ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

584

QUADRATIC FORMS AND LINEAR TOPOLOGIES

ON COMPLETIONS

BY

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HELSINKI 1974
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doi:10.5186/aasfm.1975.584

ISSN 0066-1953
ISBN 951-41-0198-7

Communicated 10 December 1973

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HELSINKI 1974

Introduction

It has often been pointed out that an infinite dimensional Hilbert space H admits no orthogonal Hamel basis. In fact, H cannot be a subspace of any inner product space with an orthogonal basis. Less trivial is that every inner product space of uncountable algebraic dimension (whether it has an orthogonal basis or not) contains subspaces without any orthogonal basis [3]. Given thus a space E without an orthogonal basis there is the question whether it may or may not be embedded isometrically into a space spanned by an orthogonal basis. In the first case E is much easier to deal with, for then there is a smallest canonical overspace $\tilde{E}$ spanned by an orthogonal basis; $\tilde{E}$ is uniquely determined up to isometry and it is the completion of E with respect to a certain *linear* topology on E canonically associated with the form [7].

Theorem 3 below will provide an answer to the question raised (corollary 1 in section 5); the topological setting enables us furthermore to deal adequately with the situation when the field of scalars is extended (corollaries 6 and 7 in section 5). The proof of theorem 3 is, however, of independent interest and we shall briefly describe here the information which can be extracted from it.

Let α be some fixed ordinal $\geqq 0$. On the vector-space E consider a Hausdorff linear topology $\varkappa_\alpha$ which admits a family $(X_\iota)_X$ of linear zero neighbourhoods with the following two properties:

(I) *the intersection $\bigcap_N X_\iota$ with $N \subset X$ and* card $N \leqq \aleph_\alpha$ *form a basis for $\varkappa_\alpha$,*

(II) *with every finite dimensional subspace $F \subset E$ there is a finite set $N \subset X$ with $F \subset \bigcap_{X \setminus N} X_\iota$.*

Let $(\tilde{E}, \tilde{\varkappa}_\alpha)$ be the completion of $(E, \varkappa_\alpha)$ and $\tilde{X}_\iota$ the closure of X_ι in $\tilde{E}$. It is not difficult to prove that the family $(\tilde{X}_\iota)_X$ enjoys the properties analogous to (I) and (II) in $\tilde{E}$.

The main result is here: If we have

(III) $\dim E/X \leqq \aleph_\alpha$ *for all $\iota \in X$,*

then there exist a partitioning of X,

$$X = \bigcup Y_\varrho , \qquad \text{card } Y_\varrho \leqq \aleph_\alpha$$

such that

$$(*) \qquad \tilde{E} = \oplus H_\varrho , \qquad H_\varrho = \bigcap_{X \setminus Y_\varrho} \tilde{X}_\iota , \qquad \dim H_\varrho \leqq \aleph_\alpha .$$

Let then Φ *be a non degenerate sesquilinear form on* $(E , \varkappa_\alpha)$ *for which orthogonality is symmetric. Assume that* Φ *is connected with the topology* $\varkappa_\alpha$ *by*

(IV) $\bigcap_N X_\iota \perp \bigcap_{X \setminus N} X_\iota$ *for all finite* $N \subset X$.

We show that if (I), (II) and (IV) are satisfied then Φ *has a natural extension* $\tilde{\Phi}$ *on* $(\tilde{E} , \tilde{\varkappa}_\alpha)$ *with*

$$\tilde{\Phi} (\lim \mathcal{F} , \lim \mathcal{G}) = \lim_{\mathcal{F}} \lim_{\mathcal{G}} \Phi(F , G) .$$

Furthermore $(\tilde{X}_\iota)_X$ *satisfies (IV) in* $\tilde{E}$ *with respect to* $\tilde{\Phi}$. *Hence, if (III) is assumed to hold the decomposition (*) is an orthogonal decomposition for* $\tilde{\Phi}$.

Questions dealing with orthogonal bases or orthogonal decompositions into *finite* dimensional subspaces are related to $\varkappa_\alpha$ with $\alpha = 0$. The authors had found it hard at the beginning to characterize in a non trivial fashion spaces which are subspaces of spaces spanned by orthogonal bases; only when they observed that when in search for certain types of bases one should not study families of lines but watch out for hyperplanes did matters become easier.

By introducing the boolean algebra of all sets $S \subset X$ with card $S \leqq \aleph_\alpha$ or card $(X \setminus S) \leqq \aleph_\alpha$ some of our results may be translated in a rather natural fashion into lattice theory. In the light of representation theorems for certain orthocomplemented lattices [6] it seems worthwhile to point out that not only the class of hermitean spaces H may be characterized lattice-theoretically but also, say, the subclass of those H spanned by an orthogonal Hamel basis. We shall treat these matters in another paper.

We finally remark that as a further corollary we obtain the so called log-frame[1] theorem (corollary 8 in section 5) which has proved to be very useful for extending results valid in spaces of countable dimension to orthogonal sums of such spaces ([1], a rather nice example is Korollar 4 zu Satz 3 in [7]). (This theorem bears no relation to the extension principle by Kaplansky in [4].)

1. The topologies

We consider k-left vectorspaces E equipped with reflexive sesquilinear forms $\Phi : E \times E \to k$. If Φ is non degenerate we say that E is semisimple (its »radical» $E \cap E^\perp$ being trivial [2]).

[1] In German: Gattersäge.

The base field k is always assumed to carry the discrete topology. For every ordinal $\alpha \geqq 0$ we define the linear topology $\tau_\alpha(\Phi)$ to have $\{ X^\perp \mid X \subset E \; \& \; \dim X \leqq \aleph_\alpha \}$ as a basis for the zero-neighbourhood filter. Besides these topologies we also consider the so called weak linear topology $\sigma(\Phi)$ with $\{ X^\perp \mid X \subset E \; \& \; \dim X < \infty \}$ as a zero-neighbourhood basis. Any of these topologies is hausdorff if and only if Φ is non degenerate. Φ is always separately continuous. If F is a subspace of (E, Φ) then the weak closure of F is the biorthogonal of F, $\bar{F} = F^{\perp\perp}$. In the sequel we often make use of the following simple facts:

(i) *If the sesquilinear space (H, Ψ) is semisimple and E, F are subspaces with $E^\perp = (0)$ and* $\dim F < \infty$ *then* $(E \cap F^\perp)^\perp = F$.

(ii) *If (H, Ψ) is semisimple and E, F are subspaces with* $\dim F \leqq \aleph_\alpha$ *and E is $\tau_\alpha(\Psi)$-dense in H, then* $(E \cap F^\perp)^\perp = F^{\perp\perp}$.

Proof. Let $\varkappa$ be any one of these topologies: If U is a 0-neighbourhood and E dense in H, then clearly $\overline{E \cap U} = \bar{U}$ (closures). Particularly, if U is of the form $U = X^\perp$ then $U = \bar{U}$ and by the separate continuity of Φ:

$$X^{\perp\perp} = \bar{U}^\perp = (\overline{E \cap U})^\perp = \overline{(E \cap U)^\perp} = (E \cap U)^\perp = (E \cap X^\perp)^\perp .$$

2. Euclidean and preeuclidean forms

Definition 1. *The sesquilinear space (H, Ψ) is called α-euclidean if it is semisimple, of dimension $> \aleph_\alpha$ and an orthogonal sum of subspaces of dimensions at most $\aleph_\alpha$.*

The case $\alpha = 0$ being the most important one we simply say »euclidean» instead of »0-euclidean». Let $H = \oplus_\iota^\perp H_\iota$ be the orthogonal decomposition of a euclidean space, $\dim H_\iota \leqq \aleph_0$. If Ψ is not skew, i.e. if not $\Psi(y, x) = -\Psi(x, y)$ for all $x, y \in H$ (possible only when k is commutative) then a multiple $\Lambda = \Psi\mu$ (for suitable $\mu \in k$) is hermitean with respect to some involution $*$ of k (antiautomorphism of period 2), $\Lambda(y, x) = \Lambda(x, y)^*$ for all $x, y \in H$ [2]. In this case each H_ι is spanned by an orthogonal Hamel basis. If, on the other hand, Ψ is skew, then each H_ι is an orthogonal sum of subspaces G_ν with $\dim G_\nu \leqq 2$. It may still happen that all G_ν are 1-dimensional, i.e. that there is an orthogonal basis. This is possible only when Ψ is not alternate ($\Psi(x, x) = 0$ for all $x \in H$); in particular we must have $\operatorname{char} k = 2$ in this case. If Ψ is alternate, then all G_ν are hyperbolic planes.

