $P \leqq Q$ and Q is completely acceptable. Let L be a normal region and f an ω-bounded function. Then*

$$\underline{u}_Q(L\,,f) = T^L_{PQ}\underline{u}_P(L\,,f)$$

and

$$\bar{u}_Q(L\,,f) = T^L_{PQ}\bar{u}_P(L\,,f)\,.$$

Proof: Let $|f| \leqq M\omega$. By proposition 2.1 and by (2.5)

$$\underline{u}_D(L\,,f) + M\omega_D(L) = \underline{u}_D(L\,,f + M\omega)\,;\ D = P\,.\,Q\,. \tag{4.3}$$

Because $f + M\omega \geqq 0$ we have by proposition 4.2

$$\underline{u}_Q(L\,,f + M\omega) = T^L_{PQ}\underline{u}_P(L\,,f + M\omega)\,. \tag{4.4}$$

On the other hand, T^L_{PQ} is linear and, because $P \leqq Q$,

$$T^L_{PQ}\omega_P(L) = \omega_Q(L) . \tag{4.5}$$

Therefore we have

$$T^L_{PQ}\underline{u}_P(L , f + M\omega) = T^L_{PQ}\underline{u}_P(L , f) + M\omega_Q(L) . \tag{4.6}$$

This, together with (4.3) and (4.4) implies

$$\underline{u}_Q(L , f) = T^L_{PQ}\underline{u}_P(L , f) .$$

The first formula is proved. For the second we notice that $-f + M\omega \geqq 0$. Therefore we have as above

$$\underline{u}_Q(L , -f) = T^L_{PQ}\underline{u}_P(L , -f) .$$

However,

$$\underline{u}_D(L , -f) = -\bar{u}_D(L , f) ;\ D = P , Q .$$

So

$$\bar{u}_Q(L , f) = T^L_{PQ}\bar{u}_P(L , f) .$$

The proposition is proved.

13. We are now ready for preliminary comparison results in the class Ω. The first is a direct consequence of proposition 4.3.

Proposition 4.4. *Let P and Q belong to Ω. If $P \leqq Q$, then*

$$N^P(\omega , R) \subset N^Q(\omega , R) \quad \textit{and} \quad N^P_\Delta(\omega , R) \subset N^Q_\Delta(\omega , R) .$$

If moreover $f \in N^P(\omega , R)$ and L is a normal region,

$$u_Q(L , f) = T^L_{PQ}u_P(L , f) .$$

Then we suppose in the same setting that the function f in question is in the class $N^Q(\omega , R)$ and ask the P-solvability. This point of view is clearly more difficult and it is necessary to require some additional properties. By taking advantage of the classification of densities and the results of [3] we are able to prove the next proposition. It concerns only functions in the subclass $N^Q_\Delta(\omega , R)$ but is quite sufficient to us as a preparatory result.

Proposition 4.5. *Let P and Q belong to the same density class N in Ω. If $P \leqq Q$, then*

$$N^P_\Delta(\omega , R) = N^Q_\Delta(\omega , R) .$$

Proof: By proposition 4.4, $N^P_\Delta(\omega, R) \subset N^Q_\Delta(\omega, R)$. Let us then take an $f \in N^Q_\Delta(\omega, R)$. Because $N^Q_\Delta(\omega, R)$ is a vector lattice we may suppose that $f \geqq 0$. Now we use the results of [3]. By theorem 4.9 in [3], $P \in N(Q)$ implies that the solution spaces AP_ω and AQ_ω are strongly isometric. By theorem 4.10 in [3] and the definition of strong isometry, T_{PQ} is an isomorphism from AP_ω onto AQ_ω and T_{QP} is its inverse mapping. On the other hand, f is ω-bounded and so $\bar{u}_P(R, f)$ exists and belongs to AP_ω. Together with proposition 4.3 all this gives

$$T_{QP}\bar{u}_Q(R, f) = T_{QP}T_{PQ}\bar{u}_P(R, f) = \bar{u}_P(R, f) .$$

Because $\bar{u}_Q(R, f) = 0$ this implies that $\bar{u}_P(R, f) = 0$. Then we see from proposition 2.2 that f is P-solvable. Because $f \geqq 0$, $u_P(R, f) = 0$ and so f belongs to $N^P_\Delta(\omega, R)$. This gives $N^Q_\Delta(\omega, R) \subset N^P_\Delta(\omega, R)$ and so the proposition.

14. The class Ω has an interesting property. If $P \in \Omega$, then all ω-bounded P-solutions are Q-solvable for every $Q \in \Omega$.