Thus, if we discard characteristic 2 for the moment, we may say that a euclidean space is an orthogonal sum of hyperbolic planes in case the form is skew and spanned by an orthogonal Hamel basis in all other instances.

Definition 2. *The sesquilinear space* (E, Φ) *is called* α*-preeuclidean if it is semisimple, of dimension* $> \aleph_\alpha$ *and if it can be embedded isometrically into an* α*-euclidean space* (H, Ψ). *(An isometry is a vectorspace isomorphism which respects the forms.)*

It has been noted repeatedly that hermitean spaces in uncountable dimensions fail in general to admit orthogonal bases. The observation to make, however, is that in the uncountable case subspaces of spaces *with* orthogonal bases fail, in general, to have orthogonal bases themselves. The situation is as follows. First, it is easy to prove that α-euclidean spaces (H, Ψ) are $\tau_\alpha(\Psi)$-complete. Less trivial are the following facts:

(iii) *a subspace* F *(not necessarily semisimple) of a* α*-euclidean space* (H, Ψ) *decomposes orthogonally into subspaces of dimensions at most* $\aleph_\alpha$ *if and only if the closure* $\bar{F}$ *of* F *with respect to* $\tau_\alpha(\Psi)$ *is a subspace of* $F + (F^\perp \cap F^{\perp\perp})$.

(iv) *If* (E, Φ) *is* α*-preeuclidean then there exists a smallest* α*-euclidean overspace* $(H, \tilde{\Phi})$, *uniquely determined up to isometry:* H *is the* $\tau_\alpha(\Phi)$*-completion of the space* $(E, \tau_\alpha(\Phi))$, *it admits a natural extension* $\tilde{\Phi}$ *of* Φ. $\tau_\alpha(\tilde{\Phi})$ *coincides with the completion topology* $\tilde{\tau}_\alpha(\Phi)$.

(v) *The properties of being non* α*-preeuclidean or non* α*-euclidean are absolute, i.e. remain unaffected under extension of the basefield.*

Proofs are carried out in detail for $\alpha = 0$ in [7].

3. Dense subspaces of euclidean spaces

Theorem 1. *If* (E, Φ) *is an* α*-preeuclidean sesquilinear space, then there exists a set* X *with* $\operatorname{card} X = \dim E$ *and a family* $(X_\iota)_{\iota \in X}$ *of semisimple proper subspaces with the following properties:*

(1) $\dim E/X_\iota \leqq \aleph_\alpha$.

(2) $\bigcap_M X_\iota + \bigcap_N X_\iota = \bigcap_{M \cap N} X_\iota$ *for all* $M, N \subset X$ *with* $\operatorname{card} N$, $\operatorname{card} M \leqq \aleph_\alpha$.

(3) $(\bigcap_M X_\iota)^\perp = \bigcap_{X \setminus M} X_\iota$ *for all* $M \subset X$ *with* $\operatorname{card} M \leqq \aleph_\alpha$.

(4) *The linear topology* μ_0 *with the finite intersections* $\bigcap X_\iota$ *as zero-neighbourhood basis is finer than the weak topology* $\sigma(\Phi)$.

(5) *The intersections* $\bigcap_M X_\iota$ *with* $M \subset X$ *and* $\operatorname{card} M \leqq \aleph_\alpha$ *form a basis for the topology* $\tau_\alpha(\Phi)$.

Proof. By (iii) of the previous section we may assume (E, Φ) to be a $\tau_\alpha(\Psi)$-dense subspace of some α-euclidean space (H, Ψ), $\Psi|_{E \times E} = \Phi$. H is $\tau_\alpha(\Psi)$-complete. Let $H = \oplus_X^\perp H_\iota$ be some fixed decomposition with $\dim H_\iota \leqq \aleph_\alpha$. We set $X_\iota = H_\iota^\perp \cap E$. Since $\tau_\alpha(\Phi)$ is finer than the weak topology we have $E^{\perp} = (0)$ in H. Using the decomposition of H we see by a crude combinatorial argument that therefore $\dim E = \dim H$. So $\dim E = \operatorname{card} X$. Furthermore, by (ii) section 1 we see that $X_\iota^{\perp} \cap X_\iota = H_\iota^{\perp} \cap H_\iota \cap E = (0)$, hence X_ι is semisimple.

(1) If U is a linear neighbourhood in a linearly topologized space, then every set $X+U$ in E is both open and closed. In particular $E+H_\iota^{\perp}$ is both dense and closed in H, hence $E+H_\iota^{\perp} = H$. Therefore
$\dim E/X_\iota = \dim E/H_\iota^{\perp} \cap E = \dim (E+H_\iota^{\perp})/H_\iota^{\perp} = \dim H/H_\iota^{\perp} = \dim H_\iota$.

(2) Since $\operatorname{card} N$, $\operatorname{card} M \leqq \aleph_\alpha$ the spaces $\bigcap_N X_\iota = (\bigcap_N H_\iota^{\perp}) \cap E$ and $\bigcap_M X_\iota$ are 0-neighbourhoods for $\tau_\alpha(\Phi)$. Consider the closure K in H of $L = \bigcap_N X_\iota + \bigcap_M X_\iota$. Since E is $\tau_\alpha(\Phi)$-dense in H we have $\bigcap_M H_\iota^{\perp}, \bigcap_N H_\iota^{\perp} \subset K$, so $L \subset \bigcap_N H_\iota^{\perp} + \bigcap_M H_\iota^{\perp} = \bigcap_{M \cap N} H_\iota^{\perp} \subset K$. Hence $K = \bigcap_{N \cap M} H_\iota^{\perp}$ as K is the smallest closed subspace containing L. Furthermore, since $U = \bigcap_N X_\iota$ is a zero neighbourhood, any linear complement D of U in E is discrete. As H is the completion of E we have $\bar{E} = H = \bar{U}+D$. Hence $K = \bar{U} + (D \cap K)$ since $\bar{U} \subset K$. Therefore $K \cap E = (\bar{U} \cap E) \oplus (D \cap K)$ since $D \cap K \subset D \cap E$. This shows that $L = K \cap E$. Thus $L = K \cap E = \bigcap_{N \cap M} X_\iota$ as asserted.

(3) By using (ii) in section 1 (with $\bigcap_{X \setminus P} H_\iota^{\perp}$ in the role of F, so $F^{\perp\perp} = F$) we obtain

$$\bigcap_{X \setminus P} H_\iota^{\perp} = (E \cap (\bigcap_{X \setminus P} H_\iota^{\perp})^{\perp})^{\perp} = (E \cap \bigcap_P H_\iota^{\perp})^{\perp} = (\bigcap_P (E \cap H_\iota^{\perp}))^{\perp}.$$

Intersection with E yields $\bigcap_{X \setminus P} X_\iota = (\bigcap_P X_\iota)^{\perp} \cap E = (\bigcap_P X_\iota)^{\perp'}$ where $\perp'$ is the operation of taking the orthogonal in E.

(4) Let F be a finite dimensional subspace of E. $F \subset \oplus_Q H_\iota$ for some finite $Q \subset X$. Hence $F^{\perp} \supset \bigcap_Q X_\iota$.

(5) We quote (iv) of the previous section.

By (2) and (5) of the previous theorem we have the

Corollary. *Let (E, Φ) be an α-preeuclidean sesquilinear space. The topology $\tau_\alpha(\Phi)$ admits a zero-neighbourhood basis $\mathfrak{A}(0)$ which is a sublattice of the lattice of all subspaces of E and distributive.*

Remark. Since by (3) for every finite dimensional subspace $F \subset E$ we have $F^{\perp} \supset \bigcap_Q X_\iota$ if and only if $F = F^{\perp\perp} \subset (\bigcap_Q X_\iota)^{\perp} = \bigcap_{X \setminus Q} X_\iota$ we see that instead of (4) we may list

(4′) For every finite dimensional $F \subset E$ there is a finite subset $Q \subset X$ such that $F \subset \bigcap_{X \setminus Q} X_\iota$. An alternate formulation is

(4″) The topology μ_0 of theorem 1 renders Φ separately continuous.

For $\alpha = 0$ theorem 1 describes preeuclidean spaces (E, Φ) where (E, Φ) is conceived as a subspace of some (H, Ψ) which decomposes orthogonally into summands of countable dimensions. If Φ is not skew we may, however, assume the space (H, Ψ) to be spanned by an orthogonal Hamel basis. This case is of particular interest:

Theorem 2. *If* $(E\,,\Phi)$ *is a preeuclidean sesquilinear space and* Φ *not skew, then there exists a set* X *with* $\operatorname{card} X = \dim E$ *and a family* $(X_\iota)_X$ *of semisimple hyperplanes* X_ι *in* E *with properties (2), (3), (4), (5) of theorem 1 with* $\alpha = 0$.

4. The main theorems

In this section we shall prove a strong converse of theorem 1 (theorem 3 below). It deals with the existence of extensions $\tilde{\Phi} : \tilde{E} \times \tilde{E} \to k$ of the separately continuous form $\Phi : E \times E \to k$ where $\tilde{E}$ is the completion of E with respect to a certain topology $\varkappa$. The proof makes use of a function M on E defined by a subbasis $(X_\iota)_X$ of $\varkappa$. In order to settle the terminology we state two rather trivial lemmas.

Lemma 1. *Let* $(E\,,\Phi)$ *be a semisimple sesquilinear space and* $(X_\iota)_X$ *a family of proper subspaces such that*

(i) $\bigcap_N X_\iota \perp \bigcap_{X \setminus N} X_\iota$ *for all finite* $N \subset X$,

(ii) *for every finite dimensional subspace* $F \subset E$ *there is a finite subset* $N \subset X$ *such that* $F \subset \bigcap_{X\setminus N} X_\iota$.

The map $M : E \to \mathfrak{P}_e(X)$ *(the finite subsets of* X *) which sends* x *into* $M(x) = \{\, \iota \in X \mid x \notin X_\iota \,\}$ *has the following properties:*

A. *For every* $\iota \in X$ *there is* $x \in E$ *with* $\iota \in M(x)$,

B. $M(x) = \emptyset$ *iff* $x = 0$,

C. $M(x) = M(\lambda x)$ *for* $\lambda \neq 0$,

D. $M(x+y) \subset M(x) \cup M(y)$,

E. *If* $M(x) \cap M(y) = \emptyset$ *then* $\Phi(x\,,y) = 0$.

If we have furthermore

(iii_β) $(\bigcap_N X_\iota)^\perp = \bigcap_{X\setminus N} X_\iota$ *for all* N *with* $\operatorname{card} N < \aleph_\beta$

then M *satisfies*

$\mathbf{E}_\beta$. *For all* $N \subset X$ *with* $\operatorname{card} N < \aleph_\beta$ *we have* $M(x) \subset N$ *if and only if* $\Phi(x\,,y) = 0$ *for all* $y \in E$ *with* $M(y) \cap N = \emptyset$.

Note that $\bigcap_X X_\iota = (0)$ since $\bigcap_X X_\iota \perp \bigcap_\emptyset X_\iota = E$ by (i) and E is semisimple. Lemma 1 has an obvious converse:

Lemma 2. *Let* $(E\,,\Phi)$ *be a sesquilinear space. If a map*

$$M : E \to \mathfrak{P}_e(X)$$

(for some set X *) satisfies* **A** *through* **E** *of Lemma 1 then* $(E\,,\Phi)$ *is semisimple, the set* $X_\iota = \{\, x \in E \mid \iota \notin M(x) \,\}$ *is a proper subspace of* E *and the family* $(X_\iota)_X$ *satisfies (i), (ii) of Lemma 1. If* M *satisfies* $\mathbf{E}_\beta$ *then* $(X_\iota)_X$ *satisfies* (iii_β) *of Lemma 1, and if, in this case, the corresponding* X_ι *are all hyperplanes, then we have*

$$\bigcap_{N} X_\iota + \bigcap_{L} X_\iota = \bigcap_{N \cap L} X_\iota$$

(for all $N, L \subset X$ *with* $\operatorname{card} N, \operatorname{card} L < \aleph_\beta$*) and*

$$\dim E \,/\, (X_{\iota_1} \cap \ldots \cap X_{\iota_n}) = n$$

for all natural n. (Cf. theorem 2.)

We only prove the last assertion as the others are quite obvious. Let $x \in \bigcap_{N \cap L} X_\iota$, $M(x) = \{\iota_1, \ldots, \iota_n\}$. Since

$$M(x) \not\subset (M(x) \cup N \cup L) \setminus \{\iota_1\}$$

there exists by $\mathbf{E}_\beta$ a vector $y \in E$ with

$$M(y) \cap [(M(x) \cup N \cup L) \setminus \{\iota_1\}] = \emptyset$$

and $\Phi(x, y) \neq 0$ and hence $M(x) \cap M(y) \neq \emptyset$. Ergo $M(x) \cap M(y) = \{\iota_1\}$ and so $y \notin X_{\iota_1}$. As we assume that all X_ι are hyperplanes there is $\lambda \in k$ such that $x - \lambda y \in X_{\iota_1}$, i.e. $\iota_1 \notin M(x - \lambda y)$. Since $\iota_1 \notin \emptyset = M(x) \cap (N \cap L)$ we have, say, $\iota_1 \notin L$. Hence $M(y) \cap L = \emptyset$ and $y \in \bigcap_L X_\iota$. Furthermore

$$M(x - \lambda y) \cap N \subset (M(x) \cap N) \cup (M(y) \cap N) \subset (M(x) \cap N) \cup \{\iota_1\} \subset M(x)\,.$$

Therefore $M(x - \lambda y) \cap N \subset \{\iota_2, \ldots, \iota_n\}$. If $\iota \in M(x - \lambda y) \cap N \subset M(x) \cap N$ then $\iota \notin L$ and the step may be repeated. We thus find finitely many $y \in \bigcap_L X_\iota$ and $\lambda \in k$ such that $M\left(x - \sum \lambda y\right) \cap N = \emptyset$, i.e. $x - \sum \lambda y \in \bigcap_N X_\iota$. We have shown that

$$\bigcap_{N \cap L} X_\iota \subset \bigcap_{N} X_\iota + \bigcap_{L} X_\iota\,.$$

The converse inclusion being trivial we have equality. Finally, consider the finitely many hyperplanes $X_{\iota_1}, \ldots, X_{\iota_n}$. If we had

$$\dim E / (X_{\iota_1} \cap \ldots \cap X_{\iota_n}) \neq n$$

then $\dim E/(X_{\iota_1} \cap \ldots \cap X_{\iota_n}) < n$ and $X_{\iota_2} \cap \ldots \cap X_{\iota_n} \subset X_{\iota_1}$ (for suitable numbering). This contradicts what we just proved:

$$E = \bigcap_{\emptyset} X_\iota = X_{\iota_1} + \bigcap_{\iota_2 \ldots \iota_n} X_\iota = X_{\iota_1}\,.$$

We now state our main theorem

Theorem 3. *Let* (E, Φ) *be a semisimple sesquilinear space which admits a family* $(X_\iota)_X$ *of subspaces* $X_\iota \subset E$ *with the following properties:*

(i) $\bigcap_N X_\iota \perp \bigcap_{X \setminus N} X_\iota$ *for all finite* $N \subset X$,

(ii) *For each finite dimensional subspace* $F \subset E$ *there exists a finite* $N \subset X$ *such that* $F \subset \bigcap_{N \setminus X} X_\iota$.

Then for every ordinal $\alpha \geqq 0$ *the form* Φ *has a natural extension* $\tilde{\Phi}$

on the space $(\tilde{E}, \tilde{\varkappa}_\alpha)$ *where* $\tilde{E}$ *is thecompletion of* E *endowed with the linear topology* $\varkappa_\alpha$ *which has the intersections* $\bigcap_A X_\iota$, $A \subset X$, card $A \leqq \aleph_\alpha$, *as a zero-neighbourhood basis. The family* $(\tilde{X}_\iota)_X$ *satisfies the analogues of (i) and (ii) in* $(\tilde{E}, \tilde{\Phi})$.

If $(X_\iota)_X$ *satisfies furthermore*

(*iii*) $\dim E/X_\iota \leqq \aleph_\alpha$,

there is a partitioning $X = \bigcup Y_\varrho$ *such that* $\tilde{E} = \oplus_\varrho^\perp H_\varrho$ *where* $H_\varrho = \bigcap_{X \setminus Y_\varrho} \tilde{X}_\iota$ *and* $\dim H_\varrho \leqq \aleph_\alpha$. *Furthermore* $\tau_\alpha(\tilde{\Phi}) \leqq \tilde{\varkappa}_\alpha$. $\tau_\alpha(\tilde{\Phi}) = \tilde{\varkappa}_\alpha$ *if and only if* $(\tilde{E}, \tilde{\Phi})$ *is semisimple. If such is the case we also have* $\tau_\alpha(\Phi) = \varkappa_\alpha$.