Theorem 4.6. *Let $P \in \Omega$. Then*

$$AP_\omega(R) \subset N^Q(\omega, R)$$

for every $Q \in \Omega$. If $u \in AP_\omega(R)$ and L is a normal region,

$$u_Q(L, u) = T^L_{PQ}u .$$

Proof: If P is ω-parabolic, then $AP_\omega(R) = \{0\}$ and the result is obvious. So we suppose that P is ω-hyperbolic. Let $u \in AP_\omega(R)$ and let L be a normal region. Suppose first that $u \geqq 0$. Then u is W-subelliptic and belongs to $\underline{U}_W(L, u)$. On the other hand, $T^L_{PW}u$ exists and belongs to $\bar{U}_W(L, u)$. So

$$u \leqq \underline{u}_W(L, u) \leqq \bar{u}_W(L, u) \leqq T^L_{PW}u . \tag{4.7}$$

Because $T^L_{PW}u$ is the smallest W-majorant of u in L (Cf. lemma 4.1 in [3]),

$$T^L_{PW}u = \underline{u}_W(L, u) .$$

Therefore u is W-solvable and

$$u_W(L, u) = T^L_{PW}u . \tag{4.8}$$

If u has also negative values, we see from the decomposition

$$u = \|u\|_\omega \omega_P - (\|u\|_\omega \omega_P - u)$$

that (4.8) is still valid. So in every case $u \in N^{W}(\omega, R)$. This fact together with proposition 4.4 gives

(4.9)
$$AP_{\omega}(R) \subset N^{W}(\omega, R) \subset N^{Q}(\omega, R)$$

for every $Q \in \Omega$. This shows the first part of the theorem.

For the second part, let $u \in AP_{\omega}(R)$ and let L be a normal region. By (4.9), u is W-solvable. Because $W \leqq Q$, this implies by proposition 4.4

(4.10)
$$u_Q(L, u) = T^L_{WQ} u_W(L, u) .$$

Furthermore by (4.8)

(4.11)
$$T^L_{WQ} u_W(L, u) = T^L_{WQ}\, T^L_{PW} u .$$

Finally, according to lemma 4.8 in [3],

$$T^L_{WQ}\, T^L_{PW} u = T^L_{PQ} u .$$

This, together with (4.10) and (4.11) gives

$$u_Q(L, u) = T^L_{PQ} u .$$

15. Let P and Q be densities in the class Ω. Our main object is to clear up when the Banach lattices $N^P(\omega, R)$ and $N^Q(\omega, R)$ are equal. We have already had some results of this kind. It can be seen that the problem has something to do with the classification of densities in Ω. Now it appears that the answer lies wholly on the classification of densities and so on the strong isometry of the corresponding solution spaces AP_{ω} and AQ_{ω}.

Theorem 4.7. *Let P and Q belong to Ω. Then the following statements are equivalent.*

(a) *P and Q belong to the same density class N.*
(b) $N^P_{\Delta}(\omega, R) = N^Q_{\Delta}(\omega, R)$.
(c) $N^P(\omega, R) = N^Q(\omega, R)$.

Proof: If either P or Q is ω-parabolic, then result follows from theorems 3.1 and 3.5. So we can suppose that densities are ω-hyperbolic.

(a) $\Rightarrow$ (b): If P and Q belong to the same class N, then the density $D = P + Q - W$ belongs, too, by theorem 5.4 in [3]. Because D is a majorant of both P and Q we have by proposition 4.5

$$N^P_{\Delta}(\omega, R) = N^D_{\Delta}(\omega, R) = N^Q_{\Delta}(\omega, R) .$$

(b) $\Rightarrow$ (c): Let $N_\Delta^P(\omega, R) = N_\Delta^Q(\omega, R)$ and take an $f \in N^P(\omega, R)$. By theorem 3.6

$$f = u_P(R, f) + v,$$

where $u_P(R, f) \in AP_\omega \subset N^Q(\omega, R)$ by theorem 4.6 and $v \in N_\Delta^P(\omega, R) = N_\Delta^Q(\omega, R)$. So f belongs to $N^Q(\omega, R)$, too. By changing the roles of P and Q we see that if $g \in N^Q(\omega, R)$ then $g \in N^P(\omega, R)$, too. Thus

$$N^P(\omega, R) = N^Q(\omega, R).$$

(c) $\Rightarrow$ (b): Let $N^P(\omega, R) = N^Q(\omega, R)$ and let $f \in N_\Delta^P(\omega, R)$. Suppose that $f \notin N_\Delta^Q(\omega, R)$. Because $N_\Delta^P(\omega, R)$ is a vector lattice we may suppose that $f \geqq 0$. We form a function $g \in B(\omega, R)$, $0 \leqq g \leqq f$,

$$g = \begin{cases} f \text{ in } \bigcup_{n=1}^{\infty} (\bar{R}_{4n-2} - R_{4n-3}) \\ 0 \text{ in } \bigcup_{n=1}^{\infty} (\bar{R}_{4n} - R_{4n-1}). \end{cases}$$