Proof. Step I. We show that the map $M : E \to \mathfrak{P}_e(X)$ of lemma 1 can be extended to a map $\tilde{M} : H \to \mathfrak{P}_e(X)$ which satisfies **A, B, C, D** of lemma 1. Let $\mathcal{F}$ be a Cauchy filter on $(E, \varkappa)$. For each X_ι there is $F_\iota \in \mathcal{F}$ with $F_\iota - F_\iota \subset X_\iota$. Consider the set $\tilde{M} = \{ \iota \in X \mid F_\iota \not\subset X_\iota \}$. For all $x \in F_\iota$ and $\iota \in \tilde{M}$ we have $\iota \in M(x)$. Assume now that $\tilde{M}$ were infinite. Choose some denumerably infinite subset $N \subset \tilde{M}$. Since $\bigcap_N X_\iota$ is a 0-neighbourhood there exists $F \in \mathcal{F}$ with $F - F \subset \bigcap_N X_\iota$. Choose fixed elements $f \in F$ and $f_\iota \in F_\iota \cap F$. If we had $\iota \notin M(f)$ for $\iota \in N$ we should have $\iota \notin M(f_\iota) \subset M(f_\iota - f) \cup M(f)$, a contradiction. So $N \subset M(f)$ which is absurd because $M(f)$ is finite. This proves that $\tilde{M}$ is finite.

We show next that $\tilde{M}$ depends only on $\lim \mathcal{F}$. Let $\lim \mathcal{F} = \lim \mathcal{G}$ and $F_\iota - F_\iota \subset X_\iota$, $G_\iota - G_\iota \subset X_\iota$ ($F_\iota \in \mathcal{F}$, $G_\iota \in \mathcal{G}$). For given neighbourhood X_ι there exists $F \in \mathcal{F}$ and $G \in \mathcal{G}$ with $F - G \subset X_\iota$. Pick $f \in F_\iota \cap F$, $g \in G_\iota \cap G$. The identity $g_\iota = (g_\iota - g) + (g - f) + (f - f_\iota) + f_\iota$ with $f_\iota \in F_\iota$, $g_\iota \in G_\iota$ shows that $f_\iota \equiv g_\iota \pmod{X_\iota}$. From this the assertion follows.

We may thus write $\tilde{M} = \tilde{M}(f)$ where $f = \lim \mathcal{F}$. Since $\tilde{M}(f) = M(f)$ when $f \in E$ we see that $\tilde{M} : H \to \mathfrak{P}_e(X)$ coincides with M on the dense subspace E. $\tilde{M}$ clearly satisfies **A, B, C, D** of Lemma 1.

Step II: We show that for $\mathcal{F}$ and $\mathcal{G}$ Cauchy filters on E the limits

$$\lim_{F \in \mathcal{F}} \lim_{G \in \mathcal{G}} \Phi(F, G) \quad \text{and} \quad \lim_{G \in \mathcal{G}} \lim_{F \in \mathcal{F}} \Phi(F, G)$$

exist and are equal.

Let $M = \tilde{M}(\lim \mathcal{F}) \cup \tilde{M}(\lim \mathcal{G})$. Choose $C \in \mathcal{F}$ with $C - C \subset \bigcap_M X_\iota$ and $x \in C$. Choose $D \in \mathcal{G}$ with $D - D \subset \bigcap_{M \cup M(x)} X_\iota$ and $y \in D$. We shall prove that

$$\lim_{\mathcal{G}} \lim_{\mathcal{F}} \Phi(F, G) = \Phi(x, y) = \lim_{\mathcal{F}} \lim_{\mathcal{G}} \Phi(F, G).$$

For arbitrary $z \in D-D$ we have $M(z) \cap M(x) = \emptyset$, so $\Phi(x, z) = 0$, i.e. $\Phi(x, D-D) = \{0\}$, in other words, $\Phi(x, D)$ is a singleton. $\Phi(x, D) = \Phi(x, y) = \lim_{G \in \mathcal{G}} \Phi(x, G)$. Let now $x' \in C$ and $G_0 \in \mathcal{G}$ with $G_0 - G_0 \subset \bigcap_{M(x-x')} X_\iota$. $\Phi(x'-x, G_0)$ is a singleton. Let $z \in G_0$. $M(z) \cap M(x'-x) \subset \tilde{M}(\lim \mathcal{G})$. On the other hand, since $x' - x \in C-C$, we have $M(x'-x) \cap M = \emptyset$. Thus $M(z) \cap M(x'-x) = \emptyset$, so $\Phi(z, x'-x) = 0$, hence $\lim_{\mathcal{G}} \Phi(x, G) = \lim_{\mathcal{G}} \Phi(x', G) = \Phi(x, y)$. Therefore

$$\lim_{\mathcal{F}} \lim_{\mathcal{G}} \Phi(F, G) = \lim_{\mathcal{G}} \Phi(C, G) = \Phi(x, y).$$

This proves half of the assertion.

There is $C' \in \mathcal{F}$ with $C'-C' \subset \bigcap_{M \cup M(y)} X_\iota$. Choose $x' \in C \cap C'$. As before we obtain

$$\lim_{\mathcal{G}} \lim_{\mathcal{F}} \Phi(F, G) = \Phi(x', y).$$

For all $\iota \in M(y) \setminus \tilde{M}(\lim \mathcal{F})$ we have $C' \subset X_\iota$ and thus $M(x') \cap M(y) \subset \tilde{M}(\lim \mathcal{F})$. Similarly $M(x) \cap M(y) \subset \tilde{M}(\lim \mathcal{G})$. $M(y) \cap M(x-x') \subset M(y) \cap [(M(x) \cup M(x'))] \subset M$. On the other hand, since $x-x' \in C-C$ we have $M(x-x') \cap M = \emptyset$, so $M(y) \cap M(x-x') = \emptyset$ and $\Phi(x-x', y) = 0$, i.e. $\Phi(x, y) = \Phi(x', y)$. Our double limits are therefore seen to be equal.

Step III. We now define $\tilde{\Phi}$ on $\tilde{E}$ by

$$\tilde{\Phi}(f, g) = \lim_{F \in \mathcal{F}} \lim_{G \in \mathcal{G}} \Phi(F, G)$$

where $f = \lim \mathcal{F}$, $g = \lim \mathcal{G}$. Φ is well defined. Φ is sesquilinear with respect to the antiautomorphism used in the definition of Φ. If we pass from Φ to a suitable multiple $\Phi\mu$ then $\Phi\mu$ is skew or hermitean. $\tilde{\Phi}\mu$ is accordingly skew or hermitean (»prolongement des identités»). Hence $\tilde{\Phi}$ is reflexive. (One may also prove reflexivity directly by the explicit construction given in step II.)

Let $\tilde{X}_\iota$ be the completion of X_ι in E. We show that the family $(\tilde{X}_\iota)_X$ satisfies the properties analogous to (i) and (ii) of the theorem, furthermore (iii) provided it holds for $(X_\iota)_X$. The last assertion follows from the remark that $\tilde{E} = \tilde{X}_\iota \oplus L$ for any linear complement L of X_ι in E. The remark also shows that $\iota \in \tilde{M}(x)$ if $x \notin \tilde{X}_\iota$. The converse being trivial we have $\tilde{X}_\iota = \{ x \in \tilde{E} \mid \iota \notin \tilde{M}(x) \}$. We now prove

$$\tilde{M}(f) \cap \tilde{M}(g) = \emptyset \rightarrow \tilde{\Phi}(f, g) = 0.$$

Indeed, in step II we had $M(x) \cap M \subset \tilde{M}(f)$ and $M(x) \cap M(y) \subset \tilde{M}(g)$. Hence $M(x) \cap M(y) \subset M(x) \cap M \cap M(y) \subset \tilde{M}(f) \cap \tilde{M}(g)$. Thus, if $\tilde{\Phi}(f, g) = \Phi(x, y) \neq 0$ then $M(x) \cap M(y) \neq \emptyset$ by (ii) of the theorem. Ergo $\tilde{M}(f) \cap \hat{M}(g) \neq \emptyset$. From this we obtain $\bigcap_{X \setminus N} \tilde{X}_\iota \subset (\bigcap_N \tilde{X}_\iota)^\perp$ for all finite $N \subset X$. Finally, as $\tilde{X}_\iota = \{ x \in \tilde{E} \mid \iota \notin \tilde{M}(x) \}$ we see that the analogue of (ii) holds. Notice that if $\tilde{M}(x) = \emptyset$ then $x = 0$ as $\tilde{\varkappa}_\alpha$ is separated.

Step IV. In order to show that $(\tilde{E}, \tilde{\Phi})$ is α-euclidean under the assumption (iii), we consider elements $x \in \tilde{E}$, $x \neq 0$ which are minimal in the following sense: For any decomposition $x = x_1 + x_2$ with nonzero $x_i \in \tilde{E}$ we have that not both $\tilde{M}(x_1)$, $\tilde{M}(x_2)$ are proper subsets of $\tilde{M}(x)$. These vectors form a set of generators for the space $\tilde{E}$. Let $(f_\iota)_{\iota \in I}$ be a basis of $\tilde{E}$ from this set of generators. Define $A_\mu = \{ \iota \in I \mid f_\iota \notin \tilde{X}_\mu \}$. The crucial point in showing that $(\tilde{E}, \tilde{\Phi})$ is α-euclidean consists in giving a proof for » card $A_\mu \leqq \aleph_\alpha$ ». We give an indirect proof.

Assume that we had card $A_\mu > \aleph_\alpha$ for some fixed μ. Hence there is a natural number $n \neq 0$ and more than $\aleph_\alpha$ among the f_ι with card $M(f_\iota) = n$. By our assumption we have $\mu \in \tilde{M}(f_\iota)$ for all these f_ι. Passing to a subfamily $(f_\iota)_C$ if necessary, we may assume that there is a finite set $M \subset X$, with $\mu \in M \subset M(f_\iota)$ $(\iota \in C)$ and for every $\sigma \in X \setminus M$ we have $f_\iota \notin \tilde{X}_\sigma$ for at most $\aleph_\alpha$ indices $\iota \in C$. We claim that card M $\leqq n-1$. Indeed, since $(\tilde{E}, \tilde{\Phi})$ is Hausdorff we have $\bigcap_X \tilde{X}_\iota = (0)$ so $\bigcap_{X \setminus M} \tilde{X}_\iota$ can be at most $\aleph_\alpha$-dimensional by (iii). If we had card M $= n$ then $f_\iota \in \bigcap_{X \setminus M} \tilde{X}_\iota$ for all $\iota \in C$. But card $C > \aleph_\alpha$ and the f_ι are linearly independent.

We consider the canonical map $\pi : \tilde{E} \to \tilde{E} / \bigcap_M \tilde{X}_\iota$. $\dim \tilde{E} >$ $\aleph_\alpha \geqq \dim (\operatorname{im} \pi)$. We try to find more than $\aleph_\alpha$ many spaces $G_\nu \subset E$, all of them spanned by vectors f_ι and pairwise disjoint, such that all these G_ν have the same image $\neq (0)$ under the map π. To this end wellorder C. If ι_0 is the first element let $I(\iota_0)$ be the shortest initial segment of C such that $(\pi f_\iota)_{I(\iota_0)}$ spans the space $k\{\pi f_\iota \mid \iota \in C\}$. If $I(\sigma)$ is defined for all $\sigma < \tau$ define $I(\tau)$ to be the shortest initial segment of $C \setminus \bigcup_{\sigma<\tau} I(\sigma)$ such that the family $(\pi f_\iota)_{I(\tau)}$ spans the space $k\{\pi f_\iota \mid \iota \in C \setminus \bigcup_{\sigma<\tau} I(\sigma)\}$. Let $G_\nu = k\,(f_\iota)_{I(\nu)}$. We obtain a decreasing nested system $\pi G_{\iota_0} \supset \pi G_{\iota_1} \supset \ldots \supset \pi G_\nu \supset \ldots$ which contains more than $\aleph_\alpha$ spaces G_ν. Since $\dim \pi G_{\iota_0} \leqq \aleph_\alpha$ we obtain a family $(G_\nu)_{\nu \in D}$ with $\pi G_\nu = \pi G_\tau$ for all $\nu, \tau \in D$ and card $D > \aleph_\alpha$. Note that all spaces πG_ν are different from (0) since $\pi f_\iota \neq 0$ for all $\iota \in C$ ($\pi f_\iota = 0$ would say that $f_\iota \in \bigcap_M \tilde{X}_\iota$, i.e. $M(f_\iota) \cap M = \emptyset$; contradiction).

Pick some $\nu_0 \in D$ and some $f_0 \neq 0$ in G_{ν_0}. For each $\nu \in D$ there is $g_\nu \in G_\nu$, $g_\nu \neq 0$, with $\pi g_\nu = \pi f_0$. Set $G_0 = \{ g_\nu \mid \nu \in D \}$. As card $G_0 > \aleph_\alpha$, $\mathcal{G} = \{ G \subset G_0 \mid \text{card}\,(G \setminus G_0) \leqq \aleph_\alpha \}$ is the basis of a filter on E. We show that it is Cauchy. To this end let $\overline{\bigcap_A X_\iota}$ be a typical 0-neighbourhood, $A \subset X$ and card $A \leqq \aleph_\alpha$. First we show that $\overline{\bigcap_A X_\iota} = \bigcap_A \tilde{X}_\iota$. One inclusion being trivial let $x \in \bigcap_A \tilde{X}_\iota$, $x = \lim \mathcal{F}$. For $\varrho \in A$ we have $\varrho \notin \tilde{M}(x)$. Thus if we pick $F \in \mathcal{F}$ with $F - F \subset \bigcap_A X_\iota \subset X_\varrho$ we have $F \subset X_\varrho$. Hence $x \in \bar{F} \subset \overline{\bigcap_A X_\varrho}$. This shows that $\overline{\bigcap_A X_\iota} = \bigcap_A \tilde{X}_\iota$. To show that $\mathcal{G}$ is Cauchy we distinguish two cases. First, $\tau \notin M$: $f_\iota \notin \tilde{X}_\tau$ for at most $\aleph_\alpha$ many ι by our choice of M; so there is a $G \in \mathcal{G}$ with $G \subset \tilde{X}_\tau$ and a fortiori $G - G \subset \tilde{X}_\tau$. Second, $\tau \in M$: As $\pi f_\nu = \pi f_\sigma$ for all $\nu, \sigma \in D$ we have $\pi\,(G - G) = 0$, i.e. $G - G \subset \bigcap_M \tilde{X}_\iota$ for all $G \in \mathcal{G}$; in particular $G - G \subset \tilde{X}_\tau$. Summarizing we have shown that for each $\tau \in A$ there exists $G_\tau \in \mathcal{G}$ with $G_\tau - G_\tau \subset \tilde{X}_\tau$. $G_A = \bigcap_A G_\tau$ is still an element of $\mathcal{G}$ and $G_A - G_A \subset \bigcap_A \tilde{X}_\iota$ so $\mathcal{G}$ is Cauchy. Let $g = \lim \mathcal{G}$. Clearly $g \neq 0$ since there is no $G \in \mathcal{G}$ with $G \subset \tilde{X}_\sigma$ when $\sigma \in M$. It is easy to see that $M(g) \subset M$: For $\iota \in X \setminus M$ there is $G_1 \in \mathcal{G}$ such that $G_1 \subset \tilde{X}_\iota$; there is $G_2 \in \mathcal{G}$ with $G_2 \subset g + \tilde{X}_\iota$ so $G_1 \cap G_2 \subset g + \tilde{X}_\iota$ and $g - g' \in \tilde{X}_\iota$ for some $g' \in G_1 \cap G_2$. $\tilde{M}(g) \subset \tilde{M}(g - g') \cup \tilde{M}(g')$, therefore $\iota \notin \tilde{M}(g)$. We shall show that the decomposition $f_{\nu_0} = g + (f_{\nu_0} - g)$ contradicts the minimality of f_{ν_0}. Since $M(g) \subset M \subsetneqq M(f_{\nu_0})$ it remains to be shown that $M(f_{\nu_0} - g) \subsetneqq M(f_{\nu_0})$. $M(f_{\nu_0} - g) \subset M(f_{\nu_0}) \cup M(g) \subset M(f_{\nu_0}) \cup M = M(f_{\nu_0})$. To prove inequality pick $G_3 \in \mathcal{G}$ with $G_3 \subset g + \bigcap_M \tilde{X}_\iota$. Since $\pi(f_{\nu_0} - G) = 0$ for all $G \in \mathcal{G}$ we have $f_{\nu_0} \in G_3 + \bigcap_M \tilde{X}_\iota$ so $f_{\nu_0} - g \in \bigcap_M X_\iota$. Hence $M(f_{\nu_0} - g) \cap M = \emptyset$. As $\emptyset \neq M \subset M(f_{\nu_0})$ we conclude that $M(f_{\nu_0} - g)$ is a proper subset of $M(f_{\nu_0})$. Thus »card $A_\mu > \aleph_\alpha$» leads to a contradiction.