Let us take an $s \in \bar{U}_Q(R, g)$, $s \geqq g$ in $R - K_s$ and an $s' \in \underline{U}_Q(R, f)$. $s' \leqq f$ in $R - K_{s'}$. If $K_s \cup K_{s'} \subset R_{4n_0-2}$, then $s' \leqq f = g \leqq s$ on ∂R_{4n-2}, $n \geqq n_0$. By the maximum principle of P-subelliptic functions (Cf. [2]), $s' \leqq s$ in R_{4n-2}, $n \geqq n_0$. Therefore the same inequality is valid on R and we have

$$\bar{u}_Q(R, g) \geqq \underline{u}_Q(R, f) = u_Q(R, f) > 0, \tag{4.12}$$

because it was supposed that $f \in N^Q(\omega, R)$, $f \notin N_\Delta^Q(\omega, R)$ and $f \geqq 0$. However, every $s \in \underline{U}_Q(R, g)$ is by the construction of g non-positive whence

$$\underline{u}_Q(R, g) \leqq 0.$$

This, together with (4.12) shows that $g \notin N^Q(\omega, R)$.

On the other hand, $0 \leqq g \leqq f$ and $f \in N_\Delta^P(\omega, R)$ imply

$$0 \leqq \bar{u}_P(R, g) \leqq \bar{u}_P(R, f) = 0. \tag{4.13}$$

So $\bar{u}_P(R, g) = 0$ and by proposition 2.2. $g \in N^P(\omega, R) = N^Q(\omega, R)$.

This is a contradiction and we must have $f \in N_\Delta^Q(\omega, R)$ i.e.

$$N_\Delta^P(\omega, R) \subset N_\Delta^Q(\omega, R).$$

By changing the roles of P and Q we get the inclusion into opposite direction. So

$$N_\Delta^P(\omega, R) = N_\Delta^Q(\omega, R).$$

(b) $\Rightarrow$ (a): Let $N_\Delta^P(\omega, R) = N_\Delta^Q(\omega, R)$ and let $u \in AP_\omega$, $v \in AQ_\omega$. By theorem 4.6

$$u_Q(R, u) = T_{PQ}u, \quad u_P(R, v) = T_{QP}v .$$

Because $T_{PQ}u \in AQ_\omega$, $T_{QP}v \in AP_\omega$, we have

$$T_{QP}\, T_{PQ}u = u_P(R, u_Q(R, u)) .$$

Now

$$u_Q(R, u) - u \in N_\Delta^Q(\omega, R) = N_\Delta^P(\omega, R) .$$

This implies that

$$u_P(R, u_Q(R, u)) = u_P(R, u) = u$$

and so

$$T_{QP}\, T_{PQ}\, u = u .$$

Similarly we see that

$$T_{PQ}\, T_{QP}\, v = v .$$

This means that T_{PQ} is an isomorphism from AP_ω onto AQ_ω and T_{QP} is its inverse mapping. The spaces AP_ω and AQ_ω are then strongly isometric which implies that $P \in N(Q)$.

The proof of the theorem is now complete.

Let especially Ω be the class of densities acceptable by the constant one i.e. the class of non-negative densities. Let moreover $N^0(R)$ be the Wiener's algebra of bounded continuous harmonizable functions (Cf. [6]). If R is parabolic, then trivially

$$N^P(1, R) = B(1, R) = N^0(R)$$

for every density $P \geqq 0$. If R is hyperbolic then according to theorem 4.7

$$N^P(1, R) = N^0(R)$$

if and only if the smallest harmonic majorant of the 1-measure $1_P(R)$ is equal to one. This happens e.g. when.

$$\iint_R P(z)\, G_0(z, z_0)\, dx dy < \infty \tag{4.14}$$

at some point $z_0 \in R$ (Cf. lemma 8.4.3. in [2]).

5. Wiener's *P*-compactification

16. Let R be an open Riemann surface and P a density acceptable by ω on R. We call a topological space $R^*_{\omega,P}$ a *Wiener's P-compactification with regard to* ω if

(a) $R^*_{\omega,P}$ is a compact Hausdorff space,
(b) R is an open dense subspace of $R^*_{\omega,P}$,
(c) every function in $N^P(\omega, R)$ can be continuously extended to $R^*_{\omega,P}$,
(d) $N^P(\omega, R)$ separates points in $R^*_{\omega,P}$.