Step V. We are now ready to prove that $(\tilde{E}, \tilde{\Phi})$ is α-euclidean. Define a symmetric relation $\mathcal{R}$ on X as follows: $\iota \mathcal{R} \iota$ for all $\iota \in X$ and $\iota \mathcal{R} \sigma$ for $\iota, \sigma \in X$ if and only if there exists $\nu \in I$ such that $f_\nu \notin \tilde{X}_\iota$ & $f_\nu \notin \tilde{X}_\sigma$. As we have seen that card $A_\mu = \text{card}\,\{ \nu \in I \mid f_\nu \notin \tilde{X}_\mu \} \leqq \aleph_\alpha$ we conclude that for all $\sigma \in X$ card $\{ \iota \in X \mid \iota \mathcal{R} \sigma \} \leqq \aleph_\alpha$. Let $\mathcal{S}$ be the »transitive closure» of $\mathcal{R}$ ($\iota \mathcal{S} \sigma$ if and only if there exist a natural n and $\iota_1, \ldots, \iota_n \in X$ such that $\iota_1 = \iota$ and $\iota_n = \sigma$ and $\iota_i \mathcal{R} \iota_{i+1}$ for $i = 1, \ldots, n-1$) and $X = \bigcup_\varrho Y_\varrho$ the partitioning of X into the equivalence classes of $\mathcal{S}$. card $Y_\varrho \leqq \aleph_\alpha$. With every class Y_ϱ we associate the space $H_\varrho = \bigcap_{X \setminus Y_\varrho} \tilde{X}_\iota = \{ x \in \tilde{E} \mid \tilde{M}(x) \subset Y_\varrho \}$. Let $\iota \in I$;

there is Y_ϱ with $\tilde{M}(f_\iota) \subset Y_\varrho$ so $f_\iota \in H_\varrho$. Since $(f_\iota)_I$ spans $\tilde{E}$ we have $\tilde{E} = \sum_\varrho H_\varrho$. Therefore each H_ϱ is the span of some f_ι and hence $H_\varrho = k\{ f_\iota \mid \tilde{M}(f_\iota) \subset Y_\varrho \}$. From this follows that $\dim H_\varrho \leqq \aleph_\alpha$. For $\varrho \neq \varrho'$ we have $H_\varrho \perp H_{\varrho'}$ since $Y_\varrho \cap Y_{\varrho'} = \emptyset$. By the same token $H_\varrho \cap H_{\varrho'} = (0)$ so $\tilde{E} = \oplus_\varrho^\perp H_\varrho$. Let $\bigcap_R H_\varrho^\perp$ be a typical $\tau_\alpha(\tilde{\Phi})$ -neighbourhood ($\operatorname{card} R \leqq \aleph_\alpha$).

$$\bigcap_R H_\varrho^\perp = (\sum_R H_\varrho)^\perp \supset \sum_{\complement R} H_\varrho = \{ x \in \tilde{E} \mid \tilde{M}(x) \subset \bigcup_{\complement R} Y_\varrho \}$$
$$= \{ x \in \tilde{E} \mid \tilde{M}(x) \cap \bigcup_R Y_\varrho = \emptyset \} = \bigcap_A \tilde{X}_\iota$$

where $A = \bigcup_R Y_\varrho$. This shows that $\tau_\alpha(\tilde{\Phi}) \leqq \tilde{\varkappa}$. Assume that $\tilde{E}$ is semisimple: if $A \subset X$ and $\operatorname{card} A \leqq \aleph_\alpha$ let A' be the saturation of A (with respect to $\mathcal{S}$). $\operatorname{card} A' \leqq \aleph_\alpha$ and $A' = \bigcup_R Y_\varrho$ for some R with $\operatorname{card} R \leqq \aleph_\alpha$.

$$\bigcap_A \tilde{X}_\iota \supset \bigcap_{A'} \tilde{X}_\iota = \bigcap_R \bigcap_{Y_\varrho} \tilde{X}_\iota \supset \bigcap_{\varrho \in R} \sum_{\varrho \neq \varrho'} H_{\varrho'} \supset \sum_{\varrho' \notin R} H_{\varrho'}$$
$$= (\sum_R H_{\varrho'})^\perp = \bigcap_R H_{\varrho'}^\perp .$$

Therefore $\tau_\alpha(\tilde{\Phi}) \geqq \tilde{\varkappa}$ in this case. Conversely if the two topologies are equal then $\tau_\alpha(\tilde{\Phi})$ is separated, hence $\tilde{E}$ is semisimple. Furthermore $\tau_\alpha(\Phi) = \varkappa_\alpha$ in this case by lemma 5 in [7]. This finishes the proof of theorem 3.

The first three steps of the foregoing proof may actually be carried out under more general assumptions:

Theorem 4. *Let (E, Φ) be a semisimple sesquilinear space which admits a family $(X_\iota)_X$ of subspaces $X_\iota \subset E$ with the following properties. There is an ordinal $\beta \geqq 0$ such that*

(i) $\bigcap_N X_\iota \perp \bigcap_{X \setminus N} X_\iota$ *for all* $N \subset X$ *with* $\operatorname{card} N < \aleph_\beta$,

(ii) *For each subspace $F \subset E$ of dimension $< \aleph_\beta$ there exists $N \subset X$ with $\operatorname{card} N < \aleph_\beta$ and $F \subset \bigcap_{X \setminus N} X_\iota$.*

Then, for every ordinal $\alpha > \beta$ the form Φ has a natural extension $\tilde{\Phi}$ on the space $(\tilde{E}, \tilde{\mu}_\alpha)$ where $\tilde{E}$ is the completion of E endowed with the linear topology μ_α which has the intersections $\bigcap_B X_\iota$, $B \subset X$, $\operatorname{card} B < \aleph_\alpha$, *as a zero-neighbourhood basis. The family $(\tilde{X}_\iota)_X$ satisfies the analogues of (i) and (ii) in $(\tilde{E}, \tilde{\Phi})$.*

If we let $\alpha = \beta$ in the previous theorem its conclusion ceases to be valid. Although the map $x \mapsto M(x) = \{ \iota \in X \mid x \notin X_\iota \}$ (see step I in the proof of theorem 3) may still be extended to a map $\tilde{M} : (\tilde{E}, \tilde{\mu}_\alpha) \to \mathfrak{P}(X)$, we have that » $\operatorname{card} M(x) < \aleph_\beta$ » does not imply any more that card $\tilde{M}(x) < \aleph_\beta$ for all $x \in \tilde{E}$. However, if we let H be the subspace of all $h \in (\tilde{E}, \tilde{\mu}_\alpha)$ with card $\tilde{M}(h) < \aleph_\beta$ then Φ may still be extended on all of H. This is of particular interest in the preeuclidean case ($\alpha = 0$) where we have the following analogue of theorem 3.

Theorem 5. *Let* (E, Φ) *be a semisimple sesquilinear space and let* $(X_\iota)_X$ *satisfy (i) and (ii) of theorem 3; assume furthermore*

(iii′) $\dim E/X_\iota < \aleph_0$ $(\iota \in X)$.

If $\tilde{E}$ *is the completion of* (E, μ_0) *where* μ_0 *has the finite intersections* $\bigcap X_\iota$ *as a zero-neighbourhood basis we let* H *be the subspace of all* $h \in \tilde{E}$ *with finite* $\tilde{M}(h) = \{\iota \in X \mid h \notin \tilde{X}_\iota\}$. Φ *has a natural extension* $\tilde{\Phi}$ *on* H. *There is a partitioning* $X = \bigcup Y_\varrho$, card $Y_\varrho \leqq \aleph_0$ *such that* $H = \oplus^\perp_\varrho H_\varrho$ *where* $H_\varrho = \bigcap_{X \setminus Y_\varrho} (\tilde{X}_\iota \cap H)$ *and* $\dim H_\varrho \leqq \aleph_0$.

Theorem 6. *Assume that in theorem 3 [theorem 5] the family* $(X_\iota)_X$ *has the following additional property*

(iv) $E = X_\iota + \bigcap_N X_\iota$ *for all* $\iota \in X$ *and* $N \subset X \setminus \{\iota\}$ *with* card $N \leqq \aleph_\alpha$ *[* card $N < \aleph_0$*]*.

Then the classes Y_ϱ *in the partitioning of* X *in theorem 3 [theorem 5] are singletons and for all* $\varrho \in X$ *we have* $H_\varrho = \bigcap_{\iota \neq \varrho} \tilde{X}_\iota$, $\tilde{X}_\varrho = \oplus^\perp_{\iota \neq \varrho} H_\iota$, $\tilde{E} = \tilde{X}_\varrho \oplus H_\varrho = \oplus^\perp_X H_\iota$ *[*$H_\varrho = \bigcap_{\iota \neq \varrho} (H \cap \tilde{X}_\iota)$, $H \cap \tilde{X}_\varrho = \oplus^\perp_{\iota \neq \varrho} H_\iota$, $H = (H \cap \tilde{X}_\varrho) \oplus H_\varrho = \oplus^\perp_X H_\iota$, *furthermore* $\dim H_\varrho < \aleph_0$ *and* $\tilde{\mu}_0 \geqq \sigma(\tilde{\Phi})$. $\tilde{\mu}_0 = \sigma(\tilde{\Phi})$ *if and only if* H *is semisimple]*.

Proof. We restrict ourselves to the case of theorem 3. Let $\iota \in X$ be fixed. We show that $\tilde{X}_\iota$ possesses a linear complement L_ι in $(\tilde{E}, \tilde{\Phi})$ with $\tilde{M}(L_\iota) = \{\iota\}$. Choose some fixed basis $(c_\gamma)_C$ of a linear complement of X_ι in E. As the sum $E = X_\iota \oplus k(c_\gamma)_C$ is topological with $k(c_\gamma)_C$ discrete we have $\tilde{E} = \tilde{X}_\iota + k(c_\gamma)_C$. Now for every $N \subset X \setminus \{\iota\}$ with card $N \leqq \aleph_\alpha$ we may, by (iv) of the theorem, pick some fixed linear complement $D(N) \subset \bigcap_N X_\iota$ of X_ι in E and decompose $c_\gamma = c_\gamma(N) + d_\gamma(N)$ with $c_\gamma(N) \in X_\iota$, $d_\gamma(N) \in D(N)$. For each γ the system $(d_\gamma(N))_N$ is a Cauchy net. Set $\tilde{c}_\gamma = \lim d_\gamma(N)$. $M(\tilde{c}_\gamma) = \{\iota\}$ is obvious from the construction of the net. We claim that $k(c_\gamma)_C$ is the required complement L_ι. Indeed, let $x = x_\iota + \sum \lambda_\gamma c_\gamma$ be a typical vector of $\tilde{E}$ $(x_\iota \in \tilde{X}_\iota)$. $x - \sum \lambda_\gamma \tilde{c}_\gamma = x_\iota + \sum \lambda_\gamma (c_\gamma - \tilde{c}_\gamma) = x_\iota + \sum \lambda_\gamma \lim c_\gamma(N) \in \tilde{X}_\iota$. Hence $\tilde{E} = \tilde{X}_\iota + k(\tilde{c}_\gamma)_C$. The sum is direct: If $\sum \lambda_\gamma \tilde{c}_\gamma = \lim \sum \lambda_\gamma d_\gamma(N) \in \tilde{X}_\iota$ then $\sum \lambda_\gamma d_\gamma(N) \in \tilde{X}_\iota \cap E = X_\iota$ for suitable N so that $\sum \lambda_\gamma d_\gamma(N) = 0$ for this N and therefore $\sum \lambda_\gamma c_\gamma = \sum \lambda_\gamma c_\gamma(N) \in X_\iota$ which entails $\lambda_\gamma = 0$ for all γ and hence $x = 0$. We have thus shown that each $\tilde{X}_\iota$ admits a complement L_ι in $(\tilde{E}, \tilde{\Phi})$ with $\tilde{M}(L_\iota) = \{\iota\}$. Since $M(\tilde{X}_\iota) \cap M(L_\iota) = \emptyset$ we have $E = \tilde{X}_\iota \oplus^\perp L_\iota$. Let $x \in \tilde{E}$ and $\iota \in M(x)$.

$x = x_1 + x_2$ with $x_1 \in \tilde{X}_\iota$ and $x_2 \in L_\iota$. Since $M(x_2) = \{\iota\} \subset M(x)$ we see that $\tilde{M}(x_1)$ is a proper subset of $\tilde{M}(x)$. This shows that $x \in \oplus^{\perp}_{M(x)} L_\iota$ and $E = \oplus^{\perp}_{X} L_\iota$. We see in particular that the »minimal» elements f_ι introduced in step IV in the proof of theorem 3 have sets $\tilde{M}(f_\iota)$ which are singletons in this case. Hence the partition constructed there has here but one-element classes $Y_\varrho = \{\varrho\}$ and $H_\varrho = \bigcap_{X \setminus Y_\varrho} \tilde{X}_\iota = L_\varrho$, q.e.d.

Finally a word on the semisimplicity of the space $(\tilde{E}, \tilde{\Phi})$ constructed in the proof of theorem 3. Since $\tilde{\Phi}$ is separately continuous and E a dense subspace with respect to $\tilde{\varkappa}_\alpha$ the radical $\tilde{E} \cap \tilde{E}^{\perp}$ of $\tilde{E}$ reduces to $E^{\perp}$. $E^{\perp}$ need not be trivial. $\tilde{E}$ is semisimple if and only if for every Cauchy filter $\mathcal{C}$ on E with $\lim \mathcal{C} \neq 0$ we have $\mathcal{U} \not\subset \mathcal{C}$ where $\mathcal{U}$ is the zero-neighbourhood filter with respect to the weak topology $\sigma(\Phi)$. It is simpler to discuss matters if we are in the situation where (iii) in theorem 3 can be assumed. Then we have that $\tilde{E}$ is semisimple if and only if $\tilde{\varkappa}_\alpha = \tau_\alpha(\tilde{\Phi})$. Now it is an immediate corollary of theorem 3 that (E, Φ) is α-preeuclidean (cf. corollary 1 in the next section). Thus by (iv) of section 2 we also see that $\tilde{E}$ is semisimple if and only if $\varkappa_\alpha = \tau_\alpha(\Phi)$. We shall obtain an independent proof of this result (cf. corollary 3 in the next section).

5. Corollaries of theorem 3

In this section (E, Φ) invariably is a semisimple sesquilinear space.

Corollary 1. *(E, Φ) is α-preeuclidean if and only if it admits a family $(X_\iota)_X$ of subspaces satisfying (i), (ii), (iii) of theorem 3 [or (i), (ii), (iii′) of theorem 5 when $\alpha = 0$].*

The next corollary gives an alternate proof for (iv) of section 2.

Corollary 2. *If (E, Φ) is α-preeuclidean then there exists a smallest α-euclidean overspace, uniquely determined up to isometry. It is the $\tau_\alpha(\Phi)$-completion $\bar{E}$ of E; the form Φ has a natural extension $\bar{\Phi}$ on E. The completion topology $\bar{\tau}_\alpha(\Phi)$ coincides furthermore with $\tau_\alpha(\bar{\Phi})$.*