The compact set $\Gamma_{\omega,P}(R) = R^*_{\omega,P} - R$ is called the *P-ideal boundary of R with regard to ω.*

It is well-known that $R^*_{\omega,P}$ exists but is unique only up to a homeomorphism (Cf. [1] pp. 96—98). Therefore Wiener's compactifications $R^*_{\omega,P}$ and $R^*_{\nu,Q}$ are said to be equivalent if there exists a homeomorphism $h: R^*_{\omega,P} \to R^*_{\nu,Q}$ so that h/R is an identity mapping. So, if P is acceptable by ω, all Wiener's P-compactifications with regard to ω are equivalent.

Let Ω be the same family of densities acceptable by ω as in the preceding chapter. Theorem 4.6 gives us immediately the following result.

Theorem 5.1. *Let P and Q belong to Ω. Then every P-solution in $AP_\omega(R)$ is continuously extendable to $R^*_{\omega,Q}$.*

Another immediate result concerning the equivalence of Wiener's compactifications is given by theorem 4.7.

Theorem 5.2. *Let P and Q belong to Ω. If they are in the same density class N, then $R^*_{\omega,P}$ and $R^*_{\omega,Q}$ are equivalent.*

Let especially Ω be the family of densities acceptable by the constant one i.e. the family of non-negative densities. Then $R^*_{1,0}$ is the harmonic Wiener's compactification of R (Cf. [6] p. 228). Now, if $P \geqq 0$, then every bounded P-solution is continuously extendable to $R^*_{1,0}$. If R is parabolic, then $R^*_{1,P}$ and $R^*_{1,0}$ are equivalent for every density $P \geqq 0$ because in this case Ω consists only of parabolic densities. If R is hyperbolic, then $R^*_{1,P}$ and $R^*_{1,0}$ are equivalent for every density P which belongs to the same class N as the density zero. This happens e.g. when the condition (4.14) is fulfilled.

17. The condition that densities belong to the same density class N was both necessary and sufficient in theorem 4.7. Now we saw that it was sufficient for the equivalency of Wiener's compactifications. It is natural to ask whether it is also necessary. We intend to show that this is not the case. In this purpose we present a situation where the family Ω generates only one equivalence class of Wiener's compactifications.

Thoerem 5.3. *Let ω be an accepting function with $\overline{\lim_{\partial R}}\, \omega = 0$ and* *let Ω be the corresponding family of densities. If P and Q are any densities of Ω, then $R^*_{\omega,P}$ and $R^*_{\omega,Q}$ are equivalent.*

Proof: The accepting function ω has always a continuous extension ω^* on $R^*_{\omega,P}$. By supposition $\omega^*/\Gamma_{\omega,P}(R) = 0$. This implies that $\Gamma_{\omega,P}(R)$ consists of only one point. Moreover, every function $v \in B(\omega, R)$ is continuously extendable to $R^*_{\omega,P}$ by defining $v/\Gamma_{\omega,P}(R) = 0$. Especially this is true for every function in $N^Q(\omega, R)$. So $R^*_{\omega,P}$ is a Wiener's Q-compactification, too. Because all Wiener's Q-compactifications with regard to ω are equivalent we have the statement.

18. Finally we show that theorem 5.3 is not a consequence of theorem 5.2 i.e. there may exist different density classes N in Ω even if $\overline{\lim\limits_{\partial R}}\ \omega = 0$.

As an example we take (Cf. [2], Ex. 6.1.7)

$$R = \{z|\ |z| < 1\}, \quad \omega(z) = (1 - |z|^2)^{1/4}$$

and

$$W(z) = \tfrac{1}{4}(|z|^2 - 4)(1 - |z|^2)^{-2}.$$

Then $\omega \in C^3$ and is a W-solution with $\lim\limits_{|z|\to 1} \omega(z) = 0$. Moreover W is completely acceptable and negative on R. We also take a positive density P_W,

$$P_W(z) = (1 + |z|^2)^{-1}.$$

Now we consider the density $P = W + P_W$. Because clearly $|P - W|\omega^2$ is integrable on R, P belongs to $N(W)$ and is thus ω-hyperbolic (Cf. [3], Corollary 5.2). So the family Ω,

$$\Omega = \{Q|\ Q \text{ a density, } Q \geqq W\}$$

is of the right type. Because $W < 0$, every non-negative density Q belongs to Ω and is ω-parabolic by the usual maximum principle. Therefore such densities do not belong to $N(W)$.

19. We have thus found a situation where Wiener's P- and Q-compactifications with regard to ω are equivalent even if densities P and Q belong to different classes N in Ω. Therefore the condition of theorem 5.2 is not a necessary one.

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