Proofs. Let (E, Φ) be α-preeuclidean. $E \subset (H, \Psi)$, $H = \oplus^{\perp}_{X} H_\iota$, with $\dim H_\iota \leqq \aleph_\alpha$, H semisimple, $\Psi|_E = \Phi$. Set $X_\iota = E \cap H_\iota^{\perp}$ ($\iota \in X$). (If $X_\iota = E$ we delete ι in X.) $(X_\iota)_X$ satisfies (i), (ii), (iii) of theorem 3. Let $(\tilde{E}, \tilde{\Phi})$ be the space of theorem 3. By the separate continuity of $\tilde{\Phi}$ we have $\tilde{E}^{\perp} \cap \tilde{E} = E^{\perp}$. $E^{\perp} \cap E = (0)$ so there is a linear complement $\bar{E}$ of $E^{\perp}$ in $\tilde{E}$ which contains E. The projection from $\tilde{E} = E^{\perp} \oplus^{\perp} \bar{E}$ on $\bar{E}$ preserves the forms, so $\bar{E}$ is at once semisimple and an orthogonal sum of subspaces of dimensions at most $\aleph_\alpha$. Let $\bar{\Phi} = \tilde{\Phi}|_{\bar{E}}$. E is $\tilde{\varkappa}_\alpha$-dense in $\bar{E}$, hence $\tau_\alpha(\bar{\Phi})$-dense in E ($\tilde{\varkappa}_\alpha|_{\bar{E}} \geqq \tau_\alpha(\tilde{\Phi})|_{\bar{E}} \geqq \tau_\alpha(\bar{\Phi})$). As an α-euclidean space $\bar{E}$ is $\tau_\alpha(\bar{\Phi})$-complete. Furthermore, by lemma 5 of [7] $\tau_\alpha(\bar{\Phi})|_E = \tau_\alpha(\bar{\Phi}|_E) = \tau_\alpha(\Phi)$ as $\bar{E} \cap E^{\perp} = (0)$. Hence $(\bar{E}, \tau_\alpha(\bar{\Phi}))$ is a realization of the $\tau_\alpha(\Phi)$-completion of E and α-euclidean. It is »contained» in every α-euclidean overspace of (E, Φ) since all these spaces are τ_α-complete. (The assertions relating to theorem 5 follow in the same manner.)

Corollary 3. *Let $(X_\iota)_X$ on (E, Φ) satisfy (i), (ii), (iii) of theorem 3 and let $(\tilde{E}, \tilde{\Phi})$ be the $\varkappa_\alpha$-completion of (E, Φ) of theorem 3. $(\tilde{E}, \tilde{\Phi})$ is semisimple if and only if $\varkappa_\alpha = \tau_\alpha(\Phi)$.*

Proof. If $\varkappa_\alpha = \tau_\alpha(\Phi)$ then we have for the completion topologies $\tilde{\varkappa}_\alpha = \tilde{\tau}_\alpha(\Phi) = \tau_\alpha(\tilde{\Phi})$ by corollary 2. We quote theorem 3.

Corollary 4. *If (E, Φ) admits a family $(Y_\iota)_Y$ satisfying (i), (ii), (iii) of theorem 3 then (E, Φ) also admits a family $(X_\iota)_X$ satisfying properties (1) through (5) of theorem 1.*

Proof. Corollary 1 and theorem 1.

Corollary 5. *(E, Φ) is α-euclidean if and only if E admits a family $(X_\iota)_X$ of subspaces $X_\iota \subset E$ satisfying (i), (ii), (iii) of theorem 3 such that E is complete with respect to the topology $\varkappa_\alpha$.*

Proof. Corollary 1 and Corollary 2.

We now discuss extending the base field. Assume that the division ring k' contains k and admits an extension (antiautomorphism) of the involution $\Theta: k \to k$ responsible for the sesquilinearity of Φ. The group $E' = k' \otimes_k E$ may be regarded as a vectorspace over k and as a vectorspace over k'. In the latter case we talk about the k'-ification E' of E. The form $\Phi': E' \times E' \to k'$ defined by

$$\Phi'\left(\sum_i \lambda_i \otimes x_i, \sum_j \mu_j \otimes y_j\right) = \sum_{ij} \lambda_i \, \Phi(x_i, y_j) \, \mu_j^\theta$$

for $\lambda_i, \mu_j \in k'$ is sesquilinear. Since a suitable multiple $\Phi \mu$ of Φ is skew or hermitean (with respect to a suitable involution which can be extended to a involution on k') the form $\Phi' \mu$ is accordingly skew or hermitean, hence Φ' is reflexive. If $\mathfrak{U} = (U)$ is a 0-neighbourhood filter for some linear topology τ on E, then $\mathfrak{U}' = (k' \otimes U)$ defines a linear topology τ' on E'. Since $(k' \otimes F)^{\perp'} = k' \otimes F^{\perp}$ for all subspaces $F \subset E$ it is clear that $\tau_\alpha(\Phi)' = \tau_\alpha(\Phi')$.

It is trivial that the k'-ifications (E', Φ') are α-euclidean or α-preeuclidean if (E, Φ) has the corresponding properties. Much less trivial is the fact that the properties »non α-euclidean» and »non α-preeuclidean» are absolute as is shown by the next two collaries.

Corollary 6. *If* (E', Φ') *is* α*-preeuclidean, then so is* (E, Φ) .

Proof. By theorem 1 there is a family $(X_\iota)_X$ on E' satisfying (i), (ii), (iii) of theorem 3 such that $\tau_\alpha(\Phi')$ has the intersections $\bigcap_A X_\iota$, $A \subset X$ and $\operatorname{card} A \leqq \aleph_\alpha$, as 0-neighbourhood basis. Since $\tau_\alpha(\Phi)' = \tau_\alpha(\Phi')$ the spaces $k' \otimes (X_\iota \cap E)$ are $\tau_\alpha(\Phi')$ neighbourhoods. (We identify E with its image under $i : x \mapsto 1 \otimes x \in E'$.) So

$$\aleph_\alpha \geqq \dim_{k'} E' / (k' \otimes (X_\iota \cap E)) = \dim_k E / (X_\iota \cap E) .$$

This shows that the family $(Z_\iota)_X$ of subspaces $Z_\iota = X_\iota \cap E$ satisfies (iii); (i) and (ii) are trivially inherited. By corollary 1 (E, Φ) is therefore α-preeuclidean.

Corollary 7. *If* (E', Φ') *is* α*-euclidean, then so is* (E, Φ) .

Proof. If (E', Φ') is α-euclidean, then (E, Φ) is α-preeuclidean by corollary 6. As E' is complete with respect to $\tau_\alpha(\Phi')$ it is easily seen that E is complete with respect to $\tau_\alpha(\Phi)$ (cf. lemma 3 in [7]). Hence (E, Φ) is α-euclidean by corollary 1.

Finally we state

Corollary 8 (Log frame theorem). *Let* (H, Φ) *be* α*-euclidean and* $H = \oplus_X^{\perp} H_\iota$ *some fixed decomposition,* $\dim H_\iota \leqq \aleph_\alpha$. *If* E *is a* $\tau_\alpha(\Phi)$ *- closed subspace of* H *then there exists a partitioning* $X = \bigcup Y_\varrho$ *with* $\operatorname{card} Y_\varrho \leqq \aleph_\alpha$ *such that* $H = \oplus_\varrho^{\perp} G_\varrho$, $G_\varrho = \oplus_{Y_\varrho} H_\iota$ *and* $E = \oplus_\varrho (E \cap G_\varrho)$.

Proof. (E, Φ) is complete with respect to $\tau_\alpha(\Phi)|_E = \varkappa_\alpha$ and the family of all $X_\iota = H_\iota^{\perp} \cap E$ qualifies for theorem 3. (Cf. theorem 3 in [7].)

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References

[1] ALLENSPACH, W.: Erweiterung von Isometrien in alternierenden Räumen. - Inaugural-Dissertation, Philosophische Fakultät II der Universität Zürich. Juris Druck+Verlag, Zürich, 1973.

[2] DIEUDONNÉ, J.: La géométrie des groupes classiques. — Ergebnisse der Mathematik und ihrer Grenzgebiete (Neue Folge) 5. Springer-Verlag, Berlin / Göttingen / Heidelberg, 1955. [§ 6: Formes sesquilinéaires refléxives.]

[3] GROSS, H.: Der Euklidische Defekt bei quadratischen Räumen. - Math. Ann. 180, 1969, pp. 95–137.

[4] KAPLANSKY, I.: Projective modules. - Ann. of Math. (2) 68, 1958, pp. 372–377.

[5] KÖTHE, G.: Topologische lineare Räume. I. - Die Grundlehren der mathematischen Wissenschaften 107. Springer-Verlag, Berlin / Göttingen / Heidelberg, 1960. [Zweites Kapitel: Lineare Räume über beliebigen Körpern.]

[6] MAEDA, F., and S. MAEDA: Theory of symmetric lattices. - Die Grundlehren der mathematischen Wissenschaften 173. Springer-Verlag, Berlin / Heidelberg / New York, 1970. [Section 34: Vector spaces with Hermitean forms.]

[7] OGG, E.: Die abzählbare Topologie und die Existenz von Orthogonalbasen in unendlichdimensionalen Räumen. - Math. Ann. 188, 1970, pp. 233–250.

Printed in January 1975